K. Costello - Topological Conformal Field Theories & the 2D Categorification

Let \( \mathcal{C}(n,m) \) be the category with objects \( \mathbb{Z}_{\geq 0} \) and morphisms \( \mathcal{C}(n,m) = \mathcal{C}(n,m) \), where a morphism \( \Sigma \in \mathcal{C}(n,m) \) is a topological symmetric monoidal category with objects \( \mathbb{Z}_{\geq 0} \) and morphisms \( \Sigma \to \Sigma' \) are natural isomorphisms compatible with the boundary conditions of \( \Sigma \) ensuring that the directed graph of \( \Sigma \) has at least one incoming and one outgoing vertex.

Field theories (G. Segal):

1. Conformal Field Theory: a symmetric monoidal functor \( \mathcal{C} \rightarrow \text{vector spaces} \)
2. Topological case: continuous symmetric monoidal functor to topological spaces or spectra.
3. Linearized version of 2: replace all spaces by their singular chain complexes.

**Def.** Let \( E \) be the category with objects \( \mathbb{Z}_{\geq 0} \) and morphisms \( E(n,m) = E(n,m) \), where a chain complex on \( \mathbb{Z}_{\geq 0} \) is \( \text{differentially graded symmetric monoidal category} \).

**Def.** A closed topological conformal field theory is a functor \( \mathcal{C} \rightarrow \text{chain complexes} \) compatible with differentiation.
\[ i.e. \text{ if } F \text{ is a TQFT, for each } n \geq 0 \]
\[ \text{have chain map } F(n) \text{ of chains} \]
\[ C(n, n) \otimes F(A) \rightarrow F(m) \]

**Example:** \( \Sigma \in M_\text{c}(n, n) \) gives a chain map
\[ F(\Sigma) : F(n) \rightarrow F(m) \]
\[ \mathbb{Z} \rightarrow [0, 1] \text{ } \text{1-parameter family of spheres} \]
\[ \text{gives: } F(\Sigma_t) : F(n) \rightarrow F(m) \text{ degree one} \]
\[ \ldots \text{chain homotopy between } F(\Sigma_{\epsilon_0}) \& F(\Sigma_{\epsilon_1}) \]

Taking \( H_0 \) gives back old definition of TQFT.

Chain complex/quiver \( \leftrightarrow \) stable \( R \)-module theory

(we don't remember the cup product so only \( \text{stably framed} \))

**How to construct these?**

1. Do open analog from ribbon graph decomposition
2. Construct closed from open

**Open version:** \( M_\text{o}(n, n) = \{ \Sigma \in \text{a R.S. with boundary} \]
\[ \Sigma \uparrow \text{ on a disk } \times [0, 1] \text{ with orientation,} \]
\[ \text{subject to boundary \& each component of \( C \) has non-empty \( \partial \)} \]

These form a symmetric monoidal category: glue all
\[ \text{outgoing open boundaries to incoming open boundaries.} \]

\( \mathcal{C} \) = dg symmetric monoidal category given by taking
chairs on \( M_\text{o}(n, n) \). (retinal reell-s Septs)
Find such: open-closed TCFT

\[ \mathcal{OC}_C(\mathcal{C}_0, \mathcal{C}_\infty), (\mathcal{M}_0, \mathcal{M}_\infty) \]

\[ \Rightarrow \mathcal{S} \]

no non-deg component has all of its boundary
closed & "degree 1"

Def: \( \mathcal{OE} \) : chain on \( \mathcal{O}_C \), objects \( \mathbb{Z}_+ \times \mathbb{Z}_+ \)

Technical point: if \( \mathcal{F}: \mathcal{C} \to \mathcal{C} \)

no non- \( \mathcal{F}(n) \subseteq \mathcal{F}(m) \)

\( \mathcal{C} \subseteq \mathcal{OE} \subseteq \mathcal{C} \)

Theorem 1: There is a homotopy equivalence of categories between open TCFTs \( \mathcal{F} \) & Frobenius A∞ algebras \( \mathcal{A}_F \)

2. If \( \mathcal{F} \) is an open TCFT

there is a homotopy universal open-closed TCFT

\( \mathcal{F} \times \mathcal{F} \), this gives an associated

closed theory \( \mathcal{F} \times \mathcal{F} \).

\[ \mathcal{H}(j \times \mathcal{M}_\infty(\mathcal{F}(n)) = \mathcal{H}(\mathcal{M}_\infty(\mathcal{A}_F)) \]

(\( \mathcal{A}_F \) = An algebra associated to \( \mathcal{F} \))

[Note: deformations compatible with Frobenius
structure \( \leftrightarrow \) cyclic topology \( \rightarrow \)

Can make Frobenius A∞ algebra into
dg Frobenius ...]

Corollary If \( \mathcal{A} \) is A∞ Frobenius

there are operators

\[ \mathcal{H}(\mathcal{M}_C(\mathcal{A})) \otimes \mathcal{H}(\mathcal{A}) \]

1. M compact simply connected \( \Rightarrow \mathcal{H}(\mathcal{M}) \)

has structure of A∞ Frobenius algebra
using homological perturbation lemma (transfer from $\mathbb{S}^1$)
$\text{HH}_* (\mathbb{H}(M)) \cong \mathbb{H}_* (L M)$

$H_* (M_{\mathbb{C}a,n}) \otimes \mathbb{H}_* (L M) \otimes \to \mathbb{H}_* (L M) \otimes$

--- this should be the only topology operation at least for M simply connected.

2. X smooth compact projective variety, $G(X) = 0$

$\Rightarrow D^b_{\text{coh}} (X)$ co-vision of derived category of sheaves on X

$\text{Hom}_2 (E, F) = \oplus \text{Ext}^i_2 (E, F)$

Frobenius structure $\Rightarrow$ Serre duality

$\text{Hom}_2 (E, F) \otimes \text{Hom}_2 (F, E) \to k$

Aoo structure

Should have $H H_n (D^b_{\text{coh}} (X)) = \oplus_{i=0}^n H^{-i} (X, \mathcal{O}_X)$

$\Rightarrow$ action of $H_* (M_{\mathbb{C}a,n})$ on $\mathcal{H}^n$

[B-model mirror to part of GW theory]

3. Fukaya category

A compact symplectic $F : \text{Fuk} (X)$ a Calabi-Yau Aoo category

Get operators $H_* (M_{\mathbb{C}a,n}) \otimes H H_2 (F) \otimes \to H H_2 (F) \otimes$

Using GW theory can construct

$H_* (M_{\mathbb{C}a,n}) \otimes H_* (X) \otimes \to H_* (X)$

There is a map $H H_2 (F) \to H_* (X)$

so obvious diagram commutes.

[Kontsevich conjectures: $H H_* (F) \to H_* (X)$; diagonal Lagrangians]
Technical point: A q-affine algebra \( A \) on \( A \) of degree \( d \), need to use chains with coefficient in a local system on \( M(1,1) \).
\[
\text{HC}^*(\mathbb{C}) = \mathbb{C}[[t]]
\]
--- Deformations of \( \mathbb{C} \) as Frobenius algebra are \( \mathbb{C} \)-algebras can make trivial but not necessarily with Frobenius structure.

Proof of Theorem - Geometric part: gen + relations
description of \( \mathbb{C} \) and quasi isomorphic model for \( \mathbb{C} \).

\[
N(r,s) = \{ \Sigma \text{ with boundary, } r \text{ and } s \text{ marked points on } \Sigma \}
\]

\[
\begin{array}{c}
\bigcirc \rightarrow \bigcirc^* \Rightarrow \bigcirc \bigcirc
\end{array}
\]

\( \Sigma \) is related to \( \mathbb{C} \).

\[
N(r,s) = \{ \Sigma \text{ as above with nodes on } \Sigma \text{ distinct from marked pts. } \}
\]

\[
\begin{array}{c}
\bigcirc \bigcirc \\
\text{real versor of node} \rightarrow \bigcirc \bigcirc \\
\text{when } r = s, \Sigma \text{ is an orbifold with corners}
\end{array}
\]

\[
\bar{N}(r,s) \text{ is an orbifold with corners}
\]

\[
\bigcirc \bigcirc \bigcirc \rightarrow \bigcirc \bigcirc
\]

\[
\text{C = } \bar{N}(r,s) \text{ form a category, } \bar{C} \text{ is to } C.
\]

Theorem: Let \( D(r,s) \subseteq \bar{N}(r,s) \) be the set of

\[
\Sigma \in \bar{N}(r,s) \text{ each of whose irreducible components is a disc}
\]

Then the inclusion \( D(r,s) \hookrightarrow \bar{N}(r,s) \) is a weak homotopy equivalence.
The opposite acts together get singular tons.

$D(r,s)$ is a cell complex (satisfies $h$ by topological type).

Each $\Sigma = D(r,s)$ maps a ribbon graph with a vertex for each closed component & edge for each node, tail for each vertex of

$\Rightarrow$ ribbon graph for a ton.

Why? Start with $\Sigma$ has canonical hyperbolic metric with geodesic boundary, from convex

$\Rightarrow$ exponential map $\Sigma \times \mathbb{R}_{\geq 0} \to \Sigma$

Let $T \in \mathbb{R}_{\geq 0}$ be the first rank where $Exp(\Sigma, T)$

is singular. Check only singular at nodes

$\Sigma \times Exp(\Sigma \times \mathbb{R}_{\geq 0}, T) \subset \mathcal{N}$

$\Rightarrow$ deformation retraction of connected surfaces

$\overline{N}(r,s)$ onto its boundary.

... breaks down precisely since for discs

Now decompose the boundary (normalise):

$\Rightarrow$ boundary is parameter""

by moduli of $\Sigma$. Normalize so apply moduli

so get discs

& cylinders, del. with every try
So inductively get weak homotopy equivalence with groups which are closed $\leftrightarrow$ ribbon graphs. Dual to usual pictures.

$D (r, s)$ topological category, with geoecdy $r$ and $s$.

generations $\rightarrow \quad \text{inner product}$

$\text{dim } \geq \text{ inner}$

$\mathcal{O} (r) \equiv$ Stasheff polytree $K_{r+1}$

d (fundamental chain of $\mathcal{O}$) $= \Sigma + \mathcal{O}$ Stability

\[ \rightarrow \text{ A relevant relation.} \]

\[ \mathcal{O} \rightarrow \mathcal{C} \rightarrow \mathcal{C} \quad \text{cobordism hoiogy of} \]

\[ j \times \Pi_{2x} F \quad \text{represent this composed functor by a bifunctor.} \]

Consider $\mathcal{C} - \mathcal{O}$ bicodule

\[ \mathcal{C} (\{0, 0\}, \{0, m\}) \]

only inners open $\&$ only outgoings closed.

\[ \text{glue over here} \quad \rightarrow \quad \text{glue close here} \]

\[ j \times \Pi_{2x} F = \mathcal{C} (\{0, 0\}, \{0, m\}) \otimes F \]

\[ \text{explicitly: augment residue } \mathcal{C} \text{ by } \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \quad \text{(bar resolution), flat regular}. \]
Need to describe \( \mathcal{OC} \) as a right \( \mathcal{O} \)-module:

\[
\overline{\mathcal{O}C} (r,s) = \{ L \text{ with boundary, } \partial L \text{ nested} \}
\]

Notes only on boundary; \( S \) adjacent boundaries are smooth.

This module space is built up from discs & annuli:

\[ \mathcal{O} A (r,s) \subset \overline{\mathcal{O}C} (r,s) \]: all \( \mathcal{O} \)-modules are discs disjoint from adjacent closed boundaries or annuli, with precisely \( 1 \) boundary component closed, adjacent.

\[ \quad \implies \quad \text{description of } \mathcal{OC} \text{ as module over disc category.} \]

relates from \( \mathcal{O} \).

Chain level: freely generated by \( a_1 \),

\[ \quad \implies \quad \text{Hochschild chain complex. } a_0 \otimes ... \otimes a_n \]

Differential: \( d \left( \otimes a_i \right) = \sum \otimes a_{i+1} + \bigotimes \delta \text{ (special) } \)

\[ d (a_0 \otimes ... \otimes a_n) : = \sum a_0 \otimes \bigotimes a_{n+1} \otimes \bigotimes (a_1, a_2, ... a_n) \]

\[ + \bigotimes \bigotimes (a_0, a_1, a_{n+1}) \otimes \bigotimes (a_0, a_1, ... a_n) \]

\[ + \bigotimes \bigotimes a_0, a_1, ... a_n \]