\[ \int_0^1 x^{n-1} \, dx = \frac{1}{n} \text{ for } n \neq 0 \quad F \text{ a field} \quad \text{integers} \]

\[ q = m \quad \text{order} \quad q = \text{negative field} \quad k \text{ field} \]

\[ q = \left| A'(k) \right| \Rightarrow L = \left[ A'_k \right] \quad \text{Lefschetz motive} \]

\[ \int_{k[[1]]} 1 \times 1 \, dx = \frac{L^n (L-1)}{n^{n+1}-1} \]

Replace p-adic integrals by such symbols, whose country parts ...

In general symbol \([X] \) for each variety \( k \)

\[ [X] = [X \times Y] + [Y] \quad Y \subset X \text{ closed} \]

\[ [X][Y] = [X \times Y] \]

be complete on \( \left( \sum_{n=0}^{\infty} L^{-n} k \text{ should be in our ring} \right) \)

\[ \Rightarrow k_0 (\text{Var}_k) \left[ \mathbb{L}^{-1} \right] \]

Theory born 12/7/95 Kohlhaush, Delo-Loes. 96-99

Geometric motive integration: \( X(C) \Rightarrow X(C[[1]]) \)

2000-2003 arithmetic motive integration (Delo-Loes)

2004-... constructible motive integration (Chodras-Loes)

\[ \begin{array}{c}
\text{Constructible MS} \\
\text{specialize} \\
\text{arith. MS} \\
\text{specialize} \\
p = 2, 3, 5, \ldots \\
p = \text{adic} \\
\end{array} \]

(\text{also rigid adic vers})
Arithmetic ms

Objects: 1st order formulas "x^3 + y^2 = 1"
"∃y: y^3 = x" rather than sets of solutions-

A Boolean combination of formulas

can interpret formulas in different rings, not universal use.

Logical symbols: A, E, E, V, N, I (revela) =

p-adic symbols: 0, 1, +, x, 8

Values Sp. 0, +, <, mi
(no product here, avoid indeterminacy results)

Ex: SL_n(G): X = (x_ij) dot X=1, val x_ij > 0

Ex: conjugacy class of X = (x_ij): ∃g s.t.
9xg^{-1} = (x_{i,j})

Quantifier elimation: ∃X: x^2 + ax + b = 0
⇔ q^2 - 4b = 0 (almost all p)
⇔ val(a^2 - 4b) ≡ 0 mod 2

∃z, z^2 = q(a^2 - 4b)

Replace p-adic quantifiers by residue field quantifiers.
1. Use quasifinite to remove pseudo
generators
\[ \text{Pretensor } \Rightarrow \text{ remove integral generators} \]
\[ \text{Galois sheaves } \Rightarrow "\ \text{ residue field }" \]

So end up with constructible sets in alg. geometry

2. resulting formula are \( \eta \in \text{AN}(k[[t]])_{\text{max}} \),

\[ \lim_{m \to \infty} \frac{[\eta_{11}], \eta}{m} = \text{ arithmetic value of } \eta. \]

[Uniform quantifier elimination: indep of underlying field.
Also in clear notice up to valued equalizers
answer indep of elimination procedure]

\[ \left\{ \xi \in \text{Arith} \mid \text{Vol}(\{x \in V \mid \xi \in F(x)\}) < 1 \right\} \]

\[ F = \mathbb{Z}_p \quad \text{almost all } p \quad \text{(Derf - Long)} \]

\[ \text{Constructible } \mathbf{M}^! \]

- Integers can have parameters
- "\over k((t)) not just \( k[[t]] \)]
- \( X_{k((t))} \times X_k \times \mathbb{Z}_p \), not just \( X(k[[t]]) \)
- First order formulas
- No completions of rings required:
- "all geometric series are summed"
Formulas for subassigns: subsets of points of \( X_k(\mathfrak{c}(1)) \times X_k(\mathbb{Z}) \) for any field extensions, not nec. subfunctors.

I do: integration = \( \int \) over \( k\)-sets.

- I dim d < time. + Fubini

Quantifier elimination breaks space into cells, on each truth value is constant (P.J. John, Dorell, Pos, Collins...)

- if can integrate over cells, can integrate a general.

Applications (pre-constructible \( \mathfrak{c}_f \))

1. Lifting orbital integrals from char \( p \) to char \( \mathbb{C} \).
   - Fundamental Lemma (G.K.N., Laurant-Lago )

Can we lift this to char \( \mathbb{C} \)?

\( \mathfrak{c}_f \) varies in all char, answers independent of field

\( \Rightarrow \) arguments for almost all \( p \), \( \Rightarrow \) (by global arguments)

get full statement.

Laurent-Lago prove for unitary \( \gamma \) in char \( \mathbb{C} \).

Heyer-Curry-Jani.

Theorem. Let \( \gamma(x,s) \) locally constant formula,

\( s \subseteq \Lambda / \mathbb{Z} \), affine

assumes projective to formula only in \( s \) variables.

(i.e. parameters over residue field).

\( \Rightarrow \) volume depends on \( \mathcal{F} \) only through \( \gamma \).

(got formula for volume)

So wherever fundamental lemma gives by such identity \( \Rightarrow \)

indep of field.
Problems: integrals of fundamental lemma depend on pradisc parameters. Show the integrals depend only through residues and $p$.

Cor: For good elts in classical groups $\Rightarrow$ fund. lemma in pos. char $\Rightarrow$ fund. in char $0$.

Waldspurger: $\text{F. L. in pos. char } \Rightarrow \text{F. L. in char } 0$ in complete generality.

2. (J. Gordon) Characters of Reductive Groups.

$G = SL(2n)$ or $SO(2n+1)$

$\Gamma = T^* \mathfrak{g}$ Deligne-Lusztig Character

$\pi$ irreducible, general position $\Rightarrow + R_{\pi} X < \text{Max}$

(reps. that exist for any field)

Ind$_k^G(\chi) \quad \text{rep. of } G(k), \text{ F. p. in rep. field } F$

$\otimes$ w character on $V$ invariant set

Theorem (Gordon) Let $\Gamma$ be a char. function of a locally closed del. set $\Rightarrow$ further char. notice Major results on the character distribution on $\Gamma$ for almost all $p$.

Fundamental Problem

$k$ field char 0, $S$ definable sub. set $\subseteq \mathbb{A}^n_k \times \mathbb{A}^n_k$

$\mathbb{T} \rightarrow S$ character regular

Claras - Lusztig: $I_{S} \mathbb{G}(\mathbb{Z}) \subseteq \mathbb{G}(\mathbb{Z})$

abelian subgroup of positive cohomology

$\mathbb{G}$ (finite) $S$-integrable

$\mathbb{G}$ (finite)
(4) Reductive theory, defined on regular $S$-sets of any reductive Lie algebra / local fields

Ex.

Steinberg characters, Shalika germs for nilpotent orbits, characters of distinguished unipotent representations of nilpotent orbits, orbital integrals, on formal groups in fields.

Problem / Conjecture: for each such $G$, $F$ constructible function $G^0$ char on regular semisimple loci is almost all $p$-specializers to actual prime value.

- i.e., all basic objects of p-adic rep theory on geometric, constructible.