M. Hopkins - Algebraic Topology & Differential Forms

$\Sigma \times S$ family of Riemann surfaces over $S$

$x \in H^3(\Sigma \times S)$

$\int_{\Sigma} x \in H^1(S; \mathbb{Z}) = \text{homology class of maps } S \to S'$

Might want to use for lagrangian etc - need to define for homology class to actual map...

Speak of more refined invariants, mixing forms & topology...

Differential Functions

$M$ smooth manifold, $X$ topological space +
choice of real cocycle, i.e. $\mathbb{Z}^n(X; \mathbb{R})$

A differential function $M \to (X, x)$ is a triple

$(c, h, w) : c : M \to X$ map

$h \in C^{n-1}(M; \mathbb{R})$

$w \in \Omega^n(M)$ s.t. $df = c - \omega$

Singular cocycle $h$ pretty discrete/combinatorial

but we want to make such functions into a space, talk about Analities ----------
take formal/combinatorial approach to spaces...

Combinatorial model for spaces:

$\text{Y space, capture most topological invariants of } \text{Y from the collection of sets:}$

$\text{sing}_0 \text{Y} = \{ \text{points of } \text{Y} \}$

$\text{sing}_1 \text{Y} = \{ \Delta^1 \to Y \}$

$\text{sing}_2 \text{Y} = \{ \Delta^2 \to Y \}$

just as sets

Tells us how to map any simplicial complex into $\text{Y}$

- gives structure of patching

- i.e. simplicial set

describe space combinatorially in terms of its simplices
Differential flasque space: \((X,\mathcal{E})^M\) with \(n\)-simplices

given by functors \((c,h,a): M \times \Delta^n \rightarrow (X,\mathcal{E})\)

No new info here beyond ordinary flasque space --
extra info: have a filtration on this space:

\[ \text{filt}_k (X,\mathcal{E})^M = \text{space with } n\text{-simplices } (c,h,a): M \times \Delta^n \rightarrow (X,\mathcal{E}) \]
\[ \text{s.t. } c_0 = c_1 1(M) \otimes \mathcal{E}^0(\Delta^n) \]

filtration by \(K\)-nerves corepresented in \(\Delta^n\).

Fundamental groupoid introduced by Reidemeister

Objects = parts
Morphisms = paths
Co-positions = 2-simplices

Auto-morphisms = \(\Pi_1\)

E.g. \(X = DP^2 = BU(1), z \in Z^2(DP^2,\mathbb{R})\)
representing \(c_1:\)

Study \(\Pi_1 \text{filt}_2 (X,\mathcal{E})^M\):

Objects = \((U(1))\text{-bundle over } M\), \([\text{with curvature}]
\]
\(c,h,a): M \rightarrow (X,\mathcal{E})\)
\(c_0 = \mathcal{E}^2(M)\)
\(h\) represents \(c_1\) of bundle \([a \mapsto \text{curvature}]
\]
\([\text{integrality properties are captured in fact that we're}
\text{pulling best forms form } X\ldots\])

Morphisms: [Parallel transport] sections

maps are just principal \(UC(1)\) bundle maps, transport along internal / boundary
Composition Horoptic maps of principal bundles are included.

Get see Tηs as just usual Maps $(M, (P^\infty))$.

But now introduce the fiber: $T\eta \times \mathbb{T}^1$, $(x; t)^M$ gives Tη Map $(M, (P^\infty))$, homotopy classes of principal bundle maps get usual topology.

Objects still same, $U(1)$ bundle + comedia
Maps = still principal bundle maps.

In $\mathbb{T}^1$, can't put arbitrary comedia over this 2-simplex to get arbitrary homotopy like we did before....

But curvature form now has at most a 1-form on $\Delta$ - so we need maps must be equal: can't integrate from on 2-simplex...

Maps now not up to homotopy, just usual morphisms of principal bundles: Tη fiber is correct category of principal bundles with comedia....

Objects principal $U(1)$ bundle + comedia.
Maps: comedia on parameter space $M^t$ internal.

but curvature form $\omega \in \Omega^2(M \times [0,1])$ must be only on $M$ part, $\omega \in \Omega^2(M) \times \Omega^1([0,1])$ & closed $\Rightarrow$ constant along $[0,1]$.

Maps are now horizontal (comedia preserving) maps of principal bundles!
(comedia preserved by this rather fairly holonomy of type can't change due to restriction on curvature $\omega$ in $\mathbb{T}^1$...
Eilenberg–Maclane space \( X = K(\mathbb{Z}, n) \)

\[ \pi_0 K(\mathbb{Z}, n) = \{ [M, K(\mathbb{Z}, n)] \} = H^n(M; \mathbb{Z}) \]

\[ \pi_0 \text{fil} H_0(X; i)^m = \text{Cheeger–Simons characteristic class} \]

\[ H^{n-1}(M) \rightarrow H_{D}^n(M) \rightarrow \left[ \mathbb{C}^{\infty}, \text{integer periods} \right] \rightarrow 0 \]

- Number closed 2-form representing class \& zero-dim info.

\[ \pi_0 \text{fil} H_0(X; i) \]

More generally \( X \) space representing some other cohomology theory \( E \).

\[ \Rightarrow \text{ differential version of } E \text{-cohomology } \overline{E}^n(M) \]

**Differential K-theory**: K-theory of vector bundles w/ 5-structure

Cheeger–Simons \( H^{n-1}(M) \text{ homomorphism } \mathbb{Z}_{n+1}(M) \xrightarrow{\chi} \mathbb{R}/\mathbb{Z} \)

\[ \chi = \frac{\omega + \omega^*(M) + \bar{\omega} + \omega^*(M)}{n} \text{ s.t. } \chi(2n) = \int \omega \]

Define \( Z(n) = \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \cdots \rightarrow \mathbb{R}^{n-1} \)

\[ H^n_{\partial}(M) = H^n(M; \mathbb{Z}(n)) \]

Chain: \( C(n)^k(M) = (c, h, \omega) \text{ s.t. } \omega \in L^n(M) \)

\[ \delta(c, h, \omega) = (dc, h - \omega + c, d\omega) \]

& \( \omega = 0 \text{ if } k < n \)
i.e. closed if \( \leq \) is closed & \( \Rightarrow \) is \( c_0 - c \)

\( m \) th cohomology group of \( (c_0)^*(M) \) is \( H^{m,n}(M) \).

Third POV see ring structure vs. non-theoretical POV

"Ordinary differential cohomology: \( H^{m,n}(M) \) come from differential forms into Eilenberg-MacLane spaces.

All usual apparatus is present: cup products, integration, pushforward/pullback.

\( G \) compact Lie group, \( \{c \in \mathbb{Z}^4(BG; \mathbb{R}) \text{ "level"} \}

\( M \) principal \( G \)-bundle w/ connection \( \rightarrow \) diff form

\( M \rightarrow (BG, \mathbb{Z}) \) Chern-Weil theory

(we're not being only formal anymore here classes
not "corrected" itself... or even curvature form itself
except in U(1) case)

Fix a principal \( G \)-bundle \( P \) on \( M \) \( \Rightarrow \) maps
\( \mathcal{A} \rightarrow \text{fib}_0(BG, \mathbb{Z})^M \) \( \mathcal{A} \) = space of cons. on \( P \).

Evaluation gives diff form
\( M \times \mathcal{A} \xrightarrow{c} (BG, \mathbb{Z}) \)
\( \dim M=2 \) \( \int (c \wedge \phi, \theta, \omega) \) integral is a differential form \( \mathcal{A} \rightarrow (CP^\infty, 2) \)

\( \text{i.e. } U(1) \) bundle with connect\( \text{ed } \) or moduli space \( \mathcal{A} \)

\( \Rightarrow \) classical Chern-Simons theory (refreen to sections
of \( \text{discretization} \) of \( \text{Gaudzi}) \).
Space of closed n-forms on $M$ as a simplicial set, k-simplices $\Delta^k (M \times \Delta^n)$:

$\Omega_c^k (M \times \Delta^n)$: Simplicial abelian group $c \mapsto$ chain complex

$\Omega_c^0 (M \times \Delta^n)$: $\sum_{i=2} c_i$; $\Omega_i (M) \rightarrow \Omega_{i-1} (M)$

$\Omega_c^1 (M \times \Delta^n)$: $\int_L$; $\Omega^1 (M)$

$\Omega_c^2 (M \times \Delta^n)$: $\int_{X^2}$; $\Omega^2 (M)$

$\Omega_c^n (M)$: $\Omega^n (M)$

Stokes theorem $\Rightarrow$ under $\int_L$, difference of face ages becomes de Rham differential. Space of closed n-forms is really the truncated de Rham complex of $M$.

Similarly, "space" of n-cocycles on $X$ with values in $A$:

$C^0 (X; A) \rightarrow \cdots \rightarrow C^n (X; A)$

Space of differential functions $M \rightarrow (K(\mathbb{Z}, n), i)$:

$C^0 (M) \rightarrow \Omega^*$

$C^* (M; \mathbb{Z}) \rightarrow C^* (M; \mathbb{R})$

$(0)^* (M) = C^*(M; \mathbb{Z}) \times C^*(M; \mathbb{R}) \times \Omega^*$

$\delta (c, h, w) = \delta c + \delta h + \delta w$

Complex of differential cocycles:

Cohomologies of all related by long exact Mayer-Vietoris.

$H^0 (\Omega^* (M)) \rightarrow H^* (\Omega^* (M)) \rightarrow H^* (M; \mathbb{Z})$

Filtration: replace $\Omega^*$ by $\Omega_{\leq n}$, $C(A)^* (M) \rightarrow \Omega^*_{\leq n}$

$C^* (M; \mathbb{Z}) \rightarrow C^* (\mathbb{R}; \mathbb{R})$
Wee n nonzero classes: \( H^{k}(M,\mathbb{R}) \quad k \geq 1 \)
\( H^{k}(M,\mathbb{R}/\mathbb{Z}) \quad k \leq n \)

\[ H(n) = \{ H^{k}(M,\mathbb{R}/\mathbb{Z}) \}
\]
\( \Lambda^{n} = \{ \{ x \circ \omega \in H^{n}(M,\mathbb{R}) : \omega_{\ast} c_{1} \} \in H^{n}(M,\mathbb{R}) \} \)

\[ H(n)^{\ast} (M) = \text{differential characters of } \mathcal{C} \text{ - class} \]

\[ H^{2}(M) = \text{class of } \mathcal{U}(1) \text{ bundles with connection} \]

Map to \( \Lambda^{2} : \text{assign curvature } + i C_{1} \),
kernel is flat line bundle.

Work with category of \( \mathcal{U}(1) \) bundle with connection →
work with cochain complex itself, not its cohomology.

Simplicial set of differential forms \( M \to (KZ, \mathbb{k}, \mathbb{Z}) \)
\( \sim \) filtration \( n \) (as simplicial abelian group)
\( \sim \) transitive chain complex

\[ \cdots \to C(n)^{k}(M) \to C(n)^{k-1}(M) \to C(n)^{k-2}(M) \to \cdots \]

- Very abelian theory, modeled on
- \( E = M \) space \( KZ, \mathbb{k} \), we'll not less abelian

\( \overline{V} \) vector bundle over space \( X \),
\( \overline{V} : \text{The complex of } \overline{V} ( = 1 \text{st cohomology of } X \text{ cplx) } \)

Classifying space for vector bundle is a manifold, so
we can talk about smoothness & transversality in the
context of maps into \( X \) by classifying maps.
Assume \( X \) of dim \( d \).

\[ \text{Map} (M \times \Delta^{d}, \overline{V}) \text{ simplicial set model for space Maps}(M, \overline{V}) \]

\[ \text{transverse map} (M \times \Delta^{d}, \overline{V}) \text{ transverse to } 0 \text{-slice} \]
Map is a homotopy equivalence of simplicial sets.
Not nec. true of local space of maps, but free in simplicial setting...

\[ \text{e.g., } M = pt, \ V = \mathbb{R} \ \mapsto \mathbb{R} \ \text{trivial} \ \text{map} (\ pt, \mathbb{R}^1) = \mathbb{R}^1 - \{0\} \]

Simplicially those have paths transverse to 0, so do corresp. right topology!

(Thm.)

Transverse map \( f \) \( \mapsto \) Submanifold \( \Sigma \subset M \times \Delta^k \)

+ map \( \Sigma \rightarrow X \) classifying the normal bundle of \( \Sigma \)

That of \( X \) as same classifying space, eg for spin bundle...

Get manifold with boundary \( \Sigma \rightarrow \Sigma M \times \Delta^k \)

Get simplicial set. 0-simplices = manifolds in \( X \) 1-simplices = cobordisms in \( X \) 2-simplices = cobordisms between cobordisms...

Contains some homotopy information as the function space again...

**Differential forms as topological field theory**

\[(X, 2) \quad 2 \in \mathbb{Z}^k(X, Z) \]

\[ M \xrightarrow{(\Omega^1, \omega)} (X, 2) \Rightarrow (c_2, \nabla, \omega) \in \mathbb{Z}(d)^c(M) \]

\( X \) takes care of integrality: forms \( c_i \) to be integral, in fact pulled back from \( X \), might store complicated relations among cohomology classes.
\[ M \rightarrow \text{map}(G) \Rightarrow \]
\[ M^{d-1} \times S \rightarrow (X, 2) \quad \text{gives} \]
\[ \mathbb{Z}(d)^{d} (M^{d-1} \times S) \Rightarrow \int_{M} (c \circ (\sigma, (1, 0))) \in \mathbb{Z}(d)^{1} (S) \]
\[ \Rightarrow \quad \text{class in } H(G)^{1} (S) = M_{\text{rep}} (S, U(1)) \]
\[ \text{class is } e^{-2\pi i \int_{S} a}, \text{ where } a \text{ is topological} \]
\[ \text{form in } \text{a Lagrangian} \]
\[ M^{d-2} \times S \rightarrow (X, 2) \Rightarrow \mathbb{Z}(d^{2} (M^{d-2} \times S) \xrightarrow{\int_{M}} \mathbb{Z}(d^{2}) (S) \]
\[ = U(1) \text{ bundle on } M \]
\[ Q \rightarrow Q \times S \rightarrow (X, 2). \Rightarrow \mathbb{Z}(2)^{2} (S \times \Delta) \]
\[ \text{is isomorphism of principal bundles} \]

Hermann line associated to any d=2 manifold
\[ 2 \text{ isomorphisms between these lines for any abelian} \]
\[ \text{can get interplay relations among thesecocycles class} \]
\[ \text{captured by } X \]

Example
\[ X = S \rightarrow \mathbb{Z}^{3} (X) \Rightarrow WZW \text{ theory} \]
\[ X = B \rightarrow \mathbb{Z}^{4} (X) \Rightarrow \text{chern simons term} \]

\[ M \rightarrow B \Leftrightarrow \text{principal } G \text{-bundle on } M \]

Can reduce into differential forms using a connection on bundle — i.e. differential forms into string classsify geometric data.

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**A Geometric Story**

\[ M = \text{Spin}^c \text{ manifold, } \not\exists \text{ Spin }^c \text{ Dirac operator.} \]
Degree 4 rapport: \((\text{index } \beta)_4 = \frac{c^2}{8} - \frac{p_1}{24} = \chi(c)\)

\(\beta = c\) of Spin\(^c\) bundle. This is not an integer cochain even though gives integers on Spin\(^c\) 4-manifold.

\(\text{not encoded in differential K-Theory cochain, will need to pass to other cochains theories that deal with reality.}\)

Variation of Spin\(^c\) structure:

\[ X(c-2x) - X(c) = \frac{x^2 - x\beta}{2} \]

\(2d\) TFT

\[ \chi(c) \quad \text{field theory on space of Spin\(^c\) structures} \]

\[ \text{input: 2-d cohomology theory on 3-mfld, writes in} \]

\(\text{Lagrangian, 2-mfld will define a line bundle on space of} \chi)\)

\[ \text{Euler class and integral} \quad \text{def of} \beta \]

Degree 6 rapoport: \((\text{index } \beta)_6 = \frac{c^3 - p_1 \cdot c}{48} = \chi(c)\)

\[ \Rightarrow 5d\text{ TFT, with fields 2-form / Spin\(^c\) structure } \]

\[ K(c+2x) - K(c) = \frac{x^3}{6} + \frac{c x^2}{2} + \frac{(3c^2 - p_1)}{24} x \]

For M-theory 5-brane encoded \(\frac{x^2 - x\beta}{2}\)

\[ \text{where } x \text{ is a 4-form (differential 4-cycle)} \]

\[ \text{- no obvious index theory interpretation must relate to 4-form} \]

\[ M\text{-theory action } \frac{x^3}{6} + \ldots \text{ on 11-manifold} \]

Diakonov - Freund - Mere, within describe these

\[ \text{field theory via Eq: index theory} \]

\[ \text{We will describe a different more topological approach.} \]

\[ \text{"Nonabelian nature of these theories"} \]

\[ \text{A. 3-d TFT assn. to Pfaffian of} \beta \]
$M^3 \rightarrow \text{point in } U(1)$

$M^2 \rightarrow \text{Kermit's line}$

cobordism $\rightarrow$ unitary map of Heegaard lines

class group of 2-nbhs $\rightarrow$ \( \otimes \) of Heegaard lines

"non linear" $\rightarrow$ product in \( U(1) \)

get symmetric monoidal structure, in fact Picard category;

groupoid with symmetric monoidal \( \otimes \) s.t. every object

invertible

C Picard category $\Rightarrow A = \text{Pic}(C)$ set of sources/ends,

\[ B = \prod \text{aut}(e) \]

automorphisms of any other object \( a \) is naturally \( \cong \text{aut}(e) \)

- tensor with identity map of \( a^{-1} \).

[ e.g. \( C \) = field groupoid of \( G / G \) with homotopy commutative group structure. ]

as \( e \) $\Rightarrow$ flip $\alpha \circ \alpha \xrightarrow{\text{flip}} \alpha \otimes \alpha$ $\Rightarrow$ end of

\[ B = \text{aut}(e) \]

\[ \Rightarrow \text{rep } A \otimes \mathbb{Z}/2 \rightarrow B \]

kinematic of \( e \)

for \( n \geq 2 \)

\[ [K(A, n), K(B, n+2)] = \text{Hom}(A \otimes \mathbb{Z}/2, B) \]

\( \text{(Eilenberg-MacLane) } \)

\[ H^{n+2}(K(A, n); B) \]

Homology fiber \( X \rightarrow K(A, n) \)

$\Rightarrow$ kinematic

\[ \pi_n(X) = A \]

\[ \pi_{n+1}(X) = B \]

Picard categories $\Leftrightarrow$ spaces with only 2 connected homotopy sets

only data is kinematic

\[ \text{eg } \mathcal{C} = \text{Kermit lines} : \quad \prod \mathcal{C} \text{ trivial, } \prod \mathcal{C} = U(1) \]
Suppose $\Sigma$ RS with non-binding spin structure --
mod 2 index of $\Theta$ not 0.

draw as $\uparrow$ spin $\uparrow$

Compute k-invariant

Compare with opposite real cobordism $\mathcal{S}$

$\Rightarrow$ opposite cobordism

$\Rightarrow$ 3-manifold, calculate

mod 2 index $\eta_\Sigma \in H_2(\Sigma \times S^1) = -1$ $g$-invar. $\cdots$

$\Rightarrow$ nontrivial k-invariant for this theory --
so not quite cohomological, nice Eilenberg-MacLane
spaces ... nonabelian.

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Were trying to calculate a nonabelian factor from
the spin cobordism category to Hamilton lines
but get contradiction: Hamilton lines have 0 k-invariant
spin cobordism category doesn't, so can't map those
two Picard categories --- so need to go to
graded Poincare lines for our field theory.

[ $k$-invariant versus spectral q-theory ]