I. Aspects of the usual CFT.

1. Standard geometric formula: $X/\mathbb{F}_p$ smooth projective curve

Complement: $\text{Pic} \ X \rightarrow \text{Gal}(\mathbb{F}_p)$

Reciprocity: If $f \in \mathbb{F}_p(\mathcal{O})$, div $f = \Sigma n_i \cdot \xi_i$, then

\[ \prod \mathbb{F}_p(\mathcal{O}) = 1 \]

\[ \Rightarrow \text{homomorphism} \quad \text{Pic} \ X \rightarrow \text{Gal}(\mathbb{F}_p) \]

\[ 0 \rightarrow \text{Pic}^0(\mathcal{O}) \rightarrow \text{Pic} \ X \rightarrow \mathbb{Z} \rightarrow 0 \]

\[ \text{(Hensel's lemma)} \]

Theorem: (1) $\text{Pic}^0(\mathcal{O}) = \text{Gal}(X_{\text{unr}}/\mathcal{O})$ not as unnatural not

(2) $\text{Pic} \ X = W^{ab} \subset \mathbb{Q}(\mathcal{O}) - \overline{\mathbb{Q}}$ well-gr "

(i) not quite canonical: instead splitting of $\overline{\mathbb{Q}}$ canonically look

at $\overline{X}/\mathcal{O}$ & construct its core ... (i) needs a choice

of an $\mathbb{F}_p$-point of $X$.

If $Y \rightarrow X$ ramified at $D = \mathcal{O}$ (ramified subscheme $\mathcal{O}$)

\[ \Rightarrow \mathbb{F}_p(\mathcal{O}) \neq \{0\} \]

\[ f \equiv 1 \mod D \Rightarrow \prod \mathbb{F}_p(\mathcal{O}) = 1 \]

\[ \text{Pic}^0(\mathcal{O}) \quad \text{relative Picard group} \quad \text{quotient at } Y \text{ as above}

\[ \downarrow \quad \text{by bundles trivialized on } Y \]

\[ \text{Gal}(X/\mathcal{O}) \]

\[ \lim_{\mathcal{O}^e/k(\mathcal{O})} \rightarrow \text{Pic}^0(k(X)) \]

2. Self-duality of Jacobian:

$k$ any field, $X$ curve over $k$, Pic$^0(\mathcal{O})$ abelian variety.

Classical: it is self-dual, $- \text{the bundles of degree } 0$ on itself.

Better: Picard stacks $\overline{\text{Pic}}(X)$ (any degree) with $\otimes$

Then this stack is self-dual: isomorphic to tensor-wraps $1$ hom $\otimes (\text{Pic } \mathcal{O}, \text{G}_n \text{-Tors})$

multilineal in bundle

\[ \text{Z of degree is dual to } \text{G}_n \text{ of automorphisms} \]

on $\text{Pic}^0$.

\[ \text{...} \]
Deligne pairs \[ \text{Pic}_X \times \text{Pic}_X \rightarrow \text{Gam}_- \text{Tors} \]
\[ L, M \rightarrow \langle L, M \rangle \] (1-dim vector space)

Construction for \( L, M \) very ample:
\[ H^0(L) \otimes H^0(M) \rightarrow \mathcal{V} = \{ (s, t) | \text{it has common zero} \} \text{ irreducible hypersurface} \]
\[ \Rightarrow \mathcal{V} \text{ has irreducible equation - a homogeneous polynomial} \]
\[ R = R(s^t) : R(C_S) = 0 \Leftrightarrow \text{has common zero}. \]

- \( R \) is defined only up to scalar
\[ \Leftrightarrow \] canonical 1-dim vector space \( \langle L, M \rangle \) where control \( R \) takes values

(True for any irreducible hypersurface in a linear space: collection of level sets of equations)

Deligne notation \( \langle s, t \rangle \) for \( R(s^t) \) by multiplicativity.

Meaning:
\[ \text{family of curves, } L, M \text{ line bundle on } X \]
\[ s \rightarrow \langle L, M \rangle \text{ line bundle on } S \]
\[ c_1(\langle L, M \rangle) = \sum_{s \in S} c_1(L) \cdot c_1(M) \]

3. Local version: Cartier-Carreere self-duality of \( \text{Gam}(R) \)

- group in scheme \( \text{Gam}(R) \times \text{Gam}(R) \rightarrow \text{Gam} \)
\[ f \mapsto \left( \log f, \text{deg} f \right) \]

(Rule resultant as Steinberg symbol: Resultant of \( f, 1-f = 1 \) no common zero)

For \( K \) a field obtain tame symbol
\[ \{ f, g \} = (-1)^{\text{deg} f \cdot \text{deg} g} / \text{gcd} f(0) \]

General \( x \in X \) \( U = X \setminus \{ x \} \), have \( \text{Pic}_C(U) \) compact support - true group scheme \( \text{triv near } x \Rightarrow \text{Pic}_C U \rightarrow \text{Pic} X \)
\[ \text{Pic} U = \text{Pic} X / \text{Pic}_x X \leftarrow \text{line bundles triv outside } x \]

"Core induced category \( \text{Pic}^e \)"

Then \( \text{Pic} U \) & \( \text{Pic}^e U \) are dual

\[ \text{exact sequence} \]
\[ k \text{[finite]}^* \rightarrow \text{Pic}^e U \rightarrow \text{Pic} X \]
\[ \text{Pic}_x X \rightarrow \text{Pic} X \rightarrow \text{Pic} U \]
\[ k \text{[finite]}^* \\ k \text{[finite]}^* \]

\[ H^2(G, \mu_n) = 0 \oplus \mathbb{Z} \]

M finite G-module. $\Rightarrow$ Disjoint union given by cup product $H^i(G, M) \times H^{2-i}(G, \text{Hom}(M, \mathbb{Z})) \to \mathbb{Z}$.

5. Local CFT: $K$ local as before.

\[ W^a_b(k) = K^* \text{ Witt group} \]

Poincaré duality explanation: $m \in \mathbb{Z}$. PD: Kumar $W_{ab} = H_1(\text{Gal}(k, \mathbb{Z}/m)) = H^1(\text{Gal}(k, \mathbb{Z}/m)) = K^*/(K^*)^m$.

II. Aspects of the higher-dimensional CFT

1. "Standard" formulation: $X/F_q$ smooth projective, dim $X = n$.

Y. unramified Abel: $n \to F_x$ as before

For every curve $C \to X$, reciprocally law $\Rightarrow$ homomorphism $CH_0(X) \to Gal(F_x/F)$. $\Gamma$ (reciprocal law, relates sim of curves).

Theorem $CH_0(X) \cong W(X)$ (unramified part of $\Gamma$).

$H^i(X, K_n)$: analog of $\text{Pic} = H^1(X, O_X)$.

$K_n = \text{Skeleta of } U \to K_n(\text{O}(U))$.

Y. ramified: $\to$ form some $CH_0(X, D)$

$D$ (hyperplane) - relations came from generalized Zariski of Cond $C(D$

(limit statement on Gal's group holds, estimating conductors is hard...).

2. "Self (n+1)-ality" of Pic $X$. $X/k$ dim $= n$.

$A_0 X \times \text{Pic } X \times \ldots \times \text{Pic } X \to \text{Gm-Top}$

$\text{lo} \to l_0 \ldots l_n \to <l_0, \ldots, l_n>$

via result of n sections.

$X$ family of n-dim varying. $<l_0, \ldots, l_n> \subset \text{Pic } S$

\[ \prod C_i(l_i) \]

Block: To $CH_0(X)$ deg 0 part have extension that gives the duality...
\[(Pic \ X)^n \rightarrow \mathbb{C}^h \ldots \text{would like to replace}
\]
\[Pic \ X \times Pic \ X^n \rightarrow \mathbb{C}^h \text{ for any pairing Pic} \& \mathbb{C}^h \ldots
\]
This pairing will EV Albanese kernel.

2. Local fields and ideals in \( n \)-dimensions (Peskun, Brillion)

Say \( n=2 \), \( X \) surface/k curve

\[
\Rightarrow \text{finite reduction on } k(X) = C_2 \Rightarrow M_2
\]

\[
\Rightarrow \text{complete } k(X)^, \text{ complete discrete valent field}
\]

with residue field \( k(C) \) - still "global".

\[
k(X)^, = \lim_{i \to \infty} M_i / M_i^2
\]

\( x \in C \Rightarrow \text{completion } k(C)^, \text{ local field} \)

Main observation: A natural replacement of \( k(X)^, \) which is complete \( D.V. \) field \( k(X)^, C \) with residue field \( k(C)^, C \)

Constructive \( M_2 \subset O_X \) sheaf of ideals of \( C \)

\[
j_2 : X - C \rightarrow X, \quad j_1 : C - \{x\} \rightarrow X
\]

\( \forall x \in \mathbb{Z} = M_i C \subset j_2 \times j_2 \subset O_X \) coherent

\[
M_i / M_i^2 = (C_{gen}, \mathcal{M}_i / \mathcal{M}_i^2)
\]

Coherent sheaf on \( X \), supported on \( C \).

\[
k(X)^, = \lim_{i \to \infty} \left( \Gamma (C_{gen}, \mathcal{M}_i / \mathcal{M}_i^2) \right)
\]

\[
= \left. \lim_{i \to \infty} \right|_{\text{coh,}} \left. \lim_{j \to \infty} \right|_{\text{coh,}} \left. \Gamma (C_{gen}, \mathcal{M}_j / \mathcal{M}_j^2) \right|_{\text{coh,}}
\]

Now if \( F \) of \( x \) is supported on \( C \)

an first compute it at \( x \)

\[
F_x^\infty = \lim_{i \to \infty} F / M_i^2 F = \lim_{i \to \infty} F / F_{i \subset C}
\]

\( \text{Supp } F / F_{i \subset C} \subset \{x\} \)

Further \( F \) restriction to punctured formal ring \( of \ x \) in \( C \)

\[
F_x^{\infty} = \lim_{i \to \infty} \left( \frac{F}{M_i^2 F} \right)_{i \subset \text{coh, Supp } F / F_{i \subset C}}
\]
So finally \( k(X) = \lim_{Y \to Y_n} \frac{\lim_{Y \to Y_n} \lim_{X \to X_n} \lim_{X \to X_n} Y}{\lim_{X \to X_n} Y} \).

\( J_2 \) may be thoroughly predefined over \( \mathbb{C} \).

**General picture (dim \( X = n \) arbitrary)**

- For any flag of irreducible subvarieties \( Y_1 \subset \cdots \subset Y_n = X \),
  \[ \dim Y_i = i \] (not necessarily full flag)
  \[ \Rightarrow \text{"convex" } K_{Y_i} \leq c_{X_i} \text{ of } O_X \text{, a ring (not necessarily) } \mathbb{C} \text{.
}\]

1. \( K_{Y_i} = O_{X_i} \quad (i = 0, \ldots, n) \)
2. \( K_X = k(X) \)
3. \( \forall \text{ for a complete flag of smooth sub-varieties} \)
   \( K_{Y_1} \leq c_{X_1} \text{ is a field complete with respect to } K_{Y_1} \text{ for } Y_{n-1} \)
4. \( \text{If a flag } Z \text{ retracts flag } Y \Rightarrow \text{embeddings } \)
   \( K_{Y_i} \leq K_Z \)
5. \( \text{A sequence of integers } 0 \leq d_1 \leq \cdots \leq d_n \leq n \)
   \( X \text{ restricted product } C_{d_1} \times \cdots \times C_{d_n} \)
\[ C_{d_i} \text{ defined directly as iterated limit } \]
\[ \text{for coherent sheaves on } X. \]
\[ \Rightarrow \text{ adelic complex } \]
\[ \mathcal{O}^* = \left\{ \bigoplus_{i=1}^n \mathcal{O}_{d_i} \to \bigoplus_{i=2}^n \mathcal{O}_{d_i} \to \cdots \right\} \]
\[ \text{cosimplicial } \mathcal{O}_X - \text{algebra.} \]
6. \( \text{For } \mathcal{V} \text{ quasi-coherent sheaf } F \text{ on } X, \)
   \( \mathcal{O}^* \otimes F \text{ calculating } \mathcal{H}^i(X, F) \)

**Examples**

- \( X \text{ curve } \)
  \[ \mathcal{O}_X = 0 \quad \mathcal{O}_{d_0} \quad \mathcal{O}_{d_1} \quad \mathcal{O}_{d_2} \quad \mathcal{O}_{d_3} \]
  \[ \mathcal{O}_X = k(X), \quad \mathcal{O}_{d_1} = k(d_1), \quad \mathcal{O}_{d_2} = k(d_2) \]

- \( X \text{ surface: } \)
  \( \mathcal{H}_X \mathcal{O}_X = \mathcal{O}_0 \quad \mathcal{O}_{d_1} \quad \mathcal{O}_{d_2} \quad \mathcal{O}_{d_3} \quad \mathcal{O}_{d_4} \quad \mathcal{O}_{d_5} \quad \mathcal{O}_{d_6} \quad \mathcal{O}_{d_7} \quad \mathcal{O}_{d_8} \quad \mathcal{O}_{d_9} \quad \mathcal{O}_{d_{10}} \)
  \[ \mathcal{H}_{d_0} \text{ Fun}(\text{fundamental of } C^d) = \bigoplus_{i=1}^n \mathcal{O}_{d_i} \rightarrow \mathcal{O}_{d_{i+1}} \rightarrow \mathcal{O}_{d_{i+2}} \]
  \[ \mathcal{O}_{d_i} \quad \mathcal{O}_{d_{i+1}} \quad \mathcal{O}_{d_{i+2}} \]

- \( X \text{ discrete } \)
  \[ \mathcal{H}_X \mathcal{O}_X = \mathcal{O}_0 \quad \mathcal{O}_{d_1} \quad \mathcal{O}_{d_2} \quad \mathcal{O}_{d_3} \quad \mathcal{O}_{d_4} \]

\[ \mathcal{H}_{d_0} \text{ Fun}(\text{fundamental of } C^g) = \bigoplus_{i=1}^n \mathcal{O}_{d_i} \rightarrow \mathcal{O}_{d_{i+1}} \rightarrow \mathcal{O}_{d_{i+2}} \]
  \[ \mathcal{O}_{d_i} \quad \mathcal{O}_{d_{i+1}} \quad \mathcal{O}_{d_{i+2}} \]

\[ \mathcal{O}_{d_i} \quad \mathcal{O}_{d_{i+1}} \quad \mathcal{O}_{d_{i+2}} \]

\[ \mathcal{O}_{d_i} \quad \mathcal{O}_{d_{i+1}} \quad \mathcal{O}_{d_{i+2}} \]

\[ \mathcal{O}_{d_i} \quad \mathcal{O}_{d_{i+1}} \quad \mathcal{O}_{d_{i+2}} \]

\[ \mathcal{O}_{d_i} \quad \mathcal{O}_{d_{i+1}} \quad \mathcal{O}_{d_{i+2}} \]

\[ \mathcal{O}_{d_i} \quad \mathcal{O}_{d_{i+1}} \quad \mathcal{O}_{d_{i+2}} \]
- Cover every for cover: all formal discs generate point of $X$ & formal nbhd of all generat pts of curves.

Picture for just one flag:

\[ U_0 = \text{form. nbhd of } x \text{ in } X \]
\[ U_1 = \text{" of } C \times X \text{ in } X \]
\[ U_2 = X \setminus C \]
\[ U_{ij} = U_i \cap U_j \text{ etc} \]
\[ U, U', U_2 \text{ in tors} \ldots \]

Berthson:

f mere function on variety: What is $f(x)$, $x \in X$?

On curve: either number, or infinity ...

On surface: can't assign value to any zero $x$.

E.g. $f = x/y$ on $\mathbb{C}^2$ - can't assign value to point zero. Need curve passing through $0$, then get angle of curve.

So need a flag to assign value!

Procedure: $f$ has pole on $C$ $\implies$ can not $\implies$ evaluate $f/c$ at $x$.

Inter of curves.

Class field theory: if covering is ramified at a point, need more data than $x$-y-a point. Plain nbhds of pts can be very complicated, branching hard $\implies$ local data hard.

But fix flag $\mathcal{U}$ - if have ramified sing curve, look in tiny nbhd of the curve, care goes to zero exponentially fast so no other curves hit its nbhd $\implies$ very simple topology just 2-dim torus - most local situation, simple.

Can't be made smaller in spectrum of 2-d loc field.
Def. An n-dimensional field is a complete DVR whose residue field is an n-1 dim local field, & 0-dim local field = finite field.

1-dim: \( \mathbb{Q}_p, \mathbb{F}_p((t)) \) (Cauchy complete)
2-dim: \( \mathbb{Q}_p((t)), \mathbb{F}_p((t))((t)) \), another type (tower semi-infinite, \( \pi \text{-convergent tail} \)

2 & higher don't have structure of topological ring — multiplication not continuous, but are ring objects in \((\text{Ind-Pro}) \) \( (\text{finite sets}) \)

4. Pairing of K-groups of a n-dim local field

n-dim local \( E = \overline{E}^n \)
\[ \overline{E} \to \overline{E}_{n-1} \to \cdots \to \overline{E}_{0} = \overline{F}_p \]

\( \overline{E}_n \to \overline{E}_{n-1} \to \cdots \to \overline{E}_0 = \overline{F}_p \)

Taylor series in \( t \) whose zeroth coeff is Taylor in \( t \) whose zeroth coeff is ...
\[
\Gamma = E^*/\overline{E}^* \overset{\sim}{\to} \mathbb{Z} \text{ not canonically.}
\]

Multiplication \( \text{ord} : E^* \to \Gamma. \)
\( \Gamma \) has a canonical filtration, quotients canonically \( \mathbb{Z} \)
\( \Rightarrow \Gamma^m \simeq \mathbb{Z}. \Rightarrow \text{have determinant of matrices} \Gamma \in \mathbb{Z}. \)

Analogy of tame splitting \( E^* \to \cdots \to \overline{F}_p^* \)
iterated to \( n \)-th \( \text{sign} \) \( (a_0, \ldots, a_n) \to (\text{det}(a_0, \ldots, a_n)) \)
\( = (-1)^c \mathbb{R} (\prod_{i=0}^{n} f_i ' (t) \text{det}(\text{ord}(a_0, \ldots, \text{ord}(a_n))) \text{mod} 2 \)

Sign \( c \) = "det" of an \( n \times (n+1) \) matrix over \( \overline{F}_p \) of \( (\text{ord}(a_i)) \) \( \text{mod} 2 \).

(Keown:3)

For a field \( F \), Milnor K-space \( K^m_n (F) = \Lambda^m (F^*) / \Lambda^0 (F^*) \times x (1 x) = 0 \)

\( F \to \mathcal{O} \) \( \text{gives boundary map} \)
\( \Theta : K^m_n (F) \to K^m_n (\mathcal{O}) \)
\( \mathcal{O} = \{ f_0, \ldots, f_n \} \) is regular, \( K^m_n (\mathcal{O}) \xrightarrow{E} \rightarrow K^m_n (\mathcal{O}) \xrightarrow{E} \rightarrow \cdots \)

\( \Rightarrow \) as in pairing a \( \mathcal{O} \)'s can write \( \text{as } E^* \otimes K^m_n (E) \to \overline{F}_p^* \)

"dual" by \( \text{id} \)
5. **n-th CERTON-CORRÉ symbol**: 
\[ \text{Gal}(\mathbb{Q}(\sqrt{p_1}), \ldots, \mathbb{Q}(\sqrt{p_n})) = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n}) \] 
should exist & provide local version for \( \langle \mathcal{L}_e, \ldots, \mathcal{L}_n \rangle \).

6. **Poincaré duality** for \( G = \text{Gal}(\mathbb{Q}(E)/E) \): 
\[ H^{n+1}(G, \mathbb{Q}(\mu_{p^n})) = \mathbb{Q}/\mathbb{Z} \] 
-- prove at \( p \), 
& for any finite \( G \)-module \( M \) we proved pairing by cup product 
\[ H_i(G, M) \times H^{n+1-i}(G, \mathbb{Q}(\mu_{p^n})) \rightarrow \mathbb{Q}/\mathbb{Z} \].

7. **Local and CERT**: 
\( W_{ab}(E) \) := some quotient of \( K_n^m(E) \)
- \( n=2 \): \( K_2^m \) can be made into a topological group,
- \( W_{ab} = \text{max.} \text{ Hasse-Witt bound} \).
- \( n>2 \) more subtle. 
  If \( E \subset F \) finite Galois extension,
  \[ \text{Gal}(F/E) = K_n^m(E)/\text{Norm}(K_n^m(F)) \]
  (can explain via Poincaré duality!
  \[ W_{ab}/m = H^n(Gal(Z/m), H^n(Gal, \mu_{p^n})) \]
- local-global: for \( n=2 \): \( X/\mathbb{Q} \) surface \( W_{ab}(X) = \text{Ch}_0(X) = H^2(X, K_2) \) 
- only possible in abelian case! don't have enough sides in non-abelian groups!
  \[ W_{ab}(X) = \frac{K_2(\mathcal{O}_{X/\mathbb{Q}})}{K_2(\mathcal{O}_{X/\mathbb{Q}}) + K(\mathcal{O}_{X/\mathbb{Q}})} \] (only thing related to 2).

\[ \text{Note } H^n(X, \mathcal{L}_n) = H^n(X, K_n^m) = \text{Ch}_0(X) \]