M. Kapranov - Higher Langlands

1. Higher dim. class field theory
   Local
   n-dim local fields
   TF_q ((t_1)) ... ((t_n))
   Q_p ((t_1), ... (t_{n-1})
   IR

We'll jump W(k) → Gal(\overline{k}/k)

W^ab = K^{top}_{M, Milw} (K → \overline{\mathbb{Q}}) \to H^n_{et} (\text{GL}_N(K))

(taking profinite completed K-theory) topological: need to take into account topology of fields, take completed tensor product as symbols.

So 1-dim reps of W ↔ continuous Steinberg symbols C (x_1, ..., x_n), x_i \in k^*

come from n-cocycles of GL_N(K)

Simplest representation: W \to \mathbb{Z}\{F_i\} \to \mathbb{C}^*

K = \prod_{i=1}^n (L_i) : tame symbol

\{x_1, ..., x_n\} \in K_0 (\mathbb{F}_q) = \mathbb{Z} \to \mathbb{C}^*

determinantal cocycle

n = 2 \Rightarrow K = L((t)) \quad \text{ordinary local field}

\{x, y\} \in L^* \to \text{canonical element in}

H^2 (GL_N(K), L^*) : Tate (determinantal) central extension

H^2 (GL_N(K), L^*) too: take \mathbb{L} \to \mathbb{C}^*

Tate symbol \{x\} : \text{ Ext}_K (K, L)

\exists \{x, y\}_\text{tor}
This is the symbol corresponding to standard 1-dim rep of $W$
$\chi \in \text{Irr}^2(G_k, k^*) \leftrightarrow$ 1-character of $W$
In this example everything comes from cohomology.

Another (more fundamental?) formulation

Let $K$ be a local field, $\Gamma = \text{Gal}(\overline{K}/K)$ satisfy Poincaré duality in dimension $n+1$: $\exists$ canonical pairing
$\delta : H^{n+1}(\Gamma, M_k^{\otimes n}) \otimes \mathbb{Z}/l \to \text{Reg}$ via $\delta_m$

$H^i(\Gamma, M_k^{\otimes j}) \leftrightarrow H^{n+1-i}(M_k^{\otimes n-j})$

\[ n=1: \quad H^1(\text{Gal}(\overline{K}/K), M_k) \otimes H^1(\text{Gal}(\overline{K}/K), \mathbb{Q}/l) \to \mathbb{Q}/l \]
\[ \text{(ordinary local field)} \quad \sqrt{K^*/(K^*)^2} \text{ by Kummer map} \]
\[ \text{true for any field.} \]

Langlands for 1-dim local field:

N-dim reps of $W \leftrightarrow$ some reps of $GL_r, K$

$H^i(W, GL_r \mathbb{C})$

when considering $\mathbb{C}$-representations $H^i(GL_k, \mathbb{C}^{\times})$

N-dim case:

Reps of $W$ is always on $H^i$ so it should
be put in correspondence with some nonabelian $H^i$ ...

N=2: Nonabelian $H^2$

$\Gamma$ a group, $A$ an abelian group $\Rightarrow$

$H^2(\Gamma, A) = \text{actions of } \Gamma \text{ on categories: } A$-Germes
A gerbe: category C with \( \text{Hom}(X, Y) \)  
\( \text{Hom}(x, y) \times \text{Hom}(y, z) \to \text{Hom}(x, z) \)  
\( \text{A-gerbe} \)

(4.5) \( A = \mathbb{C}^k \)  
\( \Pi^2(\Gamma, \mathbb{C}^k) = \text{actions of } \Gamma \)  
\( \text{on category Vector}, \text{or even } 1 \text{-dim Vector} \)

\( g \in \Gamma \)  
\( g_g : C \to C \)  
\( g_g \circ g_h = g_{g \cdot h} \)

+ 2-cocycle condition for triples \( g, h, k \).

"Candidate" for nonabelian \( H^2 \) of \( \Gamma \): actions of \( \Gamma \) on category

Classical concept of character of \( \Gamma \): \( \chi(g) = \chi(g) \chi(h) \chi(gk) \)

\[ \Rightarrow \]

2-cocycles \( \xi(g, h) \in \mathbb{C}^k \)

\[ \Rightarrow \]

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common generalization
is nonabelian 2-cocycle?
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[\( K \) is part of \( H \), things we actually "see" Core]

from homology, \( g \) the symbols here: keep the elements in homology

Analog of a character of a representation for action on category

(w. Nora Center)

Suppose have action on category: \( \Gamma \) \( \varphi(g)(C) = \varphi \)

\( \text{Tr}(\text{func}(A): C \to C) = ? \)

\[ \text{Def. } \text{Tr}(A) = \text{natural transformers}(1_c, A) \]

Ex. \( C = D^b(G \times X) \)  
\( K \in D^b(X \times X) \) kernel
\( A(F) = R\beta_{x \cdot} (\rho \cdot F \otimes K) \)

\( \Rightarrow \) \( \text{Tr} \ A = R\Gamma (X, K/\Delta) \)
If \( \Gamma \) acts on \( V \) by \( \Gamma \), we get a "sheaf" on \( \Gamma \) which is conjugation equivariant. Character sheaf (Lusztig character sheaves should be ranked as frames in this sense for action on \( \text{Db}(G/B) \) --- constructible sheaves)

If \( \gamma \in \Gamma \) commuting \( \Rightarrow \) 2-character
\[ \chi^{(\gamma)}(g,h) = tr (g / Tr \rho(h)) \]
function on pairs of commuting elements, invariant under simultaneous conjugacy. --- 2-class functions
--- appear in elliptic cohomology:
\[ \text{Ell}(\mathcal{B}T) = 2\text{-class functors} \quad (\text{Hopkins-Kuhn-Ravenel}) \]

\( \text{Ell}(\Sigma) \) in certain families of categories over \( \Sigma \), i.e.
\( K(\Sigma) \) in vector bundles over \( \Sigma \).

So \( K \text{-coh} \) in action on categories
Naive idea of matrix representation: direct sums of \( \text{Vec}_G \)
--- modules over \( \mathbb{Z} \) ring category \( (\text{Vec}_G, \oplus, \ominus) \)
\( e.g. \text{Vec}_G \oplus = \text{Coh}(\cdots) \)

\[ \Rightarrow A = (A_{ij}) \text{ matrix of vector spaces,} \]
Problem: few invertible matrices (since dim vector space > 0)

ff
Usual characters = class funs which are elementary projectors under convolution

2. \( \text{Hecke operators} \)
\[ x \text{ curve over } \mathbb{Q}, \quad x \in X \text{ Bun}_r(X) \]
\[ \Rightarrow \text{Hecke oper. } T_{x,i} : \mathcal{C} [\text{Bun}_r(X)] \to \]
\[ T_{x,i}(f) = \sum [E : 0 \to E' \to E \to k^0 \to 0] \quad f(E') \]
sum over such modifications
Let $\text{Coh}_m(X)$: purely $m$-dimensional coherent sheaves.

For $i \leq n-m$,

$$\forall F \in \text{Coh}_m, \exists E \in \text{Coh}_{m+1} \text{ can consider modifications,}$$

$$0 \rightarrow E' \rightarrow E \rightarrow F \rightarrow 0$$

$$\Rightarrow E' \in \text{Coh}_{m+1}$$

$$\Rightarrow \exists \text{ operators } T_{F'} \text{ on } \mathbb{C}[\text{Coh}_{m+1}(X)]$$

Equivalently, Hecke operators act on sheaves with $1$-dim support $\Rightarrow$

Satisfy Hall algebra relations:

$$T_{F'} \circ T_{F''} = \sum_{\mathbb{F}} \Sigma_{F_{F',F''}} T_{F_F''}$$

$$\Sigma_{F_{F',F''}} = \# \{ E \in \text{CF} : E \cong F', F/\mathcal{F} \cong F'' \}$$

Hall algebra of $\text{Coh}_m$.

On a curve $T_{F_{j,i}} \leftrightarrow \Lambda_i(F_{r_x})$ Frobenius in $\text{Galo}$ so

$X$ surface: $\text{Coh}_2$ supported at pts, its Hall alg acts on $\text{Coh}_2 = \text{Coh}_{1}$ (1-dim support), whose Hall alg acts on $\text{Coh}_2 = \text{Bun}(X)$

$s$ all $(\text{coh}) \Rightarrow \mathbb{C}[\text{Bun}, X]$ : wildly branch ed.

For non-homotopic curves operators commute

Conjecture: A point $x \in X$ shall have a class in

$$\text{HH}^2 \left( \text{Hall} \left( \text{Coh}_1 \right) \right) \leftrightarrow \text{Flavors of pts} \text{ in } \text{Cone}(X)$$
Think of point as giving relations between curve operators ... x gives a natural relation to $\mathbb{H}^2$.

Such relations become some relations in case of ADE graphs ... ... so I don't know further.

So points don't act, but give some such after cohomological objects, corresponding to Frob.

--- Look for this as basis of relation.

3. "Generalization" of elliptic modules

$X$ surface $/ \mathbb{F}_q$  
$V$ vector space  
$D$ ample divisor  
(think in terms of this embedding in rings of differential forms - 1-form polynomials)

$K =$ completion of $\mathbb{F}_q (x)$ along $D$:  
div. field, with regular field $= \mathbb{F}_q (D)$:  
i.e. semi-local field.

$A \subset K$ discrete.  
Drinfeld exponential $e_A(z) = \prod (1 - \frac{z}{a})$  
$q$-power series  
(ie of form $z + c_1 z^q + c_2 z^{q^2} + ...$)

$e_A(nz) = P_n (e_A(z))$ for $n \in A$

$P_n$ is a $q$-power series = $n u + n^q u^q + ...$

-get a formal module, not elliptic a.e. e.g.

$P_{n+m} = P_n + P_m$, $P_{nm} = P_n (P_m) = P_{nm} (P_n)$

Case of curves: $P_n$'s are finite degree polynomials. have finite order, have modular spaces.
Finiteness properties \( K/A \xrightarrow{g} K \) as abelian group

\[ g = 0 \]

\[ s = 0 \]

\[ a, b \in A \quad \text{... i.e.} \quad a, b \text{ are congruent} \]

and \( \{ a = b = 0 \} \) is an ideal subring.

\[ \text{form Koszul complex} \quad 0 \to K \xrightarrow{x} K^{\otimes 2} \xrightarrow{p_a(x), p_b(x)} K \to 0 \]

\[ \begin{array}{c}
(\mu, \nu) \mapsto P_a(\mu) P_b(\nu) \\
(\lambda, \gamma) \mapsto L_a(\lambda) L_b(\gamma)
\end{array} \]

Exact away from middle term, where cohomology

is a finite abelian group --- analog of torsion of an elliptic module --- follows from injectivity of \( K \) as \( A \)-module.

(\( K/\ker(\partial) \cong K/A \) )

K injective \( A \to \) complex calculates \( \text{Ext}^* (A/A(0), A) \)

(when considering \( A \) as the trivial \( A \)-algebra in \( K \))

\( \text{More generally} \)

\[ \text{for } L \subseteq K \text{ locally free } A \text{-module, write } \psi_L, \quad P_n^L(\mu) \]

Koszul : \[ | H' | = (rk L) \cdot A/(a, b) \]

or more canonically middle cohomology is \( M/(a, b) \otimes K^{\otimes 2} \)

Need global result: \( K \xrightarrow{\varphi} K^2, \quad x \mapsto (\mu = P_a(x), \nu = P_b(x) \}

Im(\( \varphi \)) \subseteq K^2 \text{ has unique up to rescaling analytic continued} \]

\[ R_{a, b}(u, v) \text{ \( \text{Expect} \) } P_b(u) - P_a(u) \text{ to be a polynomial of } R_{a, b}(u, v) \]
4. Eisenstein series for (Kac-Moody) groups

Usual geometric Eisenstein series: consider maps \( \overline{X} \rightarrow G/B \)
\[ \text{deg } f \in H^2(G/B) = L \text{ cone } \mathfrak{a} \text{ with } \text{Mapd } \text{ finite-dimensional}. \]

\[ E(z) = \sum_{d} |\text{Mapd}| \cdot z^d \quad z \in T = \text{Hom}(L, C^*) \]

... this series has support more or less in dominant cone, 
\( \mathfrak{a} \) gives a rational function of \( z \) satisfying

functional equation w.r.t Weyl group \( W \).

... could replace \( |\text{Mapd}| \) by a native, or
topological Euler characteristic,
Hodge polynomial etc - anything additive w.r.t
\( \text{cut & paste} \) (i.e. "measure")

\[ E(wz) = \prod_{0 < \alpha} \frac{f(z^\alpha)}{f(qz^\alpha)} \cdot E(z) \]

**p-shifted\**

**W-action\**

Now \( G \mapsto \widehat{G} \text{ Kac-Moody group} \)
\[ 1 \rightarrow \mathbb{C}^* \rightarrow \widehat{G} \rightarrow G(\mathbb{C}(t)) \rightarrow 1 \]
determined central extension from \( \text{GL}_n \) for \( n \) case
- from Sato Grassmannian

Drinfeld Poor result for construction of \( \widehat{G} \) compared with
struggle of people involved (Fetling, Lefine, Brylinski; ...)
- all fail for \( G = E_8 \) eg in families of curves
acquiring singularities

Max torus of \((\mathbb{C}^* \times \widehat{G} = \widehat{G}) \) is \( T \times \mathbb{C}^* \times \mathbb{C}^* \)

\( \widehat{W} = W \times \mathbb{C}^* \text{ acts } \)

modular variable for elliptic curves
\[
\text{let } \mathcal{E} = \{E_1 \mid \xi E_1 < 1\} \quad \text{(a)} \quad \text{there is a family} \quad \left( \frac{E \otimes L}{E} \right) / W
\]

then \( \hat{\mathcal{T}} / W = \text{total space of (a)} \).

\( \hat{\mathcal{T}} = \{1 \leq \xi E_1 < \xi\} \) - relation between characters of \( \text{Ker-Mord} \) group & theta function

"S-duality" : \( X \) projective surface / \( \mathcal{E} \)

\( \text{Bun}_\mathbb{G}(X, n) : \text{semi-stable bundles with } \xi = n \)

\( \text{Fe}(g) = \sum X(\text{Bun}_\mathbb{G}(X, n)) \cdot q^n \)

should exhibit modular behavior, for congruence subgroup

More general generality: \( Z < X \) curve

\( \text{Bun}_\mathbb{G, B}(X, Z, n, d) : \text{G-bundles on } X, \xi = n, \text{ with } B \)-reduction along \( Z \) of degree \( d \in \mathbb{Z} \)

\( \text{deg of } B \)-reduction of G-bundle on \( X \to Z \to \mathbb{P}^n \)

\( Z \otimes T^v : \text{E}_\mathbb{G}(g, Z) = \sum \mu(Bun_{\mathbb{G}, B}(X, Z, n, d)) \cdot q^n z^d \)

should have elliptic behavior in \( Z \), modular behavior in \( g \).

**Change of setup** Fix a bundle \( P_0 \) on \( X \times \mathbb{Z} \)

\( M_{\mathbb{G}, P_0}(n) : = \{(P, z) : P \text{ a bundle on } X, z : P / X \times \mathbb{Z} \to P_0, c_2(P) = n\} \)

\( M_{\mathbb{G}, P_0}(n, d) = \{(P, z) \text{ as above + parabolic structure of degree } d\} \)
Claim: If $Z \cdot Z < 0 \Rightarrow$ these spaces are finite dimensional and empty for $n < 0$.

Evidences: $\Gamma_{\text{proj}}^{n} M_{G, p_{0}} (\mathbb{A}) = H_{Z}^{n} (X, \text{ad} \mathcal{P})$

$= H_{0}^{n} (Z, H_{Z}^{1} (X, \text{ad} \mathcal{P}))$  

(quotients of $\mathbb{P}_{2} \otimes N_{1}$  

(normal bundles)... have no sections for $n > 0$.

Relation to reps into affine Grassmannians $G_{r} = G_{r}(\mathbb{C})/G_{r}(\mathbb{C})$  

Say $X = Z \times A^{'1}$. A $G$-bundle on $X = Z \times A^{'1}$  

is a map $Z \rightarrow G$.

In general have twisted standard bundle of Grassmannians  

finite dim vers. $P \rightarrow \text{Flags} (P)$  

quotient $G/B$  

$G$-bundles, $\sqrt{Z} \hookrightarrow G$-bundle over $Z$  

+ principal bundle on $\text{Tot} (L) \rightarrow 0$  

+ determinant data for rank $n$  

$\Rightarrow$ ruled surface (e.g. More generally stack)  

consider $G$-bundles on tubes around surface.

Assume $Z \cdot Z < 0$. Write $Z = Z^{*} \times C$. $\Rightarrow \text{Tot} (E) = T \times C^{*} \times C^{*}$  

Write $\sum_{n,d} \mu (M_{G, p_{0}} B (n,d)) Y_{n,d} \otimes E^{d} \otimes \Gamma_{\text{proj}}^{n} M_{G, p_{0}} (\mathbb{A})$  

$E (g, z, v)$ forms functor on $\tilde{T}$.

Thm: $E (g, z, v)$ extends to a meromorphic section of $\mathfrak{g}^{d}$  

on $(E \otimes L)/W$  

Pf: reduction to simple algebra.
\[ \tilde{\mathcal{E}}(q, z, v) = \mathcal{E}(q, z, v) \quad \text{III} \quad \mathcal{G}(\mathfrak{S}^0) / \mathcal{G}(\mathfrak{S}^0, \mathcal{L}) \]

(affine proj roots)

\[ LL = \mu(\mathfrak{A}') \quad \text{module} \quad \text{is Waff invariant} \]

\textbf{Example:} \( \mathbb{P}^1 < X = \text{ruled surface, } Z : Z = d \)

\[ x = \left[ \frac{1}{2} \left( 2x - d \right) \right] \]

\textbf{Claim:} \( \mathcal{G} \)-bundles on \( X \times Z \leftrightarrow \text{integral characters of level } \alpha \text{ for } \mathcal{G} \).

Corresponding Eisenstein series are \underline{characters}.

\'Kor-{\'a}daly bundles/\( \mathbb{P}^1 \) case from \( \text{tors} \)

\textbf{For } \mathbb{P}^1 \leftrightarrow \text{irrep of affine of } \mathcal{G} \text{ }

The \( \mathcal{E} \) is \( \mathcal{L} \)-deformation of the character

\[ LL = \mu(\mathfrak{A}') \quad \text{K points of moduli} \]

--- Hall polynomials