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definition: Topological Field Theory: $X$ manifold $\longrightarrow$ labeling group
$\text{Hom}(X,X) = \$ \quad 2\text{-category}
\overset{?}{\longrightarrow} \text{additive category}
$\quad \text{K}_0(\text{category})$

In general:
$C$ additive category $\overset{?}{\longrightarrow} \text{categorification of } C$

$\forall \ E,F \in \text{Ob}C \text{ want to construct new additive category } \tilde{\text{Hom}}(E,F)$ s.t. $\text{Hom}_{\tilde{C}}(E,F) = \text{K}_0(\text{Hom})$

Conversely: decategorification:
$C^{(2)}$ 2-category, $E,F \in C^{(2)} \Rightarrow$ category $\text{Hom}_{C^{(2)}}(E,F)$ additive, triangulated et al.

$\overset{\sim}{\longrightarrow} \text{K}_0(C^{(2)}): \text{ some objects, }$
$\text{Hom}_{\text{K}_0(C^{(2)})}(E,F) = \text{K}_0(\text{Hom}_{C^{(2)}}(E,F))$

Alternatively could take $K(C^{(2)}),$
$\text{K} \text{ is K-theory spectrum of } \text{Hom}_{C^{(2)}}(E,F)$

$\Rightarrow$ category enriched / spectra, can get triangulated version.

Example 1: (R. Meyer, R. Nest)
Category with objects = (nonunital) C*-algebras.
$\text{Hom}(A,B) = KK(A,B)$ Kasparov (bivariant) K-theory in mixes $K^0(A) \& K^0(B)$.

... an abelian group (2 K-theory of Kasparov bimodules)
In fact this is a triangulated category.

Shift: \( \otimes_{\mathbb{C}^{\rightarrow}} (IR') \) functor vanishing at \( \infty \) at \( \infty \).

2-period: \([2] \sim [0]\). "Noncommutative topology."

Toy model in pure algebra:

- \( k \)-field. 2-category w/ objects:
  - assoc. unital \( k \)-algebras
  - \( \text{Hom}(\cdot, \cdot) \) = subcategory of \( A\text{-mod}-B \) which are fin. gen. & projective as \( B\text{-mod} \).

E.g., if \( f: A \rightarrow B \Rightarrow \text{assoc. graph} \)

\[ \text{graph}(f) = B \text{ as } A_f\text{-mod}-B. \]

This is an exact category & can pass to \( k\)-theory

\[ \Leftrightarrow \text{Hom}(\cdot, \cdot) = \text{additive functor:} \]

\[ (\text{f.g. projective}) \mapsto (\text{f.g. projective}) \]

\[ \Leftrightarrow \text{functor } B\text{-mod} \rightarrow A\text{-mod} \text{ preserving all } \lim \text{ & } \varprojlim. \]

Left & right adjoints

\( A, B \text{ commutative: } \text{module is graph of correspondences wisted on it} \)

\( \text{Spec} A \rightarrow \text{Spec} B \)

\( \text{Spec} \otimes A \rightarrow \text{Spec} B \)

\( \text{Spec} \otimes A \rightarrow \text{Spec} B \)

\( \text{Spec} \otimes A \rightarrow \text{Spec} B \)

\( A\text{-mod} \rightarrow \text{B-mod} \text{ w/ vector bundles on correspondences.} \)
Example 2  "Non-commutative motives"

Reminder: Grothendieck pure motives $/k$

$\text{Ob} =$ smooth proj. varieties of pure dimension $[X]$

$\text{Hom} ([X],[Y]) = \sum \text{cycles in } X \times Y \text{ of dimension } d = \dim X$

$\text{Num. equiv. : } Z; n = 0$ if $\text{cycle } Z'$

$\text{of dim } \leq \dim Y \leq n$ \text{ for which } \exists Z'$

(Numerical equivalence)

-believed to be $\Leftrightarrow$ homological equivalence

Next must add several things: $[P'] \to pt \to [P']$

-projector in this category

-formally add the direct summand

$[P'] = [pt] \oplus \text{Lefschetz motive}$

-Fundly add Tate motive $= \text{Lefschetz}^{-1}$

-plus it's tensor powers.

In fact take Karoubi closure of all summands coming from projectors.

Could also look at $\mathbb{Q}$-cycles or $\overline{\mathbb{Q}}$-cycles; get more & more projectives $\rightarrow$ more objects.

Believed to be rigid semi-intr functor categories.
A little bit different case: \((\eta, -\text{coefficients})\)

\[
\{\text{pure motives}\} / \text{action of } \mathbb{Z} \xrightarrow{\cong} X(n) \quad \text{Tate twist}
\]

\[
\text{Hom}_{\text{gr coh}} ([X], [Y]) = \text{cycles of all possible}
\text{classes in } X \times Y / \text{Numere}
\text{(Korabi connection)}
\]

Give another description: first look at cycles mod rational equivalence

\[
(\text{cycles in } X \times Y / \text{rat. equiv}) \otimes \mathbb{Q}
\]

\[
\uparrow \quad \text{(Chen character)}
\]

\[
K_0 \left( D^b (\text{Coh} (X \times Y)) \right) \otimes \mathbb{Q}
\]

\[
\Rightarrow \quad 2\text{-category: } \mathcal{C}^\text{op} = \text{smooth proj. scheme}
\]

\[
\text{Hom}_{\text{Coh}} (X, Y) = \text{triangulated category } D^b (\text{Coh} (X \times Y))
\]

\[
\text{that of } \mathbb{C}_{\text{Functors}}
\]

\[
D^b (\text{Coh} X) \to D^b (\text{Coh} Y)
\]

Numerical equivalence: on \(K^0 (D^b (\text{Coh} X \times Y))\)

have bilinear form with values in \(\mathbb{Z}\)

\[
\langle [\mathcal{E}], [\mathcal{F}] \rangle = \chi (R\text{Hom} (\mathcal{E}, \mathcal{F}))
\]

\text{Euler characteristic}

Left kernel of this form = right kernel
\text{(can though not symmetric). Use some functor}

\[
R\text{Hom} (\mathcal{E}, \mathcal{F})^* = R\text{Hom} (\mathcal{F}, \mathcal{E} \otimes K_{X \times Y}, \text{Id} \in X \times Y)
\]

\[
\leftarrow \text{kernel of numerical equivalence}
\]

\text{(up to Todd class)}
Larger class of spaces \( \supset \) smooth proj. varieties

"Noncommutative smooth proper varieties"

Definition: A unital dg algebra

"proper": \( \Sigma \text{rk } H^i(A) < \infty \)

"smooth": look in dg category of dg bimodules,
its \( H^0 \) is triangulated (= derived category of \( A \)-bimodule). Smoothness means:
look at bimodules \( A, A \otimes A \).
A itself is a direct sum of finitely extensible of \( A \otimes A \) \([n]\).

... look at smallest triangulated category containing \( A \otimes A \)
\( \iff \) thick subcategory generated by \( A \otimes A \)

Smoothness: this subcategory should contain diagonal bimodule \( A \).
(Theory: M. Van den Bergh & A. Bondal, explaining
Thomason: any scheme \( \iff \) dg algebra)

These smooth proper \( A \) are our objects

Morphism: category of \( A \)-mod-\( B \) bimodules with finite dim total cohomology.

\( \rightarrow \) "NC motives" includes gerbe objects,
NC projective spaces etc.

NC mixed motives \( \rightarrow \) triangulated category.

Stay with same class of algebras. Instead of
\( K^0 \) take K-theory spectrum of category of \( A \)-dimm. dg bimodules \( A \text{-mod-} B \)
What does this mean for finite simplicial complex \( X \) describe maps into this \( K \)-theory spectrum.

Approach: replace each simplex by \( \Delta^n \), glue together get singular affine scheme \( \mathbb{Z}[X] \), \( X^\text{ad} \).

Look at category of perfect complexes on \( X^\text{ad} \).

Can now look at \( \text{Funct}_{\text{perf}}(X^\text{ad}, \mathbb{A}^n) \) now mod out by homotopy equivalence, \( n \mapsto \text{describes } K \)-theory type of this spectrum. \( T \)

So get category enriched over spectra,

Add cones & take Kan extension \( \Rightarrow \) NC mixed motives.

This \( X, Y \) smooth proj varieties

\[
\text{RHom}_{\text{NCMMot}}([X], [Y]) = K_*(X \times Y) \otimes \mathbb{Q}
\]

\( K \)-theory of motives

In usual category of mixed motives, we use Adams grading on \( K_*(\otimes \mathbb{Q}) \),

\[
K(X \times Y) \otimes \mathbb{Q} = \bigoplus K_i^J (X \times Y) \text{ graded}
\]

As \( \mathbb{Z}/2 \) graded vector space get same from this Adams grading as before, but \( \mathbb{Z} \) grading is slightly different.
What are cohomology theories for NC pure motives?

Étale cohomology: \[ H^i (X \times \text{Spec } k, \mathbb{Q}_l) \] as \( \text{Gal } k \rightarrow \mathbb{Q}_l \) doesn't work: we're quotiented out by Tate motives \( \mathbb{Q}(1) \sim \mathbb{Q}(0) \)

so don't get Galois motives.

(problem: use cycle on \( X \times Y \) of arbitrary dimension.)

Turns out that can't generalize (as far as we know) pure-dimensional version with keeps Tate motives around.

Instead introduce on the Galois side

\[ L = \left( \bigoplus_{n \in \mathbb{Z}} \mathbb{Q}_l(n) \right) \text{ or rather a Laurent cyclotomic of it } (L^\cdot (\mathbb{Z})) \]

commutative algebra in Galois motives.

Guess: NC étale cohomology should be

\[ H^{non} (X) \cdot H^i (X) \otimes \mathbb{Q}_l = \text{free finite rank } \mathbb{Q}_l \text{-module} \]

For NC pure motives expect to get a free \( \mathbb{Q}_l \)-module, with tensor product being \( \otimes \).

If \( \text{Char } k = 0 \) can make de Rham cohomology: natural homological dga algebra \( A \) \( \rightarrow \mathbb{C} \rightarrow A \) Hochschild complex

\[ A_{\otimes \mathbb{Q}_k} \otimes (A/k)_{[1]} \]

has differentials \( b+1 \)

\[ B = -1 \]

If \( A = \mathcal{O}(x) \) \( \Rightarrow \) \( (H^i (C(AA), b), B) = (\mathbb{Q}^\cdot, 1) \)
Negative cyclic complex: \( C^{-}(A) = C((A, A)[[y]], b + \nu B) \)
\( \text{deg } u = +2 \)
\( C_{\text{per}}(A) = C((A, A)[[y]]) \)

If \( A \) descends \( \text{deg} \) vary \( X \) then \( H^0(C_{\text{per}}(A)) \) is a finite rank module \( / R[[u]] \)
\( \mathbb{Z} \)-graded.

\( \text{deg } 0: \quad H^0(C_{\text{per}}) = H_{\text{even}}(X) \)
\( H^1(C_{\text{per}}) = H_{\text{odd}}(X) \)

... again get bundle over Laurent series in \( u \)
(like Tate expectation).
& Gal action \( \leftrightarrow \) Con action \( \mathbb{Z}_2 \)-graded

Conjecture: Hodge to Kähler ...

Suppose have \( \otimes \) dg category \( (\text{symmetric monoidal}) \)
... e.g. usual commutative variety.
Maybe with this extra structure can define
Adooz operating? (they break down for
\( \text{non } \otimes \) equivalence)

Expect NC Gerhardtide varieties \( \leftrightarrow \) commutative varieties.

Example 3 ... ? higher-dimensional long-lived correspondence? \( (\text{clear } \rho > 0) \)

Remark: \( k \)-field \( \rightarrow \) Mot \( (k) \)
\( \otimes \)-linear category of profunctor motives, con? sensible
\( k = \mathbb{F}_2 \Rightarrow \text{Milne (using Tate conjecture)} \)

gives complete description of \( \text{Mot}(k) \):

Simple objects = simple objects in a category \( \mathcal{V} \) over \( \mathbb{Q} \), f.d.

\[ \begin{align*}
\mathcal{V} & \text{ vector space over } \mathbb{Q}, \text{ f.d.} \\
F : \mathcal{V} & \text{ semisimple,} \\
& \det (1 - \lambda \mathbb{G}) \in \mathbb{Z} \left[ \frac{1}{2} \right] [t] \\
& \text{all eigenvalues have norm } \in \mathbb{Z}/2 \mathbb{Z}
\end{align*} \]

\( \mathcal{Q} \) this category is the same as \( \text{Mot}(k) \) over \( \mathbb{Q} \) over \( \mathbb{Q} \) they're different.

Langlands conjecture: gives conjectural description of motives of 1-dim fields, e.g.

\( k = \mathbb{F}_2 (C) \) = curve.

Simple motives of \( r_k = N \) with coeffs in \( \mathbb{Q} \)

\( = \) cuspidal automorphic representations of \( GL_n \mathbb{A}_k \).

\( C = \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \)

\( \text{Irrep} \mathbb{C}^{\text{geom}} (C, \{ 0, 1, \infty \}) \xrightarrow{\text{continuation}} \mathbb{C}^{\text{geom}} (GL_2 \mathbb{A}_k) \)

\( \mathbb{Z} = \text{Gal} (\overline{\mathbb{F}_2} / \mathbb{F}_2) \rightarrow \text{Art } \mathbb{C}^{\text{geom}} \)

acts on this space,

consider \( (\text{Irrep}) \mathbb{F}_r \) invariant.
Get set with \( \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \) acting.

(in fact finite set)

\[ \Longleftrightarrow \text{ look at arithmetic } \mathcal{T}_1, \text{ which map to } \hat{\mathbb{Z}}^\times, \text{ so } \text{can tensor } \mathcal{T}_1, \text{ by } \hat{\mathbb{Z}} \text{ module. Above problem: } \]

\[ \text{ look at geom. irreducible reps. of } \mathcal{T}_1, \text{ up to transitive by } \hat{\mathbb{Z}} \text{ module.} \]

Langlands tells us how to give an automorphic description of this set, which is elementary.

Langlands: Galois spectrum = Hecke spectrum.

Look at Hecke spectrum on automorphic forms = here functions on finite sets.

Take Hecke operators only at \( T_{\mathbb{Q}} \)-points be (moduli of bullets is 1-dim here, so this is rough)

Spreading - Aut forms: finding \( T_{\mathbb{Q}} \rightarrow C \)

Hecke operators \( T_x \in \text{Mat } (T_{\mathbb{Q}} \times T_{\mathbb{Q}}, \mathbb{Z}) \)

Make one code for \( \gamma, \mathcal{Z} \in T_{\mathbb{Q}} \):

\[ (T_x)_{\gamma, \mathcal{Z}} = 2 - \chi \left\{ W \in T_{\mathbb{Q}} : W^2 = f_{\gamma}(x, y, z) \right\} \]

\[ - \left\{ \begin{array}{c}
    q + 1 \quad x = y \in \{0, 1\} \\
    1 \quad x = y \not\in \{0, 1\}
  \end{array} \right. \]

\[ 0 \]

\[ 2 \]

\[ x \neq y \]

\[ y = \frac{1}{x} - x \]
\[ + \begin{cases} \begin{align*} z = 0 & \text{if } x \in \{0, 1, t\} \land \left\{ \begin{array}{l} y = \frac{t-x}{1-x} \quad z = 0 \\ y = \frac{t-x}{t-x} \quad z = 1 \\ y = \frac{t(t-x)}{t-x} \quad z = t \end{array} \right. 
\end{align*} \end{cases} \]

where
\[ f_t(x, y, z) = (xy + yz + zx + t)^2 + 4xyz(1 + t - (x + y + z)) \]

common eigenvalues of these matrices \((T_x)\)

(which commute and are semisimple)

\(\Rightarrow\) set of irreps before.

\(G = \text{char. polynomial decomposes into product of 4 polynomials of order } 25, 25, 25, 25). \]

because \(T_0, T_1, T_t\) are involutions (Frobenius involution)

\(\{T_0, T_1, T_t, I\} \subseteq \mathbb{Z}/2 \otimes \mathbb{Z}/2. \)

\(\circ\) In this case \(T_x \cdot T_y = \sum c_{x,y,z} \cdot T_z\) close.

\(\&\) in fact \(c_{x,y,z} = \text{T}_x(y, z)\) ( motives entry )

\(\circ\) Observation: these calculations over \(F_3\)

have a simple structure, with answer motivic,

no finite fields involved.

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Guess: Jacquet-Langlands Automorphic Forms for \(GL_2\) have description independent of fields
Formalism \& any field
(poor man's version: decategorification).

Category $C_k$ : $Ob = \text{constructible sets} / k$ (subset of $\mathbb{A}^n_k$)

$\text{Hom}(X,Y) = \text{formal linear combination of constructible sets} \ Z \rightarrow X \times Y$

with $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$

\[ \sum_{\mathcal{Z}} \mathcal{Z} = \left[ \sum_{\mathcal{Z}_1} \mathcal{Z}_1 \right] + \left[ \sum_{\mathcal{Z}_2} \mathcal{Z}_2 \right] \]

Briefer:

Define motive number, & motive functions

(poor man's version)

Grothendieck group of varieties

Motive number: \([X]\) variety / k st. if \(Y \subseteq X\) closed

\([X] = [Y] + [X \setminus Y]\). Call its elements motive numbers. If \(k\) is finite

can get morphism to \(\mathbb{Z}\) by counting points.

Motive function $S$ algebraic variety

$X$ varieties / $S$, Play sage game rel $S$

\[ \downarrow \]

$S \Rightarrow$ Motivic functions on $S$.

Again for \(k\) finite motive function give usual function. \(\text{Form a ring}\)

Maxim's version: $\text{Hom}(X,Y) = \text{motive function on} \ X \times Y$

composition = motivic multiplication.
Jacquet-tempered $L$-functions for function fields can be rewritten in this language.

To get sophisticated notes: look at Falin of variety $k$ — project $k$.

(Keber $k$) This category is a rigid category,

$$[X] \otimes [Y] = [X \times Y].$$

Rigidity: all $X \cong X^*$, where $\left( X \times X^* \right) \rightarrow \text{Id}_pt$,

is just $\text{diag} : X \times X \times pt$.

$k = \mathbb{F}_q$, $\forall n \geq 1 \Rightarrow$ tensor functor

$$\phi_n : \mathbb{F}_q^n \rightarrow \text{fin dim vector spaces}/\mathbb{Q}$$

$$X \rightarrow \mathbb{Q}^X(\mathbb{F}_q^n) = \mathbb{Q}-\text{valued functions on } X(\mathbb{F}_q^n).$$

$$Z \downarrow$$

matrix $\phi_n( \begin{pmatrix} \mathbb{F}_q^n \\ X \times Y \end{pmatrix} ) (x, y)$

$$= \# \left\{ z \in \mathbb{Z}(\mathbb{F}_q^n) \mid y \right\}$$

Proof of above formulas which works in this category.

Conclusion of above formulas: get a comm. assoc. algebra in $\mathbb{C}^{\text{inf}}^+$

= global Hecke algebra, gives concrete algebras by applying $\phi_n$'s.
Langlands \{ \text{irreps } \pi \mapsto \text{GL}_N(\mathbb{Q}) \} \\
with \text{bounds on refinement}

\text{look at } F^n \text{ fixed points}

\text{Conjecture} \{ \text{Irr } \pi, \rightarrow \text{GL}_N \} \overset{?}{=} \text{Hom}(\varphi_n(A), \overline{\mathbb{Q}})

\text{where } A \text{ is a commutative algebra in } C_F^N

\text{(commutative with } \text{GL}(\mathbb{Q}/\mathbb{Q}) \times \mathbb{Z}/n\mathbb{Z} \text{ symmetric)}

\text{(Bdell)} \text{ Langlands conjectures (for } \text{GL}_N \text{ & a curve } X/k \text{ or } X/S)

\Rightarrow \text{ algebra } A \in C_S \text{ canonically defined.}

\text{easy to define, commutative.}

\text{Main property: if } S = \text{Spec } F^i_{\mathbb{Q}}

\text{I apply } \varphi_n \text{ to } A \text{, then Spec } \varphi_n(A)

\text{bijectively correspond to } \text{Frob fixed points}

\text{in representation.}

\text{Key features: }

\begin{align*}
& \text{A very explicitly defined} \\
& \text{Kontsevich conjectures can still define such } A \text{ (commutative)}
\end{align*}

\text{for any diagonal variety instead of curve.}

\text{Describe } A \text{ explicitly by structure constants } C(i,j,k)

i, j & k \in \mathbb{X} \text{ variety whose order is } A
(Katz) "Why to believe this? \( \mathbb{G} \) a projective surface, \( C \subset S \) curve, ample \( \Rightarrow H^1(C) \to H^1(S) \)
so there are fewer local systems on \( S \).
--- expect quotient algebra of curve \( C \).

(Katz) One problem: suppose \( C \subset S \) over \( \overline{\mathbb{Q}}_p \).

\[ H^1(C) \to \text{GL}_2(\overline{\mathbb{Q}}_p) \]
\text{adic reps

the notion of compatible families of such: all eigenvalues of Frobenius are \( p \)-ad integers (mod \( p \))
so if \( \text{character} \) \( \text{admissible} \).

Can ask if there are \( \ell \)-adic reps with same eigenvalues? A: yes (follows from Langlands description: have same automorphic spectrum, over \( \overline{\mathbb{Q}}_p \)).

Given \( \pi : \pi \to \pi \) (\( \text{GL}_2(\overline{\mathbb{Q}}_p) \))

Does this compatible rep also factor through \( \pi \) ?

\( \text{Katz:} \) \( \pi_1, \pi_2 : \pi_1 \to \pi_2 \) (\( \text{GL}_2(\overline{\mathbb{Q}}_p) \))

Can calculate Ext groups!
Let $(S, p^*, e_0)$ over $\mathbb{F}_p$:

Directions of these jump on Brill-Noether loci.

coroutine they come from bimodules in $\mathcal{C}_2$

Rehn's versus form 2-category,

$X, Y$ schemes, $\text{Hom}(X, Y) = \text{contravariant morphisms}$

above should care by taking $K_0$.

(Ext conjecture above close to Langlands functionaly:

$A^m$ describes n-dim reps of $\pi_{\text{geo}}(S)$

$A(m)$ mod. n reps exact $\text{Ext}$

to relate $\to A(m) = \text{nuc}(A(m))$

Example 4: Lattice models.

$X \in \mathcal{O}_b \mathcal{C}_1$, $M \in \text{End}(\mathcal{C}_1, X)$ with $\text{inertia}$ in 2 variables

$\Rightarrow$ family of finite matrices $g_n(M)$ $n \geq 1$,

with more linear, explicit spectrum. What if??

Understand $\text{Spec } g_n(M) \in \mathcal{C}_1$ as $n \to \infty$

... study char. polynomials (size $2 \times X(\mathcal{F}_p)^2$)

$Z(n) := \text{Trace } (g_n(M)^m) ~ n, m \geq 1$

$m=1$ : ignore zero eigenvals of $g_n(M)$
Observations

1. Fix $n \Rightarrow \exists$ finite-collapsing \{ $\lambda a$ \} $\subset \mathcal{C}$
   (obvious)
   s.t. $\mathcal{Z}(n,m) = \sum_{\lambda=a}^\infty \forall m$

2. Fix $m \Rightarrow \exists$ finite-collapsing \{ $\mu_a$ \} $\subset \mathcal{C}$
   $\exists a = \pm 1$ s.t. $\mathcal{Z}(n,m) = \sum_{\mu=a}^\infty \forall n$

Why? Trace $\varphi^*_n(M)^m = \text{Trace} (\varphi^*_n(M^m))$ $\Rightarrow$ have variety $Y(m) \rightarrow X^m \times X^m$ corresponding to $M^m$.

$\text{Trace} = \text{Trace} (Y(m) \cap \text{Diag}) (\text{Frob}_n)$

$\Rightarrow$ trace of Frobenius on $\mathcal{C}$

-- so behavior in $m,n$ symmetric (both cases $\times X$, $\times m$ grows exponentially)

2-dim lattice models

$\mathbf{k} \in \text{Vec}_{/k}$ $\dim \mathbf{k} < \infty$

$R : \mathbf{v}_1 \otimes \mathbf{v}_2 \rightarrow \mathbf{v}_1 \otimes \mathbf{v}_2$. Want to calculate traces of powers of $R$.

Choose bases of $\mathbf{v}_i \Rightarrow$ tensor with $4$ indices

\begin{align*}
\Rightarrow & \ ij, \ ji \text{ basis of } \mathbf{v}_i, \ jj', \ basis of \mathbf{v}_2
\end{align*}

\begin{align*}
fixed \text{ boundary values of indices, take sum over all ways to complete inside.}
\end{align*}
Look in doubly periodic settings (Kdef). Calculate PWS partition function. Get function of \( \nu, m = \text{trace}(T_{(m)}) \).

Operator \( T_{(m)} : V_{i}^{\otimes m} \rightarrow V_{i}^{\otimes m} \) (transfer matrix)

with matrix coeff.

\[
T_{(m)}_{i_1 \cdots i_m} = \sum_{j_1, \ldots, j_m} T^{ij} R_{j_1 \cdots j_m}^{i_1 \cdots i_m}
\]

If \( V_{i} \) super vector spaces, \( R \text{ even} \Rightarrow \) sums as above but with signs.

\[Q:\text{ Given } X \in \text{Fg}, \ M \in \text{End}(X)\]

Does there exist a lattice model producing \( X \) space number?

Given a lattice model, could also formulate. For \( \forall \Gamma \subset \mathbb{Z}^{\mathbb{R}} \) of finite index, get \( \mathbb{Z}(\Gamma) \subset C \).

\( \Gamma \) lattice \( \Gamma \) depends on three parameters \( n, m > 1 \), \( k \in \mathbb{Z}/m \mathbb{Z} \).

Geometric form: \( \text{Tr} \left( \phi_{n}(M)^{m} \cdot \phi_{n}(F_{r_{x}})^{d} \right) \)

\( F_{r_{x}} \) = Frobenius operator on \( X \).
Claim: In both cases, \( \forall v_1, v_2 \in \mathbb{Z}^2 \) noncollinear \((v_1 \wedge v_2 \neq 0) \) \( \exists \) finite collection \( \lambda_\alpha \) & signs \( \pm \) s.t.

\[
\Sigma \text{ s.t. } N \geq 1 \quad \mathbb{Z}(\mathbb{Z}v_1 \ominus N \cdot \mathbb{Z}v_2)
\]

\[
= \sum \pm \lambda_\alpha^N
\]

(generalization of above property for \( n, n \))

Say two lattice models are isospectral if same \( \lambda_\alpha \)'s. \( SL_2 \mathbb{Z} \) acts on isospectral lattices up to isospectral equivalence, can reduce quest to simpler lattices...

d-dim lattice model, \( d \geq 0 \):

\( V_0, \ldots, V_d \in \text{Vect}_C \) \( \implies \) \( \text{dim } V_i = \cos \)

\( R \): \( V_0 \ominus \ldots \ominus V_d \)

\( \Rightarrow \) partition function on all finite index lattices \( \Gamma \subset \mathbb{Z}^d \).

Similar behavior of \( \lambda_\alpha \)'s . . . .

-- Can formulate such in any rigid tensor category

(partition function will be endomorphism of identity object)

-- In particular in \( \mathbb{C}G_{\mathbb{F}_2} \).

Q: Given a d-dimensional lattice model \( M_d \), does there exist a \((d-1)\)-dim numerical lattice model \( M_{d-1} \) s.t.:

\( \forall n \geq 1 \quad \varphi_n(M_d) \sim K_{\text{Kahler under isometries}} \)
d = 0 case: need morphism \( A \rightarrow \), i.e.

a variety, and we're calculating its number
of points over finite fields.

statement follows from lalil conjecture.

Hecke operators: commuting operators \( \hat{A} \)

commuting transfer matrices (R matrices,
depending on variable \( z \), with commuting coefficients)

Consider 2-category \( \mathcal{A} \),

\( \text{Obj} = \text{f.d.
super vector spaces} / G \)

\( \text{Hom}_\mathcal{A}(U, V) = \text{finite dimensional tensor categories} \)

\( C^*(U \otimes V) \)

\( \text{ie morph } U \otimes V \otimes E \rightarrow E \)

\( \Leftrightarrow U \otimes E \rightarrow V \otimes E. \)

Composition

\( V \otimes F \rightarrow V \otimes E \otimes F \)

"biadditive factor", \( \mathcal{A} \) is a \( \otimes \)-category.

\( \Phi_n : K_0(\mathcal{A}) \rightarrow \text{f.d.
super tensor categories} \)

\( V \mapsto V^{\otimes n} \)

given morphism \( U \otimes E \rightarrow V \otimes E \)

can build \( \rightarrow \) like before
Frobenius Fro : $V \otimes V \to V \otimes V$ for $V \otimes V$
action of permutation.
On $V \otimes v$ get action of cyclic permutation.

Conjecture $\exists \circ$ functor $(F_q \to k_0(A))$
($\&$ category)

$\phi_n \to \phi_n$
$\text{Vec}$

..... implies lattice model conjectures above.

A world give canonical construction of finite fields $\circ$

hire object $A' \subset C_k$, which is a commutative algebra (multipli- 
by diagonal $\Delta_3 \subset (A')^3$.

..... in fact this is a ring scheme object in $C_k$
(i.e. $A' \otimes A' \to A'$ & two connect $A' \to A' \otimes A'$)

Now if above factor exists apply it to $A'$, get vector space $C^n$, with
operators satisfying some rules . . . .

on $\phi_n(A')$ get $C^{2^n}$ with commutative alg. study
simulates structure of finite field $F_{q^n}$.

Carrying in addition of digits
is 2d lattice model $\circ$