Character Sheaves

Suppose we want a formula for characteristic is that given $\mathfrak{g}$, $B \subset G$ and $\mathfrak{g} = \text{Lie} G$. Just a permutation representation. Have a set $X(\mathbb{F}_2)$, $\mathbb{F}_2$ points and $\text{tr} \mathfrak{g}$.

Trace of a permutation matrix is just the fixed points.

$\text{Tr}(g, B) = \{x \in X(\mathbb{F}_2) \mid gx = x\}$

Let $X^0$ closed subset of $X$.

If we want its number of pts, we can find it as:

$X^0 \subset \mathbb{F}_2 = \{(y, x) \in G \times X \mid gx = x\}$

Consider $\pi : g \in G$.

$\pi(y, g) = y^g$. Proper: $g \in G$.

where $X^0(\mathbb{F}_2)$.

$X^0(\mathbb{F}_2) = \sum_{\iota : 0} \text{Tr} (F; H^i(X^0, \mathfrak{g}))$

$g \rightarrow H^i(X^0) = (R_i \pi, \mathfrak{q}_i)$ by proper base change.

$H^i(X^0) = (R^i \pi, \mathfrak{q}_i)$

Furthermore, complex of sheaves $\pi_\mu: \mathfrak{q}_i : \text{take } \mathfrak{g} \in \text{Tr} (F, \mathfrak{g})$

Other characters

Y verify our $\mathfrak{g}$. $F$ constructed $Y(\mathfrak{g})$

$F : Y \rightarrow Y$ & $F : F \rightarrow F$.

$F \rightarrow$ & $F \rightarrow F \Rightarrow$ & $F \rightarrow F$

If $y \in \mathfrak{g}$, $\Rightarrow F_y : F \rightarrow F$

S similarly $G_y$ on $Y(\mathfrak{g})$.

$F$ bundle complex & $\text{H}^i(F)$ construct

$\Rightarrow \mathfrak{g} \in \text{Tr} (F, H^i(F))$

\[ \{g \rightarrow \text{Tr} (g, B) \} \text{ sum over characters of the reps } \\pi(e)(E) \quad (E \in W) \text{ with coefficients of } \pi(e) \]
(For $E_7, E_8$ need to choose $P \in \Phi$ to notch up runs of $\mathfrak{w} \not\leq \mathfrak{h}$.)

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{m} \oplus \mathfrak{m} = \mathfrak{t} \oplus \mathfrak{m} \oplus \mathfrak{m},$$

where $\mathfrak{t}$ is the center of $\mathfrak{h}$.

Now for $g \in G, T \in \mathfrak{h} \Rightarrow (g, T) \in \mathfrak{g} \times \mathfrak{h}$

$$\text{Tr}(g \cdot T; \mathfrak{g}) = \sum_{\xi(T)} \text{Tr}(g; \xi(T)) \cdot \text{tr}(T; \mathfrak{m})$$

$$= \sum_E \xi(T(E)) \cdot \text{tr}(T; \mathfrak{m})$$

$E \in \mathfrak{m} \leftrightarrow E$.

So for fixed $T$ get linear combo of the irreducible characters, with some coefficients --- vary $T$ get lots of stuff.

Obvious choice for $T$ the the standard $T_u$ of $\mathfrak{h}$.

$$\Rightarrow \text{tr}(g \cdot T_u; \mathfrak{g}) = \sum_E \xi(T_u(E)) \cdot \text{tr}(T_u; \mathfrak{m})$$

Let's compute $\text{tr}(g \cdot T_u; \mathfrak{g})$.

$g \in G \Rightarrow g : G^F/B^F \to G^F$.

$$(g \cdot T_u)(x) = \varphi(g^{-1}x),$$

where $x \in \mathcal{G}$.

$$\text{Tr}(T_u \cdot \varphi)(x) = \sum_{x \in G^F/B^F} \varphi(y) \cdot T_u(x^g y)$$

$x^g y^{g^2} = \sum_{x \in G^F/B^F} \varphi(y) \cdot T_u(x^g y)$.

- So trace is given by sum of diagonal elements:

$$\text{tr}(g \cdot T_u; \mathfrak{g}) = \sum_{x \in G^F/B^F} \varphi(x^g x)$$

- orbital integrals of $T_u$.

$T_u$ is $B^F$-biinvariant on $G^F$ so the expression $T_u(x^g x)$ makes sense for $x = \text{const}$.

$$\left| Y_{w, g}(T_u) \right|$$

where $Y_{w, g} = \{ x \in X : (x^g x) \in \mathcal{G}(w) \}$

- "Denote locally with $g$ instead of $F$"

- locally closed subvariety of $X$ as functor of $g$.
Case of Bred: \( k_3 = K_1 = \oplus E \otimes E(1) \)

Gives \( \text{dim of perverse sheaves } E(1) \) character sheaves indexed by reps of \( W \).

Claim \( T_{p} \) is small ...
- \( G_p \) non-regular, of see, dimesions \( G \):
  - \( \widehat{G_p} \to G/p \) fiber over a point in \( G/p \times X \)
  - \( \tilde{\alpha}^{-1}(x) \) is \( X \times \tilde{\alpha}^{-1}(x) \).
- \( RiT_{p} \tilde{Q} \) is proper w. to shift \( i \) is middle exten.
- of its restriction to regular semisimple locus \( G_{rs} \subset G \).
  - to see this take \( G_p \times \tilde{G} \)

\[
\begin{array}{c}
\text{Ccc} \quad \text{strata getting under in } G \\
\text{degre of dj} \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\
\text{cdd} \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\
H^i(A) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
\end{array}
\]

To be perverse need to be better.
- diag. to be an IC complex.
- can't have compact a diag. 1.
- except in one slot.

In our situation, holes reflect cohomology of fiber:

\[
\text{Stalk of } \left( H^i RiT_{p} \tilde{Q}_x \right)_x = H^i \left( T_{p}^{-1}(x), \tilde{Q}_x \right)
\]

- so need these to vanish for \( i \) not in closed region
- one way to arrange this is to use
  \[
  H^i \left( T_{p}^{-1}(x), \tilde{Q}_x \right) = 0 \quad \text{for } i > 2d \dim T_{p}^{-1}(x)
  \]
- so want stratification whose inverse of fibers behaves well.

Lusztig's trick! look at fiber product \( \tilde{G}_p \times \tilde{G} \)}
- exactly doubles fiber dimension:
  \[ \dim (\text{fiber of } \mathbb{P} \circ \mathbb{P}) = 2 \dim \mathbb{P} \circ \mathbb{P} \]

For generality:
Need \( (\dim \text{ of } \mathbb{P} \circ \mathbb{P}) + \text{(codim statum)} \leq \dim G \)

\( \Rightarrow \) \( \dim \text{ of } \mathbb{P} \circ \mathbb{P} \leq \text{codim} \)

Can guarantee this if dimension of whole fiber product is \( \leq \dim G \)

To prove this use Bruhat decomposition:

\[
\mathbb{G}_P \times \mathbb{G}_P \overset{\sim}{\rightarrow} \mathbb{G}/P \times \mathbb{G}/P = \bigcup_{\mathcal{W}} \mathcal{W} \circ \mathcal{L}(w), \quad w \in \mathcal{W}
\]

\( \{(y^1, y^2)/y_1 = x_1, y_2 = x_2\} \)

- Fibers over each of these \( \mathcal{G} \)-orbits has exactly \( \dim = \dim \mathcal{G} \)
- \( \dim \text{ of fibers} = \dim \text{ of } \mathcal{L}_\mathcal{W} \circ \mathcal{L}(w) = \text{codim of } \mathcal{G} \circ \mathcal{W} \)

For middle extension want no cohomology on diagonal for \( \text{codim } > 0 \)...

Suppose \( H^l \) on \( \text{codim } 1 \) \( \Rightarrow \) contribution from

a local system on a \( \text{codim } 1 \) statum (if known to be proper)

\( \mathbb{G}_P \times \mathbb{G}_P \) not irreducible (bad components for \( P = B \) ...)

Need all \( (\dim \text{ of } \mathcal{G} \circ \mathcal{W}) \) components to map dominantly:

If we have \( \dim \mathbb{G}_P \circ \mathbb{G}_P = \dim \mathcal{G} \) and every irreducible component of \( \mathbb{G}_P \circ \mathbb{G}_P \) dominates \( \mathcal{G} \) \( \Rightarrow \) our statum is a middle extension

- Uses that \( \mathbb{G}_P \) is nonsingular
- Need support codimension both for stack \( \mathcal{L} \) and its Verdier dual.
- But \( \text{ID}(\mathcal{L}_\mathcal{W}) = \mathcal{L}_\mathcal{W} \text{ (shifted) on smooth } \mathbb{G}_P \)

This proves is proven by noting any relative rational

\( \mathcal{L}/\mathcal{W} \) can be achieved over reg ss details so all our irreducible components map dominantly.

What does \( \mathbb{R}^{\mathcal{L}_B}_* \overset{\sim}{\rightarrow} \) look like over \( \mathbb{G}_P \)?

- Let \( P = B \), \( \mathbb{G}_P \) a reg ss.
\( \Pi^{-1}_B(x) = \{ x \in G/B : g \cdot x = x \} : \) translt of core 
\( X = \mathfrak{t} \subset B : \quad \Pi^{-1}(1) = "W = G/B" \) stacked \n\( (G) \)rs Galois over, \( G \)-pair \( W : \) pullback of \( GIT \) 
\( \bigwedge \) 
\( G \)rs 
So \( \Pi \)T, \( G \)rs) \( \rightarrow W \) via this covering space.

So our local system comes from the regular rep of \( W \)
\( \mathcal{R}T_{B \times \mathcal{Q}L} / G \)rs = loc system comes for \( \mathcal{Q}L[W] \).

\( \mathcal{Q}L[W] = \oplus \bigwedge^r E \otimes E \)
Right action of \( W \) commutes with \( \Pi \)T, action via \( G \)rs action.
Here all self-conjugated so \( E^{\bullet} = E \).

\( \mathcal{R}T_{B \times \mathcal{Q}L} = \oplus _{E \in \mathcal{W}} \bigwedge^* E \otimes C(E) \), \( C(E) = \langle \zeta \rangle \) \( E \)-cofree of \( E \)s free

End \( (\mathcal{R}T_{B \times \mathcal{Q}L} / G \)rs) = \( \mathcal{Q}L[W] \) acting on right

End \( (\mathcal{R}T_{B \times \mathcal{Q}L}) \) Springer representation.

Since \( \mathcal{R}T_{B \times \mathcal{Q}L} / G \)rs = \( (\zeta) \) \( \otimes (\mathcal{R}T_{B \times \mathcal{Q}L} / G \)rs) 

\( \Rightarrow W \) acts on \( \mathcal{R}T_{B \times \mathcal{Q}L} \), hence \( \mathcal{Q}L[W] \) \( \otimes \) \( \zeta \) goes \( G \).

\( \Rightarrow W \otimes \mathcal{H}(X, \mathcal{Q}L) \quad \forall \mathcal{Q}L \).

(Prop. base change \( X \) \( \mathcal{H}(X, \mathcal{Q}L) = \mathcal{H}^*(X, \mathcal{Q}L) \))

See story for \( P : \quad W/P \subset G/P \) as fixed points.
Monodromy is \( \mathcal{Q}L[W] \mathcal{H}[W/P] \).

\( \mathcal{Q}L[W/P] \) is covering repn but not Galois: \( \mathcal{Q}L/W \) \( \mathcal{G}P / \mathcal{G}B \)

\( \mathcal{Q}L[W/P] = \mathcal{Q}L[W \mathcal{G}B / \mathcal{G}B ] \)

\( \Rightarrow k = \oplus _{E \in \mathcal{W}} \bigwedge^* E \otimes C(E) \).
Example: $GL_2$

<table>
<thead>
<tr>
<th>$w$</th>
<th>$K_w$</th>
<th>$K_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$e(1)+e(3)$</td>
<td>$e(1)+e(3)$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$(+\omega)e(1)$</td>
<td>$g_e(1)-e(3)$</td>
</tr>
</tbody>
</table>

in $K$-group of paraise stens, pairs of $\omega$ for shifting of abelian $(\omega$ match $)$

$W^p = \{l, e \}$ fixed by $\omega$.

\[ \overline{\chi_{\omega}} = \overline{G \times P^1} \]

\[ \mathbb{L} = \mathbb{C} \times \mathbb{C} \]

$G$ degree $2$, take first $1 \mapsto$ Frob acts by $\omega$

... illustration of decomposition theorem - paraise stens with different such ...

$K_1 + K_{\omega_0} = K_0$, in this case: build up via distinguished triangles.

$GL_3$

\[
\begin{array}{ccc}
\times & 2 \times 3 & 1 \times 3 \\
0 \times 3 & 1 \times 3 & 1 \times 3 \\
\end{array}
\]

$P = \left( \frac{a^2x}{a+b} \right)$

$G/P = \{1, \sigma \}$

$W_\omega = \{g, e \}$

$W^p = \{l, e \}$

$\overline{\chi_\omega} = \overline{G \times P^1}$

$\overline{\chi_{\omega}} \ast \overline{G \times P^1}$

Spurvo stack $\frac{3}{2}$ (paraise $\frac{a}{b}, \frac{c}{d}$)

Spurvo for $G/P$:

\(\chi_B = \chi_{\omega}, \chi_p = \chi_{\omega}$ match $\frac{a}{b}$, from last time $\frac{1}{2}$

Spurvo $\chi_p$: count of $\in (-)$ with multiplicity given by dim of $W_p$-fixed vecs

Pouese shifts: $g, A = A \cdot [2](-1)$ where trace locked on $\frac{a}{b}$ or $\frac{c}{d}$

What is $\overline{\chi_\omega}$?

$\mathbb{L} = \mathbb{C}$

\[ \overline{\chi_{\omega}} = \{ g \in \mathbb{C}^* : gl = 1 \} \]

\[ \chi_{\omega} \ast \overline{G \times P^1} \]

\[ \overline{\chi_{\omega}} \ast \overline{G \times P^1} \]

$R \bar{\chi}_{\omega} = \{ g \bar{\chi}_{\omega} : gl = 1 \}$ (Spurvo)

($1, \bar{\chi}_{\omega}$ is character of $P^1$.

\[ \bar{\chi}_{\omega} \]
\[ K_{13} : (Y_{13})_g = \begin{cases} \{ l \in p : g_l = 5 \} & \text{if } g_l \neq 0 \\ \{ l \in p : g_l = l \} & \text{if } g_l = 1 \end{cases} \]

\[ q^* (l) = \begin{cases} p^* & \text{if } g_l \neq 0 \\ \text{codimension } 1 + 2 + 2^2 & \end{cases} \]

\[ C(1) = \text{just compact stack} \]

So \( \overline{Y_{13}} \) is \( G \times \mathbb{P}^2 \) blown up along \( \delta_p \). 

\[ \overline{Y_{13}} = \{ (g, l, p) : l \in p : g_l = p \} \]

\[ \overline{G_p} = G \times \mathbb{P}^2 = \{ (g, l, p) \} \]

\[ \text{ch } \overline{Q} \rightarrow \mathbb{R} \times \overline{Q} \rightarrow (\overline{Q} \times G) \]

\[ G/B : 1 \ 2 \ 2 \ 1 \Rightarrow (1/2) (1 + 8 q_2) \]

Gleaming for \( \text{K}^n \)

Correlated with exactly character table for \( H \) of \( G_2 \). 

Exactly same for \( GL_n \), similar behaviour for general groups.

\[ \text{GL}_n : \text{funcun on } G^n \quad \text{virtual cohomology of character sheaves} \]

\[ i_p = \sum \dim E^l_p \cdot \Pi(E) \quad j_p = \sum \dim E^l_p \cdot C(E) \quad \text{really labeled by reps of } \Pi \text{ (Sp-aug)} \]

Special \( \text{GL}_n : \Pi(E) \leftarrow C(E) \)

\[ \text{GL}_n \text{ have enough parabolics } \ni \text{ip, } \ni \text{jp} \text{ to separate out } \Pi(E), C(E)'s \]

- labeled by partitions of \( n \) in both (unordered!)
- conjugate parabolics give same \( i_p, j_p \)
- invertible square matrix to pass from \( j_p \rightarrow T(E) \)

\[ \text{Sp}_4 \text{ : 5 principal series, got 4 ip's...} \]

\[ \text{ip} \leftrightarrow \text{jp} \text{ via functor-sheaf correspondence always} \]

\[ \Pi(E) \leftrightarrow C(E) \]

for GL thanks to above space within!

Deligne-Lusztig for \( G_2 \)

Notation: \( \mathbf{\tau}, \mathbf{\xi} \) ch. reps of \( H \) corresponding to \( 1, 3 \) ch. reps of \( H \)
<table>
<thead>
<tr>
<th>W</th>
<th>$\hat{\mathbb{E}}$</th>
<th>$\hat{\mathbb{E}_1}$</th>
<th>$\hat{\mathbb{E}_2}$</th>
<th>$\hat{\mathbb{V}}$</th>
<th>$\hat{\mathbb{V}'}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>s</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>t</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>st</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>ts</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>stst</td>
<td>3</td>
<td>-2</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>tsst</td>
<td>3</td>
<td>-2</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>ststs</td>
<td>3</td>
<td>-2</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>tssts</td>
<td>3</td>
<td>-2</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>bstst</td>
<td>3</td>
<td>-2</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Curtis Imanishi, R. L. W. Blies.

Can read off table certain cosets which are true reps from socle e.g., $q^2$ suggests arising from dim 4. It yields --- buildable for $K-L$ basis for $(A, M)$.

- for $G_2$ everything is natural homology manifold, $A_w = \sum_{w \in \mathbb{E}} \mathbb{E}_w$.

... to each $E \in \mathbb{H}$ we assign $R_E = \sum_{w \in \mathbb{E}} \text{tr}(wE) R_w$.

(2) (3) All roots of $R_E$ are actual reps.

take table for $A_w$, separate out tors reps from given power of $E$ and use to form columns of $R_E$.

\[ \text{tr}(A, M) : 1_{E_2} 0 1_{E_2} 0 1_{E_2} 1_{E_2} \]

\[ \Rightarrow R_{E_1} \cdot R_{E_1} \cdot R_V \cdot R_V \text{ is a true rep. of } G_F \]

\[ \Rightarrow R_{E_1} \cdot R_V \cdot R_V \text{ also a rep } \text{(R}_{E_1} \text{ is irreducible }) \]...

so does \( E_{12} \).

End up with a number of linear \(*\) of $R_E, R_E$ use row seqs.

congruence of $E_2, E_2, V, V'$ - core in one family.

\[ \Rightarrow \text{two reps:} \]

\[ R_{E_2} + R_V + R_V = 7_{E_2}, 7_{E_2}, 7_{E_2} \] (use \( \Delta \times \times \) to sort out irreps)

\[ R_{E_2} + R_V + R_V = 7_{E_2}, 7_{E_2}, 7_{E_2} \]

\[ R_{E_2} + R_V + R_V = 7_{E_2}, 7_{E_2}, 7_{E_2} \]

\[ -R_{E_2} + 2R_V = \alpha + 13 + 7_{E_2} + 7_{E_2} + 7_{E_2} \]

\[ -R_{E_2} + 2R_V = \alpha + 13 + 7_{E_2} + 7_{E_2} + 7_{E_2} \]

\[ \Rightarrow 8 + 2 \text{ min reps } \]

4 of the principal series

4 cuspidal uniprops.
These 8 reps come in a natural fashion. First let's make some calculations.

\[ R_{\nu} = \frac{1}{2} \left[ 3 \tau_{1} + 2 \tau_{2} + 2 \tau_{3} + 3 \tau_{4} + 7 \tau_{5} + 2 \tau_{6} + 2 \tau_{7} \right] \]

\[ R_{\nu'} = \frac{1}{2} \left[ 7 \tau_{1} + 3 \tau_{2} - 7 \tau_{3} - 7 \tau_{4} - 7 \tau_{5} - 7 \tau_{6} - 7 \tau_{7} \right] \]

\[ R_{\varepsilon_{1}} = \frac{1}{3} \left[ 7 \tau_{1} - 7 \tau_{2} - 2 \tau_{3} + 7 \tau_{4} - 7 \tau_{5} + 7 \tau_{6} + 7 \tau_{7} \right] \]

\[ R_{\varepsilon_{2}} = \frac{1}{3} \left[ 17 \tau_{1} + 2 \tau_{2} - 17 \tau_{3} - 7 \tau_{4} - 7 \tau_{5} + 7 \tau_{6} - 7 \tau_{7} \right] \]

**Explanation:** Murnaghan-Foner theorem on group \( S_3 \) --- conjugacy group of stabilizer of a certain element in \( G_2 \).

Of finite groups (e.g., \( S_3 \))

Consider pairs \((x, \rho)\), \(x \in G\), \(\rho \in \text{Aut}(G)\). Inner automorphism \(g \mapsto \rho \circ g \circ \rho^{-1}\)

Equivalence: \((x, \rho) \sim (g x g^{-1}, \rho)\) (conjugacy)

\[ M(G) = \{ (x, \rho) / \sim \} \]

\[ M(S_3) \text{ has 8 elements} \]

**Symplectic pairing:** [abelian case]

Define \(\{ (x, \rho), (y, \sigma) \} = \langle g \cdot x, g^{-1} \cdot y \rangle - \sum_{\sigma = 1}^{3} \tau_{\sigma} \cdot (g \cdot x, g \cdot y) \)

\[ x, y \in G \text{ commute} \]

\(\Rightarrow\) symmetric matrix of size \(|M(G)|\)

4 out of the 8 columns where our considerations above.

Other 4 will correspond to other character tables (above 4 come just from Spinor theory) - almost characters - cuspidal series of character tables.

**Sp4 Character Tables**

\[ G = SP(V, <, >) \]

\[ G/ P = \text{isotropic planes} \quad V \]

\[ H^3 = G/P = \text{lines} \quad V \]

\[ W = \{ \psi \} \quad W_0 = \{ \psi \} \quad \text{Assume characteristic } \neq 2 \]

We get our cuspidal rep \( \psi \) from \( GP \). Similarly here we'll get everything else out of \( G/ P \).

\( G \)-orbits on \( G/ P \), \( G/ P : \text{possibles for } (e, e') \) are

\( l = l' \}\quad \text{orbit closures} \]

\[ H \]
$G \times G/\mathcal{O} = \{(g,x) : (x, gx) \in \text{one of the odd classes}\}$

$\mathcal{L} : \mathcal{L}^* \to \mathcal{O}$ is \text{Sp(2)}  \to \text{Sp(1)}$ \text{Sp}(-1) \to \text{Sp(2)}\\
\text{Group orbit closure} \Rightarrow \text{just get} \ G \times G/\mathcal{O}$

\text{Interestingly case:} \text{closed} \ {\{ll \neq l^* \} 0 \neq a, b}$

$G \times G/\mathcal{O} = \{(g, l) / g \in \mathcal{L}^*: Z \ (\overline{g} \ i \in \mathcal{L} \ or \ \mathcal{L}^*) \}$

Look at \ \mathcal{L} := R \times \mathcal{G} \text{ = direct sum of shifts of character sheaves}$

\text{Stalk} \ \mathcal{L}_g = H^0(\mathcal{L}/g \ll \mathcal{L}) = H^0(\mathcal{Z}_g)$ \text{Fiber cohomology}$

$\mathcal{Z}_g \in P^3 \text{ closed, hyperplane, quadric} \quad g \ll \mathcal{L} \text{ is quadratic conic, but written in terms of skew form}$

\text{Choose} \ v \in \mathcal{L} \text{ nonzero :} 0 = \langle g \cdot v, v \rangle = \langle v, g^{-1} \cdot v \rangle = -\langle g \cdot v, v \rangle \text{ (fiber)}$

\text{[char2]} \quad \frac{1}{2} \langle (g \cdot v), v \rangle = \frac{1}{2} \langle g^{-1} \cdot v, v \rangle \quad \text{symmetric bilinear form} \quad v \cdot v(\mathcal{E}_{4,4}, \mathcal{E}_4)$

\text{Classification over quadratic forms over alg closure: just dim of kernel of}$
\text{the symmetric bilinear form (null vectors)}$

\Rightarrow \text{need dim ker} (g^{-1} - 1) = \dim \ker (g^2 - 1)$

- this will single out certain conjugacy classes as being special ----

\text{5 cases:} \quad \begin{array}{c|c c c c c}
\dim \ker (g^2 - 1) & Z_3 \\
\hline
4 & P^3 \\
3 & P^2 \ (\text{null 2) } \\
2 & 2 \ P^2 \ \text{ needed in } P^1 \\
1 & \text{ 8 cone in } P^3 \text{ with vertex, our cone in } P^2 \ (\text{null 1)} \\
0 & \text{smooth} \ P \times P \hookrightarrow P^2 \ (\text{null 0)} \\
\end{array}$

\text{cone vertex is a line bundle over } P^1$

\begin{array}{ccccccc}
\text{1} & 0 & 1 & 2 & 3 & 4 & 5 \\
\text{6} & & & & & & \\
4 & \begin{smallmatrix} 9^2 \\
3 & 9^2 \end{smallmatrix} & \begin{smallmatrix} 29^2 \\
2 & 9 \end{smallmatrix} & \begin{smallmatrix} 9 \\
1 & \\
0 & \end{smallmatrix} & \begin{smallmatrix} 9^3 \\
3 & \end{smallmatrix} & \begin{smallmatrix} 9^2 \\
2 & \end{smallmatrix} & \begin{smallmatrix} 9 \\
1 & \end{smallmatrix} & \begin{smallmatrix} 9 \\
0 & \end{smallmatrix} \\
\end{array}$

\$ = H^0(\mathcal{Z}_g) \text{ for } \dim \ker (g^2 - 1) = 1$

\text{Our one case is } \text{"fake } P^2"$

- \text{same cohomology as } P^2$

\text{line bundle over } P^1 \to (g \cdot v, v)$

+ 1 \text{ for vertex
This has to be IC sheaves of P^3,...

On open subset of E get fiber E \to \text{get local system over } \mathfrak{m}^1, \mathfrak{m}^2

in degrees 4, 2, 0.

\Rightarrow its middle ext will be summand of our L.

Degrees 0, 4, get constant local gys (contradict constant line in P^3 - constant line in 6 \times P^3)

\Rightarrow same for "half" of the 2g in deg 2...

Take fiber in P^1 \times P^1 \Rightarrow get line \times \text{line in } P^3, \text{line comes from boundary of } P^3.

So class in P^3 goes to diagonal pt \times P^1 - P^1 \times pt \Rightarrow \text{in constant sub local system}.

Now subtract these constant sheaves [heavilyagy emphasizes!]

\text{L} = \overline{\mathcal{L}} + \overline{\mathcal{L}}[-2](4) + \overline{\mathcal{L}}[-4](2) + \{2\} \text{ in derived category}

(in k-gap) = C(i), (1 + 2 + q^2) + \{2\}

Shells of \{2\}:

\begin{array}{c|c|c}
\text{cell} & \text{1} & \text{2} \\
\text{1} & \frac{q^3}{2} & \frac{q^2}{4} \\
\text{2} & \frac{2q}{4} & \frac{q}{2}
\end{array}

Have to be careful: "2" stratum is not irreducible, needs finer decomposition. Has a component of codimension 2 \times 2, \leq \frac{1}{2} smaller pieces.

\text{Can a stratum turn out to be non-constant --}

\text{case from a 1-dim character of } W \Rightarrow \text{i.e. on \mathfrak{m}_3 \text{ locus (not finite or sign!)}}

\text{Does its middle ext explain the degree 4 component?}

\text{Can't go up by 2 in degree & \text{by 2} in codimension - right on edge of diagonal!}

\text{So this is perverse but not IC!}

\text{So } \{2\} \text{ is middle part of } \mathbb{Q} \oplus \text{middle Ext of a locus in deg 4 on codim 2}

\text{and component } \mathcal{E}, \oplus \text{ maybe wrong else?}
Recall the Frobenius twist $M(g)$: the linear groups that we get here for $Sp_4$ is $g = Z/2$, $M(g) = (Z/2)^2$.

- In general, we need to take a quotient of this finite group as
  
  “Usuki’s quotient groups”

(csp: actually comes from cuspidal characters stable on $Sp_4$)

- It is important, but not important, but the cuspidal characters still.

Parameterization of characters on $T^F$

First case: $T = G_m \cdot G_m(F_{q^d}^*) = (F_{q^d})^*$

\[
\text{Hom}(\frac{T^F}{T^F_1}, \overline{Q}_x^*) \rightarrow \text{Hom}(\frac{\lim_{\rightarrow d} F_{q^d}}{T^F_1}, \overline{Q}_x^*) \rightarrow \frac{T^F}{T^F_1} \rightarrow \overline{T}_1^* \rightarrow 1
\]

Now note:

\[
\text{N}(x) = x^T(x) \cdot x^2(x) \cdots x^{2^n}(x) = x^{1+2+4+\cdots+2^n} = x^{2^{n+1} - 1}
\]

\[
\lim_{\rightarrow} \overline{T}_{q^d}^* = \frac{1}{N} \sum_{x \in \overline{Q}_x} \overline{T}_1^*, \quad \overline{Q}_x^* = \frac{1}{N} \sum_{x \in \overline{Q}_x} \overline{T}_1^*, \quad \overline{Q}_x^* = \frac{1}{N} \sum_{x \in \overline{Q}_x} \overline{T}_1^*
\]

\[
(\ast) \Rightarrow \text{Hom}(\lim_{\rightarrow} \overline{T}_{q^d}^*, \overline{Q}_x^*) = \left\{ s \in \overline{Q}_x^* \mid F(s) = s \right\}
\]

So up to the twist $\text{Hom}(\lim_{\rightarrow} \overline{T}_{q^d}^*, \overline{Q}_x^*) = \overline{Q}_x^*

\Rightarrow 1 \text{Hom}(\overline{T}_{q^d}^*, \overline{Q}_x^*) = \{ s \in \overline{Q}_x^* \mid s^2 = s \} = M_2(T(\overline{Q}_x^*))

Now consider any torus $T$

$\text{Hom}(T(T(\overline{Q}_x^*)), \overline{Q}_x^*) = \text{Hom}(\lim_{\rightarrow} T(T(\overline{Q}_x^*)), \overline{Q}_x^*)$

\[
\left\{ s \in \text{Hom}(\frac{L(T(\overline{Q}_x^*)), \overline{Q}_x^*) \mid F(s) = s \right\} \quad F = F_T \quad Frobars k tors
\]

\[
\left(\ast\right) \frac{L(T(\overline{Q}_x^*))}{(T(\overline{Q}_x^*))} \cong \frac{X_T}{\overline{Q}_x^*} \cong F \quad F = \epsilon
\]

\[
\{ s \in T(T(\overline{Q}_x^*)) \mid F^V(s) = s \} \quad \frac{L(T(\overline{Q}_x^*)}{\overline{Q}_x^*} \cong F = \epsilon
\]
\( T/F \) \to \text{L-adic and } \overline{T}/\overline{\mathbb{Q}} \), so we \( X_\mathbb{F}(T) = X^{\text{st}}(T) \).

\[ F : T \to T ', \quad F^*: T' \to T \]

\( g/F \) split \( \Rightarrow \) for \( \mathfrak{w} \in W \), \( F_\mathfrak{w} : T \to T ', \quad F^\mathfrak{w} = \mathfrak{w} \circ F \).

\( \text{Hom}(T, \overline{\mathbb{Q}}^*) = \{ s_\mathfrak{w} \in T^* (\overline{\mathbb{Q}}) : (\mathfrak{w} \circ F^*) (s) = s \} \)

\[ \text{Hom}_F (s) = s_{\mathfrak{w}} = \mathfrak{w}(s) \]

Relation to reps: Keep over local fields \( F = \mathbb{F}_p \to \mathbb{F}_p^* \)

\( G(F) \supset G(F) \text{ compact open} \)

\[ \downarrow \]

\( G(F) \)

So can induce up reps of \( G(F) \to G(F) \).

For cuspidal reps of \( G(F) \) the compact induced rep will be an irreducible rep of \( G(F) \).

Basic conj: irreps of \( G(F) \) parameterized by homomorphisms

\[ W_F \to G^*(\overline{\mathbb{Q}}) \quad + \text{additional data} \]

Weil-Deligne group = \( W_F \times K(\cdot) \)

\[ W_F \leftarrow \mathbb{G}_m \to \mathbb{G}_m \]

\[ I \rightarrow \mathbb{I} \rightarrow \text{Gal}(\overline{F}/F) \rightarrow 
\]

\[ \hat{\mathbb{I}} \rightarrow \frac{\mathbb{Z}}{\mathbb{Z}} \rightarrow \frac{\mathbb{Z}}{\mathbb{Z}} \]

\[ 1 \leftarrow \mathbb{I} \rightarrow I/F \rightarrow \hat{\mathbb{I}} \rightarrow 1 \]

\( \text{wild inertia} \quad \text{maximal} \quad \text{inertia} \)

For containing all \( p \)-rational prime roots of unity: get from residue field extensions.

Take root \( \sqrt{\mathfrak{a}} \) such that \( \mathfrak{a}^{N} = N^k \mathfrak{a} \), \( N \) root of unit in \( F \).

\[ \mathfrak{a} \in I/F \quad \text{with} \quad \mathfrak{a}^{N} = N^k \mathfrak{a} \quad \text{for some } \mathfrak{a} \quad \text{root of unity in } F \]

So \( \text{Hom}(T, \overline{\mathbb{Q}}^*) \) contiains

\[ \{ s \in \text{Hom}(I/F, T^*(\overline{\mathbb{Q}})) : F^\mathfrak{w}(s) = s \} \]

\( F \) acts on \( I/F \) and \( \mathfrak{a} \) as finite order automorphism of \( T(\overline{\mathbb{Q}}) \) (for \( \mathfrak{a} \)).
\[ K \]

\[ L' + (g, s) = \text{Same as } L \]

\[ L + (g, s) = (1 + i)(q, s) \]

Recall from D-L theory:  
\[ R_E := \frac{1}{1+i} \sum_{w \in W} Tr(wE) R_w \]

Analog:  
\[ K_E := \frac{1}{1+i} \sum_{w \in W} Tr(wE) K_w \]

Miracle before:  
\[ R_w = \sum_{E \subseteq G^F} \frac{E}{E} \to \text{as } w \in G^F \]

\[ E \to \text{character table of } H/E \]

\[ [K_w := \text{have to use purely } K_w \text{ character tables}] \]

\[ E \text{ trivially } \]

\[ R_w = \sum_{E \subseteq G^F} \frac{E}{E} R_E \]

\[ E^* = \text{I} \text{H imp important to } E \]

Analog:  
\[ K_E = \sum \frac{E}{E} \]  

\[ K_{E'} = C \]

\[ K_{E'} = \frac{1}{2} [C_2 + C_2 - 2 \cos \theta] \]

\[ K_{E'} = \frac{1}{2} [C_2 + C_2 + 2 \cos \theta \]  

\[ K_{E'} = \frac{1}{2} [C_2 + C_2 + C_2 - 2 \cos \theta] \]

\[ \text{Suggested by pattern above} \]

\[ \text{Function associated to } \phi \text{ is } K_{E'} \text{ gives character of principal series rep } \]

\[ \text{Not irreducible any more } \]

\[ \text{Irreducible character, but gives irreducible character} \]

\[ \phi \text{ gives character of cuspidal unipotent of } G \]
$C(\mathfrak{g}) = C_1 + C_2 - C_0$, comes from Steinberg rep - so we're only counting of Steinberg rep - but only for conjugacy class that split over our field already (those listed are all classes over the alg. closure).

Non-semisimple alts have vanishing Steinberg character!

There is actually a single conjugacy class $\begin{array}{c} \mathfrak{g} \\ \mathfrak{g} \end{array}$ is actually a single conjugacy class

D is middle extension of this piece is actually just extension by zero.

$D = \mathfrak{g}^{2Cusp} \oplus \mathfrak{g}$ (cusp is IC of the sysm $\mathfrak{g}$)

$D$ is actually an IC complex! once we account for piece $C^n C^n$

everything left satisfies property to be an IC complex...

$D = \mathfrak{g}$, middle extension of $\mathfrak{g}$.

$k_0 = C(1) + C(K_2^0) + C(\mathfrak{g}) \Rightarrow k_0 - k_0 = C(K_2^0) - C(\mathfrak{g})$

$k_0 = C(1) + C(K_2^0) + C(\mathfrak{g})$

$C(\mathfrak{g})$ comes from local system on $\mathfrak{g}$, by fact comes from a character of Weyl groups (though $\Pi, C(point) \neq 0$) => so it's $C(K_2^0)$.

Recall: $G = SL_2, \ G/\mathfrak{p} = SL_2$, $G/\mathfrak{p} = SL_2$

$\{g, \tilde{g} : g \mathfrak{p} = \tilde{g} \mathfrak{p}\}$ = $Z = \mathfrak{g} \cap G/\mathfrak{p}$

$R_{\mathfrak{g}}^G (\mathfrak{g}^{IC(2)}) = L$

$L$ We write $L$ as direct sum of irreducibles $L = (1 + 2^0 \mathfrak{g}^2) (C(1)) + 2^0 \mathfrak{g}^{2Cusp} + 2^0 C(K_2)$

Let $Z' = \{ (g, \tilde{g}) : g \mathfrak{p} \neq \tilde{g} \mathfrak{p}\}$

$L' = R_{\mathfrak{g}}^G (\mathfrak{g}^{IC(2')})$
<table>
<thead>
<tr>
<th>Codim</th>
<th>Typical elt. in stratum $a, b, f \neq 1$</th>
<th>$a^{16}$</th>
<th>$a^{17}$</th>
<th>$a^{18}$</th>
<th>$a^{19}$</th>
<th>$a^{20}$</th>
<th>$a^{21}$</th>
<th>$a^{22}$</th>
<th>$a^{23}$</th>
<th>$a^{24}$</th>
<th>$a^{25}$</th>
<th>$a^{26}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>2</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>3</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>4</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>5</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>6</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Codim</th>
<th>Typical elt. in stratum $a, b, f \neq 1$</th>
<th>$a^{16}$</th>
<th>$a^{17}$</th>
<th>$a^{18}$</th>
<th>$a^{19}$</th>
<th>$a^{20}$</th>
<th>$a^{21}$</th>
<th>$a^{22}$</th>
<th>$a^{23}$</th>
<th>$a^{24}$</th>
<th>$a^{25}$</th>
<th>$a^{26}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>2</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>3</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>4</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>5</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>6</td>
<td>$a^{1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
As we saw for $\text{Sp}_4$ (keeping track of shifts),

\[ K_\nu = \sum_{E \in W} \nu(E) K_E \]

(Lusztig: Character Sheaves III
Cor. 14.11 p299)

\[ K_E := \frac{1}{|E|} \sum_{E \in W} \nu(E) K_E \]

The computation we did in the beginning of this section discusses shifts

For associate to $K_\nu$ in $\sum_{E \in W} \nu(E) K_E$ (Case of $\Pi(E)$)

$\Rightarrow$ An associated to $K_E$ is $\Pi(E)$.

Last time:
$G = SL_2 \rightarrow \mathcal{U} = \{\text{unipotents}\} = G_0 / Z \cdot G_0$
- Stabilizer of $(0,1)$ is $\pm (1,0)$.
- Class $\nu_2$:
  $\mu_2 \times \text{Lang}
\Rightarrow 2 \cdot G_0$-equivalent
- Local systems on $U$, $\mathcal{O}_L$ and $L = L_\mathcal{O}$.

- Extend by $0$ of $\mathcal{O}_L$ to $G$ is by a middle extension

$\mathcal{O}_G \rightarrow G \rightarrow \text{Rk}_Z L$.

Clue: this extension is a character stack on $SL_2$.

$W = \{e, w_0\}$

\[ G \rightarrow T / T^2 \]

General: $G \rightarrow \text{(w-1) T}$

Start quotient of $T$ by $T$, acting on itself by x-ray
we get

$N \gg 0$ prime to characteristic $\Rightarrow$ the local action
which is $T^N \& (w-1) T$-equivariant
- Pull up and push forward,
- Get local sys on $G \rightarrow \text{character stack.}$

In our case $w - 1 = 2 \Rightarrow T / T^2$, $\exists$ 1-dim rational
$T^2$-equivariant local sys on $T$.

Our $C$: $\Pi_1(T) = \mathbb{Z}$, our loc sys is 1-dim map $Z^2 \rightarrow \mathbb{A}$

$R_{T_1, \nu}$, $G \rightarrow T / T^2$

\[
\begin{array}{c|c|c}
\nu & h(c) = h(\mathcal{O}_L) & \text{empty} \\
\hline
1 & 0 & \text{empty} \\
\hline
-1 & 0 & \text{empty} \\
\hline
\end{array}
\]

Our loc sys is trivial (as...)

$\text{not simp connected}$

$\text{Pic}^0 \simeq \mathbb{G}_a$.

$\mathbb{P}^1 \times 0 \Rightarrow \mathbb{A}$.
we get exactly our local system on unipotent s & mobyle extract:

\[ \text{RT} \times \mathbb{C} \cong k^+ \left( \text{SL}_2 \left[ k^+ \right] \right) \]

Relate this cuspidal character start 

\[ \text{conagrey class } j = \left( \begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right) \quad g = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) = s \in \text{Jord} \]

Centralizer \( G_s = \left( \begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array} \right) = \text{SL}_2 \times \text{SL}_2 \)

\( u \in G_s \) is reg \( \text{unipotent} \times \text{reg \ unipotent} \).

Write \( G(x) = \{ x'g x \mid x \in G \} \) 

What is \( G(x) \subset G \)?

\[ G \times G(x) \overbrace{\longrightarrow}^{G(x)} G \]

\( y \in u \), \( x' \) \( \in G(x) \) \( \in (x, y, y') \) \( \in G(x) \) \( \in G \)

\( G(x) \) is closed.

In general, this describes closure of orbits of elements using Jordan form:

\[ G \times G(x) \overbrace{\longrightarrow}^{G(x)} G \]

\( G \)-equiv \( \langle \text{loc sys on } G(x) \rangle \) \( \iff \) \( G \)-equiv \( \langle \text{loc sys on } G(x) \rangle \)

So reduces our question to \( \text{SL}_2 \times \text{SL}_2 \):

\( \langle \text{SL}_2 \rangle \) on \( \text{SL}_2 \times \text{SL}_2 \) \( \iff \) \( \text{cusp def } G_{\text{sl}} \) for \( G_4 \).

Springer theory

\[ G = \{ (g,x) \in G \times G \mid g \times x \} \]

\[ K = \text{RT}_+ \left[ \mathbb{C} \right] = \text{RT}_+ \left[ \mathbb{C} \right] = \mathbb{C}^+ \left[ \mathbb{E} \right] \otimes \mathbb{C} \left( E \right) \]

\( \hat{u} \in \hat{G} = \hat{G}_x \)

\( \hat{u} \) nonsingular & \( \hat{t} \) small

\[ \text{Tu} \]

\[ \text{Tu} \]

\( u = G \) \( = G_x \)

\( \langle u, \text{Tu} \rangle \) small:

\( \text{Tu} \) is small enough:

look at \( u \) \( \hat{u} \) \( \mathbb{E} \)

\( \hat{u} \) nonsingular \( \left( -T \times G / B \right) \) \( \dim \hat{u} = 2 \cdot \dim G / B \)

\( \mathbb{E} \) large enough.

\( \text{Tu} \) is small enough:

look at \( \hat{u} \) \( \hat{u} \) \( \mathbb{E} \)

\( \hat{u} \) is small enough:

look at \( \hat{u} \) \( \hat{u} \) \( \mathbb{E} \)

\( \hat{u} \) large enough.

\( \text{Tu} \) is small enough:

look at \( \hat{u} \) \( \hat{u} \) \( \mathbb{E} \)

\( \hat{u} \) is small enough:

look at \( \hat{u} \) \( \hat{u} \) \( \mathbb{E} \)

\( \hat{u} \) large enough.

\( \text{Tu} \) is small enough:

look at \( \hat{u} \) \( \hat{u} \) \( \mathbb{E} \)

\( \hat{u} \) is small enough:

look at \( \hat{u} \) \( \hat{u} \) \( \mathbb{E} \)

\( \hat{u} \) large enough.

\( \text{Tu} \) is small enough:

look at \( \hat{u} \) \( \hat{u} \) \( \mathbb{E} \)

\( \hat{u} \) is small enough:

look at \( \hat{u} \) \( \hat{u} \) \( \mathbb{E} \)

\( \hat{u} \) large enough.
Only one proper belongs $U$! But all maps are done for smallness.

Proper small maps from zero module nonsingular $(XY$ nowhere $) \Rightarrow$ in codata category $\text{End} (R_f * Q_e) = D^X_C (Y * Y, Q_e)$

$Q_e = \sigma _{x = x \in Y} \Rightarrow$ a codega $(\text{then } 0 \text{ smallness})$

$\text{Right : } R_f \{ Y * Q_e \}^B = A \Rightarrow (X * X) A * A \\
\text{Left : } D(A * B) = D(R_f * A, D B)$

$\Rightarrow R_f (R_f \{ A * B \}, D B) = D R_f \{ A * B \}$

\text{proper : } f = _f \\
\text{Right : } R_f (R_f \{ A * B \}, D B) = D R_f \{ A * D A \} \Rightarrow \text{non-singular : } D\bar{Q}_e = \bar{Q}_e [E * E]$  

Typically get $D^X_C (Y * Y, Q_e)$ middle codega, but the smallness $= \dim X = \dim Y$ aka, modulo is actually the map $\Rightarrow (0 \text{ rel. dim}) \Rightarrow \text{just count vectors}$

$\Rightarrow \dim \text{End} R_f (\{ Q_e \} ) = |W|$ unital twist

In end $\Rightarrow \text{End} R_f (\{ Q_e \} ) = D\{ W \}$ using $\text{NS} S$ (say)

But $R_f (\{ Q_e \} ) = R_f (\{ Q_e \} )$ (prop. base change)

So $\{ W \} = \text{End} (K_f) \xrightarrow{\text{res.}} \text{End} R_f (\{ Q_e \})$

Claim it is an isomorphism (easy to show injective!)

In fact, enough to look at sheaves over identity $1 \in U$

$\Rightarrow \text{map to } \text{End} (\text{sheaf of } Z) = \text{End} (H^* \{ W \})$

\text{get faithful rep of } W

(our $\sigma < \mathcal{O} / \bar{W}$ is $K / T$ correct rep)

So by perverse sheaf $R_f (\{ Q_e \} [w: 2c]) = 0$ imm $\text{end} I$ contains by dora lemma (cover alg curve), with $\text{Endo} = \{ W \}$
\[ K \backslash U = \bigoplus_{E \in \mathcal{E}} E \mathcal{E}(E) \backslash U, \quad \text{each component can't break up further since endomorphism are the same.} \]  
(End doesn't get bigger on \( U \)) \( \Rightarrow \mathcal{C}(E) \backslash U \) instead perseveres sheaf (as in \( G \text{-stuff} \).) \( \mathcal{C}(E) \text{-equivariant} \) (again de bods.)

\( U = \text{finite union of univalent classes}, \quad G \text{-equivariant} \)

So \( \mathcal{C}(E) \backslash U = \text{middle extension of local systems on univalent orbits.} \)

\[ \Rightarrow \text{ Springer map } \quad W \longrightarrow \{ [U, L] / [U, L] \text{ univ. class } L \text{ for } g \in G \} \]

\[ u \longmapsto \mathcal{C}(E) \backslash U = \mathcal{C}(E) \mathcal{E}_{\text{univ}} \]

\( \text{and } L \rightarrow \text{ rep of stabilizer group } T \text{ of } (G, U) \)

---

**Miracle's theorem on character of Hecke**

\( \mathcal{H}^{c}(X_{W}) \text{ as } G^{x}(F) \text{-linear combo of } \mathcal{B}^{x} \text{ with roots chosen of } \mathcal{B}_{W} \ldots \)

First study \( (g \in G^{x}) \) \( \mathcal{H}(g; F_{x}^{\infty}, H^{c}_{x}(X_{W})) = \) \( r \geq 1 \)

\[ = \ast \{ h : G^{x} / g^{x} : h \rightarrow g^{x}(F_{x}) \in \mathcal{B}^{x} w B \} \text{ (twisted)} \text{ (integral)} \]

- use Lefschetz formula = count \( \mathcal{B}^{x} \)-eigenparts of \( g^{x} \) on \( X_{W} \)

\( \text{(left } G \text{-module, category commut.)} \) + Lang's tech for \( B, F_{x}^{\infty} \) + Lang's tech for \( G, F_{x}^{\infty} \) \ldots \text{ Need } r \geq 1 \text{ to control Lefschetz !}

Here \( g^{x} \in G^{x} \) is defined as follows: choose (by Lang)

\[ x \in G^{x} \backslash F^{x}, \quad g = F^{x}(x), x^{-1} \text{ (not unique...)} \]

\( g^{x} = x^{-1} F^{x}(x) \). Ambiguity in \( x \) is somehow fixed by \( F^{x} \).

\[ \Rightarrow g^{x} \text{ well defined up to } F\text{-conjugacy} \]

\( g^{x} = x^{-1} g^{x}(x), \quad x \in G^{x} \)

1. \( g^{x} \in G^{x} \): work in \( G^{x} X(F) \),

\[ g^{x} F_{x} = x^{-1} F_{x} \times \quad F_{x} = x^{-1} (g^{x} F_{x}) x \]

ie. pair \( g^{x} F_{x} \) are taken to \( F_{x}, g^{x} F_{x} \) by \( x \)-conjugation. So \( g^{x} F_{x} \text{ commute iff } F_{x} \text{ commute} \Rightarrow g^{x} \text{ commutes with } F_{x} \leftrightarrow g \text{ commutes with } F_{x} ! \]

but \( \mathcal{B}^{x} \neq \emptyset \) !

2. The \( F \)-equiv- class of \( g^{x} \) is well defined.
get bijection $G^F/\text{F-conj}_u \rightarrow G^F/\text{F-conj}_u \quad (\text{Shintani})$

3. Moreover, cardinality $|\text{conj class of } g| = \frac{|\text{F-conj class of } g^*|}{|G^F|}$

(came up when composing inner product of class function.)

Consider induced rep $I(\psi) = C[G^F/\text{F conj} \rightarrow C$ rep of $G^F/\text{F conj}$:

For $g^* \in G^F$, acts by $(g^* \phi)(x) = \psi(g^* x^{-1} x)$

$H(\psi) = C[B^F \backslash G^F/\text{F conj}] = \text{End}_C I(\psi)$

$\psi: g \mapsto \psi_g = \phi_{\text{F conj}}$, $\epsilon_f = \text{F conj}$, $\phi_g = \epsilon_f$

$\left( T - \psi(g) \right) \phi_g = \sum_{x \in G^F/\text{F conj}} \frac{\chi_{(x)}(g)}{\chi_{(x)}(g^*)} \phi_g$

Calculate: $\text{tr} (g, F \cdot T_w, I(\psi)) = \# \{ x \in G^F/\text{F conj} : xg^*x^{-1} (g^*) \}$

Note: Frobenius acts trivially on $V \Rightarrow$ hence on $H^{-1} : F = \text{id}$.

So $g F, T_w$ commute.

Save formula we saw for trace on $H^{-1} \chi_{(\psi)}(x) \cdots$

\[
\begin{bmatrix}
\text{Upshot} & \forall \psi, \ g^* x = x \Rightarrow \text{tr} (g, F \cdot T_v, I(\psi)) = \text{tr} (g^* F \cdot T_w, I(\psi))
\end{bmatrix}
\]

Lemma G affine, group, $\sigma$ automorphism of G, $G^\sigma = G^\sigma/\langle \sigma \rangle$

V f.d. rep/C of G which is fixed as $\sigma$-module $G^\sigma$

Then \( \langle \sigma, \theta \rangle \sigma = 1 \)

saying: $g, h$ in $G$, $\langle g, h \rangle = \frac{1}{|G|} \sum g^i(h^jg^i \overline{g^j})$

$s = \langle \sigma^i \rangle \sigma^i$, $\langle \sigma^i \rangle = \frac{1}{|G|} \sum g^i(h^jg^i \overline{g^j})$

Hint for proof: Choose bijective char from $\langle g \rangle = \mathbb{Z}/n \rightarrow \mathbb{Z}^*$

Get homomorphisms $G \chi(\sigma) \rightarrow (\sigma^i) \rightarrow \mathbb{Z}^*$, $i = 1, \ldots, n,$

$V \otimes \chi \cong V \otimes \chi$ is $\sigma$-invariant $ightarrow$ since $V$ isn't $G$-equivariant.

use usual orthogonality relations on $G \chi(\sigma)$.
Goal: express trace of $H_c(X^w)$ as $\sum_{E \in \mathcal{T}(q^r, \mathbb{Q})} \text{tr}(g_s^*F, T_E)$

for see $\mathbb{K}^r$ (following Asgeir Tórnadóttir)

\[
\forall w \in \mathcal{W}, \quad \text{tr}(g_s^*F; H_c(X^w)) = \text{tr}(g_s^*F; \mathbb{I}(q^r)) = \sum_{E \in \mathcal{W}} \text{tr}(g_s^*F, T_E(q^r)) \cdot \text{tr}(T_E, E(q^r))
\]

System of equations labeled by $w$.

\[
\text{End}_{\mathcal{W}} \mathbb{I}(q^r) = [\text{End}_{\mathcal{W}} \mathbb{I}(q^r)]^F = H(q^r)^F = H(q^r)
\]

$\Rightarrow$ invert for $g_s^*F$ as $\text{End}_{\mathcal{W}} \mathbb{I}(q^r)$ is not changed when we restrict from large group to smaller group, so decomposition into irreps is same, so $T_E(q^r)$ extended canonically to reps of $G^F \times F$.

Principal series reps carry canonical $F$-action.

Have $g_s$-analog for $H_c(X^w)$ of orthogonality relation for $w$.

For rational reps $f_w, E \in \mathcal{G}$ (induced $\sigma$) s.t.

\[
\forall E \in \mathcal{W}, \quad \sum_{w \in \mathcal{W}} f_w(E(q^r), \mathbb{I}(q^r)) = \sum_{E \in \mathcal{E}} f_w(E(q^r), \mathbb{I}(q^r)) = 0
\]

- Let's invert (conjugate) matrix of character table.

- Multiply $\sigma$ by $f_w, E'$ and sum over $w$ $\Rightarrow$ get

\[
\text{tr}(g_s^*F, T_E(q^r)) = \sum_{w \in \mathcal{W}} f_w(E(q^r), \mathbb{I}(q^r)) \cdot \text{tr}(g_s^*F, H_c(X^w))
\]

This is one linear combo of class reps on $G$, must be a combo (for fixed $\sigma$) of linear combo of the unipotent characters.

\[
\Rightarrow \text{tr}(g_s^*F, T_E(q^r)) = \sum_{\text{unip. rep} \ of \ G^F \times F} c_{E, \sigma}(r) \cdot \text{tr}(g_s^*F)
\]

Will show the $c_{E, \sigma}(r)$ are very simple $\sigma$ of $r$.

We know $\sigma$-vals of $F$ on unipotent reps appear in different cocharacter classes of $G$, and pairs of $q$.

\[
\Rightarrow c_{E, \sigma}(r) = \chi^r \cdot \text{val}_r \text{nd } \text{ord } q \text{ val } \text{p} \text{ of } \sqrt{q^r}
\]

\[
\chi^r \text{ nd } q \text{ val } \text{p} \text{ of } \sqrt{q^r}
\]
2. (normalize $\lambda_0$ using Riemann hypothesis: compare $H^i_{\mathbb{C}^+}(K, \mathbb{C})$ to $\mathcal{IC}$: all these will also appear in $\text{see } \mathcal{IC}$, so points of $F$ are algebraic numbers $\lambda_0 \in \overline{\mathbb{Q}} \subset \mathbb{C}$ with $\text{L}(\zeta_0) = 1$ for $\forall \lambda_0 : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$: all these numbers are seen and one $\mathbb{C}^2$.

- $C_{\mathbb{P}, P}(r) \in \mathbb{Q}(\lambda_0, \mathbb{C})$ number field.

- $16^2 \cdot C_{\mathbb{P}, P}(r)$ is an algebraic integer: i.e. LHS of $\text{XXX}$.
  - write class $F \to (5^t, \mathbb{Q}(\zeta_0^n))$ in terms of characters:
  - get algebraic integer divided by order of group: there are alg integers.

- Fix $\zeta_0 : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Take $\zeta_0$ on $G^F$ be even $\text{XXX}$:
  $$\sum_{r \leq P} |C_{\mathbb{P}, P}(r)|^2 = 1$$

Applying (3) plus known about abelian groups: inner of $G \times G^F$
- inner product of $0 + r(5^t, \mathbb{Q}(\zeta_0^n)) = 1$.

Now fix $E, P$ and consider $\{C_{\mathbb{P}, P}(r) \}$ for $r = 1, 3, 3, \ldots$:
- all in see number field, alg integers $/16^2$, and all their complex absolute values $|C_{\mathbb{P}, P}(r)| \leq 1$ $\forall r$.

So by number theory this set is finite!

- but there are of form $\lambda_0^r \cdot$ (not $\zeta_0$ of $\zeta_0^n$).

=> if not identically zero, must have different values of $r$ where these numbers are seen, so see more of $\lambda_0$ is real.

$\therefore$ see pair of $\lambda_0$ is $\pm 1$. i.e. $\text{max } |C(P)| = 1$.

So $\lambda_0$ is a root of unity (if $C_{\mathbb{P}, P}(r) \neq 0$).
  - for given $P$ need to show $\forall E \neq \mathbb{E}$ were $k_1 \neq 0$ -
  - would contradict fact that might not have all appear in see cohomology ... $\forall E \neq \mathbb{E}$ $C_{\mathbb{P}, P}(r) \neq 0$

$\Rightarrow \lambda_0$ is not of $\mathbb{Q}$. 

So now vary our $r$. We $\lambda_0^r = 1 = \text{root of } \mathbb{Q}$.

Taking are value as many times: so this root $\mathbb{Q}$ is only a rational number.

- $C_{\mathbb{P}, P}(r) = \zeta_{E, P} \lambda_0^r$, $\zeta_{E, P} \in \mathbb{Q}$.
We now get
\[
\text{tr}(gF, \mathcal{H}^E(\mathcal{X}_w)) = \sum_{E \in W} \sum_{\rho \in \rho} \chi^\rho \rho(g) \cdot \text{tr}(g, E(g^\rho))
\]
\(r \geq 1\)

both sides depend coherently on \(r\) via the powers of eigenvalues
-- so by linear algebra this formula must also be valid
for \(r = 0\)? (can't really use this for \(r = 0\))

\[
\text{tr}(g, \mathcal{H}^E(\mathcal{X}_w)) = \sum_{E \in W} \sum_{\rho \in \rho} \chi^\rho \rho(g) \cdot \text{tr}(g, E(g^\rho))
\]
\(G \in (T_w \times W)\)

apply orthogonality relations to \(W \Rightarrow\)
\[
\sum_{\rho \in \rho} \chi^\rho \rho(g) = \text{tr}(g, E(g^\rho))
\]

So \(\chi^\rho \rho\) are just multiplicity \(a_{E, \rho} = \langle \rho, E \rangle\).
Taking \(r\) that kills the root of unity \(\chi^\rho\), set \((\sum_{\rho \in \rho} \chi^\rho \rho(g) = 0)\)
\[
\text{tr}(gF, \mathcal{H}^E(\mathcal{X}_w)) = \sum_{E \in W} \text{tr}(T_g, E(g^\rho)) \cdot \text{tr}(g, E(g^\rho))
\]

We also have now \(\chi^\rho \chi^\sigma: \text{tr}(g^{\# F}, \mathcal{H}^E(g^\sigma))\)
\[
\sum_{\rho \in \rho} \langle \rho, E \rangle \chi^\rho \chi^\sigma \rho(g) \cdot \text{tr}(g, E(g^\rho))
\]
so twisted character \((F-\text{class} F\sigma)\) on \(L(F)\)

can be written in terms of twisted characters:
\[
\rho_a \chi^\rho = 1 \Rightarrow \langle \rho, E \rangle = \delta_{\rho_a, E}
\]

\(\Rightarrow\) Shintani descent for \(G_a\):
\[
\text{tr}(g^{\# F}, \mathcal{H}^E(\mathcal{X}_w)) = \text{tr}(g, E(g^\rho))
\]

(but e.g. explicit in \(g \in \mathcal{X}_w\))