D. Bogarchenko: Character sheaves and the orbit method (Drinfeld) 8/26/08

Orbit method: we'll work with unipotent groups.

One possible goal of the theory of character sheaves is to construct an algebraic group $G_0$ over $\mathbb{F}_q$, & try to produce a collection $\mathcal{C}(G)$ of irreducible perverse sheaves on $G = G_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ when $G_0$ enjoys a suitable property.

Let $Fr: G \to G$ denote the Frobenius endomorphism. (For absolute Frobenius on $G_0$.)

Let $\mathcal{C}(G)^{Fr} = \{ M \in \mathcal{C}(G) \mid Fr^* M = M \}$

For every $M \in \mathcal{C}(G)^{Fr}$, choose $\psi_M: Fr^* M \to M$ (any two proportional).

Thus, for $M \in \mathcal{C}(G)^{Fr}$, define

$$\psi_M: Fr^* M \to M$$

by

$$\psi_M(g) = \sum_{i \geq 0} (H_i)^i Fr (\psi_M, H_i(M_g))$$

where $g \in G_0(\overline{\mathbb{F}_q}) = G(\overline{\mathbb{F}_q})^{Fr}$.

Property: The irreducible characters of $G_0(\overline{\mathbb{F}_q})$ can be "reconstructed" from the $\psi_M$'s.

Lusztig: solve this problem for reductive groups.

Today: solve for unipotent groups (w/ some restrictions).

Reminder about the orbit method:

$\Gamma = \text{finite group}, \quad \text{Fun}(\Gamma) = \text{functions } \Gamma \to \overline{\mathbb{Q}_e}$

is an algebra with convolution.

Class functions $\text{Fun}(\Gamma)^\Gamma$ form subalgebra $= \mathbb{Z} \text{Fun}(\Gamma)$.
Key point: irreducible characters of \( \Gamma \) are (up to scalars) the same as the minimal idempotents in \( \text{Fun}(\Gamma) \).

If we had a Fourier transform relating convolution to multiplication, easy to read off idempotents.

Orbit method works when \( \exists c \in \mathbb{N} \) s.t. \( \Gamma \text{Nil}_c \)

\[ = \{ \text{(possibly infinite) abstract groups } N \text{ s.t. } 1. \text{ nilpotent class of } N \text{ is } \leq c \]

\[ 2. \text{ the map } N \to N \times \cdots \times N \text{ is bijection for } 1 \leq k \leq c \} \]

M. Lazard defined isomorphisms of categories

\[ \text{Nil}_c \xrightarrow{\exp} \text{nil}_c = \{ \text{Lie algebras } \mathfrak{n} \text{ over } \mathbb{Z}\mathbb{Z} \text{ with nilpotent class } \leq c \} \]

\[ \text{Nil}_c \xrightarrow{\exp} (\mathfrak{n}) \text{ is } \mathbb{N} \text{ as set with multiplication given by Baker-Campbell-Hausdorff} \]

\[ \log (\exp(x) \exp(y)) = \log (\exp (x+y)) \]

Now let \( \Gamma \in \text{Nil}_c \) be finite, \( \omega = \log (\Gamma) \).

\[ \exp: \omega \to \Gamma \text{ the identity map on underlying sets.} \]

Have action of \( \Gamma \) on \( \omega \) & on \( \omega^2 = \text{Hom} \mathbb{C} (\omega, \mathbb{C}^2) \)

Key result: \( \exp^* : (\text{Fun}(\Gamma))^\mathbb{C} \to (\text{Fun}(\omega))^\mathbb{C} \)

is an isomorphism of \( \mathbb{C} \)-algebras (RHS: convolution for addition).

Easy: Fourier transform \( \mathcal{F}: \text{Fun}(\omega) \to \text{Fun}(\omega) \)

\[ \mathcal{F}(f) (x) = \sum x \omega (x) f(x) \text{ is } \Gamma \text{-regular isomorphism of } \mathbb{C} \text{-algebras} \]

\[ (\text{Fun}(\omega, x)) \to (\text{Fun}(\omega, x)) \]
So irreducible characters of $\Gamma \to \Gamma'$-orbits on $\mathfrak{g}^*$, i.e. Kirillov character formula: irreducible characters are Fourier transforms of coadjoint orbits.

**Geometric setting**

$\Gamma' = \text{finite nilpotent group}$

$G = \text{unipotent group}$

over $k = \mathbb{C}$, char $k = p$

$\Gamma \in \text{Nilp}$

for some $\leq N$

Nilpotent class of $G$ is $\leq p$

$\text{Fun}(\Gamma') \to D_c(G) = D_c^G(G, \mathbb{Q}_p)$

constructible derived object

$\text{Fun}(\Gamma') \to D_b(G) = G$-equivariant objects in $D_c(G)$

convolution $\ast$

Convolution: let $\mu, \nu, \psi : G \times G \to G$

$M \ast N = \mu_\ast (\psi \circ \nu)$

$\text{g} = \log \Gamma' \to \text{g} = \log G$ is the Lie ring scheme representing the functor $k \times \Gamma \to \text{nilp}_G$, $S \to \log(G(S))$

won't in general come from a Lie algebra.

Remark: even if $G^p = 1$, $\text{g}$ may NOT come from a Lie algebra over $k$ (as a set functor it is the Pontrjagin dual) 

For example if $p \geq 3$ and noncommutative 2-dimensional $G$'s so can't come from a Lie algebra.

Pontrjagin dual $\to$ Serre dual:
Given a connected unipotent group \( G \) over \( k \), connected \( \mathcal{A}_G \) and a central extension

\[
0 \rightarrow G_p / \mathbb{Z}_p \rightarrow E \rightarrow \mathcal{A}_G \times G_p \rightarrow 0
\]

such that

\[
0 \rightarrow G / \mathbb{Z}_p \rightarrow E^\text{per} \rightarrow \mathcal{A}_G^\text{per} \times G_p \rightarrow 0
\]

is a universal central extension of \( G^\text{per} \) by \( G_p / \mathbb{Z}_p \).

Fix an isomorphism \( \mathbb{Q}_p / \mathbb{Z}_p \cong \mathbb{Q}_p^\times \).

Let \( L \) be a local system on \( \mathcal{A}_G \times G_p \) induced from \( E \) via this.

**Fourier Transform**

\[
\text{Fourier-Deligne transform} \quad F : \mathcal{D}(\mathcal{A}_G) \longrightarrow \mathcal{D}(\mathcal{A}_G)
\]

\[
F(m) : \text{pr}_1^!(\text{pr}_2^*M) \otimes L \otimes L^\vee \left[ d \right]
\]

\[d = \dim \mathcal{A}_G, \text{ takes } \ast \text{ to } 0.\]

**Key properties:**

i) \( F \) preserves perverse sheaves.

ii) \( F(M \boxtimes N) = F(M) \boxtimes F(N) \left[ d \right] \)

If \( \mathcal{A}_G = \text{Log} G \Rightarrow \)

\[
F : \mathcal{D}_c(\mathcal{A}_G) \longrightarrow \mathcal{D}_c(\mathcal{A}_G)
\]

**Remark:** all of this makes sense for \( k = \overline{\mathbb{Q}_p} \), in this case \( \mathcal{A}_G : \mathcal{A}_G(\overline{\mathbb{Q}_p}) \times \mathcal{A}_G(\overline{\mathbb{Q}_p}) \rightarrow \overline{\mathbb{Q}_p}^\times \)

is a perfect pairing.

Analog of minuscule weights \( \rightarrow \) the functions of codimension 0's should be \( \mathcal{A}_G \) constant (fixed on arch) \& \( \mathcal{A}_G \) (constant orbits are closed for minuscule pairs).
Conjecture to prove: \( \mathfrak{P} \lhd \mathcal{Q} \) in effect.

Def: An object \( \mathfrak{P} \) in a monoidal category \( \mathcal{M} \) is an ideopotent if \( \exists \mathfrak{P} \to \mathfrak{P} \) which becomes an isomorphism after tensoring by \( \mathfrak{P} \) on either side.

Note: guess: if \( k = \mathbb{F}_q \) and \( G = G_0 \otimes k \) then the character spaces on \( G \) should be minimal ideopotent in \( \mathcal{D}_0(G) \).

--- not good enough in general, because if \( \mathfrak{P} \subset \mathfrak{g} \) is defined over \( \mathbb{F}_q \), then \( \mathcal{M}(\mathfrak{P}_q) \) may not be in \( \mathcal{D}_0(G_0(\mathbb{F}_q)) \).

**Theorem:** Assume that the \( G \)-stabilizers of points in \( \mathfrak{g} \) are connected. For every minimal ideopotent \( \mathfrak{P} \in \mathcal{D}_0(G) \) there exists \( 0 \leq \eta \leq \dim G \) such that \( \mathfrak{P} \in [\eta] \) is proper, \& if \( G^*(\mathfrak{P}) = \mathfrak{P} \) there is a canonical choice of \( \mu : G^*(\mathfrak{P}) \rightarrow \mathfrak{P} \) such that \( \mathcal{D}_0(G_0(\mathbb{F}_q)) \) is an indecomposable character of \( G_0(\mathbb{F}_q) \) in the reduced character is obtained this way.

**General setting (nilpotent class \( \mathfrak{g} \))**: recall minimal ideopotent:

\[ \mathfrak{P} \in \mathcal{D}_0(G) \text{ minimal ideopotent} \]

\[ \{ \mathfrak{P} \in \mathcal{D}_0(G) \mid \mathfrak{P} \text{ is a } \mathfrak{g} \text{-stable subobject of } \mathfrak{g} \text{ stable} \} \]

Def: a character \( \mathfrak{P} \) of \( \mathfrak{P} \) is an irreducible object of \( \mathcal{M} \).

**Theorem:** this construction achieves desired goals.

\& \( \mathcal{M} \) is semisimple. Closedness of orbit and identity.