Hodge Theory - T. Panet

\( X \) smooth projective \( / \mathbb{C} \).

\[ H^*_\text{Del}(X, \mathbb{C}) = \bigoplus H^*(X, \Omega^*_X) \]

\( H^*_X = H^*(\Omega^*_X, \mathbb{C}) \),  \( H^*_X(X, \mathbb{C}) = \text{sing}\text{. cohomology} \)

Source of Hodge theory - relate HCH cohomology.

Then (de Rham): \( H^*_X \cong H^*_0 \) In fact true over alg. closed field char 0, local fields etc. - comes from Riemann Hilbert.

Then (Hodge) \( H^*_0 = H^*_1 \) (Kähler)

These come with different linear algebraic data, combine them all.

Hodge structure: \( V \) - in \( \Omega^* \mathbb{R} \) rebr. some.

Extra structures: \( \omega \in \Omega^{2*} \) symmetric form

- metric \( \langle x, y \rangle = \omega(x, y) \), nondegenerate.
- almost complex structure \( J:V \to V \), \( J^2 = -1 \).

\[ \{ V, \omega \} = V \otimes \overline{V} \]

\[ V \xrightarrow{\sim} \overline{V} \]

Kähler structure - these forms with compatibility: \( V \to \mathbb{C} \)

\( H \) hermitian nondegenerate form on \( V \).

The structure is positive if \( H > 0 \).

\( H = h + i\omega \)

\[ \omega(x, y) = h(x, y) \]

On \( \Lambda^*V \): \( J \Rightarrow \text{type decomposition} \).

\[ \Lambda^{*,2} V = \Lambda^{0*,2} V \otimes \Lambda^{2*,0} V \]

Kähler \( \Rightarrow \) \( \omega \) is of type \( (1, 1) \)

\( \omega \) - sl(2) action on \( \Lambda^*V \), \( N^+ \to \mathbb{C}^* \rightarrow \Lambda^*V \).

\( N^+ \mapsto \Lambda^{1,1} \), contraction with Hodge structure \( \chi \): element in \( \Lambda^2V \) corps to \( \omega \in \Lambda^2V \) under the isomorphism induced by \( \omega \).

\( H \to D = (n-1)\text{id} + \Lambda^*V \)

Exercise: check this is sl(2) rep, that \( D \) s a derivation.

\( \Rightarrow \) if \( V = V \otimes V \), \( \omega = \omega_1 \otimes \omega_2 \) check that \( \text{sl(2) action is the tensor product action} \).

\( \omega \) \( \text{Hodge operator} \):

\[ \iota_\omega : \Lambda^m V \to \Lambda^{2n-m} V \]

\( \text{Interchanges degrees} \).

\( \text{Int. charges} \).

\( \Lambda \)
Now let $(V, \omega, J)$ be a Kähler space.

Def: Primitive k-forms are low-weight k-forms ($\Lambda^k = 0$).

Since $\Lambda^k$ is of pure bidegree $(-k, -k)$, $\Lambda^k \omega$ makes sense.

Prop (Lefschetz decomposition)
1. $\Lambda^k \omega = \text{ker } L^{n-k} \Lambda^k V \oplus \text{ker } L^{n-k} \Lambda^k V$
2. $\Lambda^k V = \bigoplus \Lambda^k \omega$

Note that $\Lambda^k = \Lambda^1 (V_0, \omega)$ acts on $V$, hence on $\Lambda^k V$.

Exercised: Check that $\pi^* \omega$ is an inner of $\Lambda^k$ for $\pi^* \omega$ in $\Lambda^k V$.

Note also that $\Lambda^k = \text{End } (\Lambda^k V)$ normalizes each other
$\Rightarrow$ Lefschetz gives $\Lambda^k$ decomposition

There are two natural operators from $\Lambda^k$ to $\Lambda^{n-k} V$:
1. $L^{n-k}: \Lambda^k \omega \rightarrow \Lambda^{n-k} V$
2. $\Lambda^k \omega \rightarrow \Lambda^{n-k} V$

Proposition: Take $\zeta \in \Lambda^k \omega$. Then
$\zeta \leq \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)}$

Exercised: Both sides commute with $(\zeta, \zeta)$, check on irreducible, hence proportional by $\omega(\zeta, \zeta)$, check on highest weight vector. (See Wells)

Def: Define a Hermitian inner product (polarization) on $\Lambda^k V$ by setting $\langle \zeta, \zeta \rangle = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)}$

Claim: $(-1)^k \zeta$, $\zeta$ is positive definite on $\Lambda^k \omega$.

Proof: $\langle \zeta, \zeta \rangle = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)} = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)} = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)} = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)} = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)}$

$\Rightarrow \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)} = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)} = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)} = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)}$

$\Rightarrow \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)} = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)} = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)} = \frac{1}{k!} \frac{1}{n-k-2 \omega(\zeta, \zeta)}$

Thus (Newlander - Nirenberg) If $C$ is integrable if $\omega$ is closed, (C) is integrable if $\mathbb{T} = \omega$ is an integrable distribution.

Thus (Kähler) If $M$ is Kähler, $\omega$ is integrable to second order to a flat metric

$L, M, D$ act on $A^{p, q}, A^{p, q}$

Manifolds for which $L, M, D$ descend are called Hodge $\equiv$ Kähler.
Integrability - extra condition to impose linearity on field / sheaves of sections \( \mathcal{O}(M) \) - compatibility
with giving certain structures are locally products.

Def. A tensor structure \( T = (\mathcal{V}, \mathcal{W}, \mathcal{U}, \mathcal{T}) \) is called integrable if for every \( x \in M \),
there exists a neighborhood \( V \ni x \) in \( \mathcal{O}(M) \) of \( \mathcal{U} \) such that \( T \) becomes constant in this trivialization.

In symplectic, almost complex cases have easy sufficient
conditions for integrability.

Then (Darboux) \( T \) is integrable if \( \omega = 0 \).

Then (Newlander-Nirenberg) \( T \) is integrable \( \iff \bar{T} \) is integrable where

Equivalently, decomposing \( T = d \omega \), sections to
\( \text{Kähler} \) and \( \text{symplectic} \).

Def. A manifold \( M \) is equipped with a Kähler structure if
it's equipped with an integrable complex structure and a
Kähler metric it's integrable

Free.

Let \( M \) be a complex Hermitian manifold. Then:

1. \( M \) Kähler, \( T \) flat. \( \forall \psi \in \mathcal{T} \) each local hol. coordinate \( z \), \( \psi \), \( \bar{z} \), \( z \), \( \bar{z} \), \( \psi \), \( \bar{\psi} \)

so that if \( \psi = H (\frac{z}{2z}; \frac{z}{2z}) \), \( \psi = i \frac{d\psi}{d\bar{z}} + O(\|z\|^2) \)

If. Write \( \omega = \omega + i \ar{z} \frac{d\psi}{d\bar{z}} \). For \( z \), \( \bar{z} \), \( \psi \), \( \bar{\psi} \), \( \psi \), \( \bar{\psi} \), \( \psi \), \( \bar{\psi} \)

Conversely, can choose \( \omega \) such that \( \forall \psi \), \( \bar{z} \frac{d\psi}{d\bar{z}} \) \( \omega = i \frac{d\psi}{d\bar{z}} + O(\|z\|^2) \)

Look for quadratic change of \( \omega \) in \( \psi \), \( \bar{\psi} \), \( \psi \), \( \bar{\psi} \), \( \psi \), \( \bar{\psi} \), \( \psi \), \( \bar{\psi} \)

\[
\frac{\partial}{\partial \psi} = \Xi (\frac{\partial}{\partial \psi} + \frac{\partial}{\partial \bar{\psi}}) \frac{\partial \psi}{\partial \bar{\psi}}
\]

\[
\bar{\psi} = i \left( \frac{\partial}{\partial \bar{\psi}} + \frac{\partial}{\partial \bar{\psi}} \right) \psi
\]

\[
\begin{pmatrix}
\frac{\partial}{\partial \psi} & \frac{\partial}{\partial \bar{\psi}} \\
\frac{\partial}{\partial \bar{\psi}} & \frac{\partial}{\partial \psi}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\bar{\psi} & \psi \\
\psi & \bar{\psi}
\end{pmatrix}
\]
Thus \( a^{ik}_{ij} = a^{ik}_{ji} + \frac{j^2 \xi_j}{2x_k} \), so set \( a_k = -\frac{1}{2} \xi_k \) in \( \mathcal{W}_k \).

But \( d\omega = 0 \Rightarrow a^{ik}_{ij} = a^{ik}_{ji} \) so this is conserved.

\[
A^k(m) = 0, \quad A^{ik}(m), \quad \text{and} \quad \alpha^k \rightarrow \alpha^k \text{to} \quad 2/\alpha.2.
\]

Then \( \mathcal{L} \mathcal{G}(m) \subset H^1(m) \) (all \( \alpha^k \) are fine, by Poincaré; it is a resolution.)

Then (Deligne) set \( \mathcal{H} \mathcal{G}(m) \subset H^1(m) \).

Then in \( \mathcal{H}^0(\mathcal{G}^n) \subset H^0(\mathcal{G}^n) \).

\[ p.1 \]

Kähler Manifolds

1. \( \mathcal{K} \mathcal{M} \) manifolds: \( V_{\alpha} : \mathcal{H}^0 \subset \mathbb{C}^n \equiv \mathbb{C}^n / \Gamma, \)

\[ \Gamma = \{(a_i, o) \mid a_i = 1 \text{ or } 2 \} \]

\( \mathcal{H}^0 \subset \mathbb{C}^n \equiv \mathbb{C}^n \setminus S \)

2. Gadd-Frank \( \mathcal{C} \mathcal{E} \mathcal{M} = S_{2n+1} \times S_{2m+1} \)

with fiber at zero glue to \( \mathbb{C}P^n \times \mathbb{C}P^m \)

a complex structure:

Choose \( T \) in \( \mathbb{C} \) plane, \( E = \mathbb{C} / 2 \pi \alpha \)

\( \mathbb{C}^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid \bar{z} = \bar{z} \} \)

\( \mathbb{C}^{2m+1} = \{z \in \mathbb{C}^{m+1} \mid \bar{z} = \bar{z} \} \)

\( \text{coordinate charts: } x_0(x, \bar{z}) = \frac{x_0}{x_0} \)

\( x_0(x, \bar{z}) = \frac{x_0}{x_0} \quad y_0 = \frac{x_0}{y_0} \quad z_0 = \frac{z_0}{z_0} \)

\( (x_0, x_1, \ldots, x_{2n+1}) : V_{\alpha} \rightarrow \mathbb{C}^n \times \mathbb{C}^n \times E \)

\( \text{transitions: } \alpha_1, \alpha_2 \rightarrow \alpha_2 \)

\( x_{i+1} = \frac{x_{i+1}}{x_i}, \quad y_{j+1} = \frac{y_{j+1}}{y_i} \quad z_{k+1} = \frac{z_{k+1}}{z_i} \)

\( \text{transitions: } \alpha_1, \alpha_2 \rightarrow \alpha_2 \)

\( x_{i+1} = \frac{x_{i+1}}{x_i}, \quad y_{j+1} = \frac{y_{j+1}}{y_i} \quad z_{k+1} = \frac{z_{k+1}}{z_i} \)
Regard for $M$ compact Kähler, $b_2(M) = 1$ from $\omega$. ($w^0$ when $6m$)

$E_{\infty}$ can't be killed: $b_2 = 0$.

Even if we start drawing up points on $E_\infty$, get $b_2 = 0$ but $H^{\infty}$ it's still not zero.

$(\Lambda^0, 2 \mathbb{J})$: diagonal complex is de Rham $(\Lambda^0, d)$

the horizontal complex is $(\Lambda^0, 2) \mathbb{J}$.

The fact that $(\Lambda^0, d)$ comes from a double complex gives us a filtration from horizontal filtration of $\Lambda^0$.

$E_0^0 = \mathbb{B} \mathbb{J}^0$.

$E_0^{p, 0} = H^q(M, \mathbb{S}_{M}^p)$.

$E_1^{p, 0} = H^q(M, \mathbb{S}_{M}^p)$.

Each version for $dR$, try to substitute $(\Lambda^0, d)$ with $(\Lambda^0, 2 \mathbb{J})$ except not fine.

But we have $\mathbb{B} \mathbb{J}^0 \rightarrow \mathbb{B} \mathbb{J} \rightarrow \mathbb{J}$.

Think of $\mathbb{J}$ as more of a category $!: \mathbb{J} \rightarrow \mathbb{J}$.

Get now on cohomology $H^0(M, \mathbb{J}) \rightarrow H^0_{\mathbb{J}}(\mathbb{J}^0)$.

Hope he is isomorphism.

Theorem is an isomorphism.

If $\mathbb{J}$ may be: is a quasi-isomorphism, i.e., $H^0(\mathbb{J}^0, \mathbb{J}) = \mathbb{J}$.

- the $\mathcal{O}$-Poincaré lemma...$

$E_1^{p, 0} = H^q(M, \mathbb{S}_{M}^p)$.

Spectral sequence still makes sense: $H\mathcal{O}$ - de Rham is.

Algebraically we have $H^p(M, \mathbb{S}_{M}^q) \rightarrow H^p(\mathbb{J}^0, \mathbb{J})$.

$i_{\mathbb{J}}$ (Grothendieck) The natural map $H^q(\mathbb{J}^0, \mathbb{J}) \rightarrow H^q(\mathbb{I}, \mathbb{J})$ is an iso.

In proper case, tautological (etale). In general need a resolution of singularities.

Let $M$ be Kähler. Goal: find HIR degenerates at $E_i$.

Disregard bimeromorphic. Need to construct canonical maps of dR to de Rham classes. First way to measure sizes of elements. Take positive definite metric on $M$.

$M$ compact oriented (for integration).
A form is harmonic if closed if of minimal size in its cohomology class. 

For a local form, find formal adjoint of derivative. Use \( \Delta \) as \( \Delta^k \) is formal adjoint of \( \Delta^{k-1} \). 

For the natural map \( \ker \Delta^k \to \Omega^k(M) \) is injective. 

\[ \Delta = \Delta^k + \Delta^{k-1}. \]

\[ \Delta \text{ harmonic } \iff \Delta^k \neq 0 \quad \Delta^{k-1} = 0. \]

Theorem (Hodge) The map \( \Omega^k \to H^k(M) \) is an isomorphism.

Weil proof for Riemann surface: use conformal flatness. 

Reduce to case of disc whose we have Green's function & Poisson kernel.

For complex Hermitian manifold \((M, H)\), do similar for \( \overline{M} \).

Laplacian \( \Box = \partial\overline{\partial} + \overline{\partial}\partial \), \( \Box = 5 \ast + 5 \ast = 5 \ast + 5 \ast = 5 \ast \)

still elliptic. A form \( \omega \in \Omega^k(M) \) is harmonic \( \iff \)

\[ \overline{M} \text{ harmonic } \iff \overline{M} \text{ harmonic } \iff \frac{\partial^2 \omega}{\partial z \partial \overline{z}} \equiv H^2(\overline{M}) \]

For \( M \) Kähler, have Kähler identities:

1. \[ [\wedge, \Box] = 0, \quad [\wedge, \Box^*] = 0 \]
2. \[ [\wedge, \Box^*] = 0 \]
3. \[ [\wedge, \Box] = 0 \]
4. \[ [\wedge, \Box^*] = 0 \]
5. \[ \frac{1}{2} \Delta = \Box = \Box^* \]
6. \[ [\wedge, \Box] = [\wedge, \Box^*] = 0 \quad \Rightarrow \quad H^2 = 0 \]

Remark: On cohomology, \( \Delta \) doesn't depend on anything but the class of the Kähler metric. \( \Delta = [M] \) only depends on the class. \( \Delta \) seems to depend on metric but doesn't.

i.e., if we have two metrics with same [\( M \)], we get two

\( \Delta \) which agree on \( H^2 \).

\[ \Rightarrow \text{ Let's take decomposition} \quad H^2(M) \text{ has Levi-Civita's} \]

Hodge-Kähler relations:

1. \[ \Omega^{1,0} = H^{1,0} \]
2. \[ \Omega^{0,n} = \overline{H^{0,n}} \]
3. \[ \Omega^{n,0} = \overline{H^{n,0}} \]
4. \[ \Omega^{0,1} = H^{0,1} \]

\[ \text{Hodge index} \quad n \text{ of even complex dim } n, \quad \Omega^k \ast \overline{\Omega^k} \to \text{c.c.} \text{ for } \Omega^k \text{ prime} \]

For \( q, \overline{q} \): 0 for \( q \in H^{1,0}, \overline{q} \in H^{0,1} \), unless \( p, \overline{p} = g, \overline{g} \),

\( p \neq \overline{p} \).

Hodge index \( M \) of even complex dim \( n, \quad \Omega^k \ast \overline{\Omega^k} \to C, \quad \Omega^k \ast \overline{\Omega^k} = \sum (\ast \Omega^k) \overline{\Omega^k} \)
Hodge-de Rham spectral sequence:
\[ f: X \rightarrow S \text{ smooth } \Rightarrow (\Omega^\bullet_X, d, \lambda) \Rightarrow f^! \mathcal{O}_S \text{ - linear}. \]
\[ E^{p,q}_1 = H^p_X(\mathcal{O}_X) \Rightarrow H^{p+q}_X(S) = H^{p+q}_X(X/S) \]

- degenerate for proper morphisms of schemes via $\mathcal{O}$.

Deligne, IHES 68

There is a filtration \( F^p H^{p+q}_X(X/S) \), Hodge filtration.

\[ X \text{ smooth proper } \Rightarrow \text{ set } H^{p,q}_X(X) = \bigcap F^p H^{p,q}_X(X). \]

\[ h^{p,q} = \dim H^{p,q}(X, \mathcal{O}_X) \]

Remarks:
1. Complex conjugation comes from characteristic.
2. \( H^{p,q}(X) = H^{p,q}(X, \mathcal{O}_X) \) (Hodge theory) \( \Rightarrow \)
3. \( H^{p,q}(X) = H^{p,q}(X) \)

Prop. \( X \) smooth, proper. \( \Rightarrow \)

1. \( H^{p,q}(X) \) degeneration at \( E_1 \)
2. Canonical isomorphism. \( F^p H^{p,q}_X(X) \)

In particular, \( H^{p,q}_X(X) = \bigcap H^{p,q}_X(X) \).

\[ h^{p,q} = h^{p,q} = \dim H^{p,q}_X(X) \]

Proof:

Case 1: \( X \) projective:

\[ E_1 \Rightarrow E_1 \]

\[ H^{p,q}_X(X) \Rightarrow H^{p,q}_X(X) \]

\[ [\xi] \rightarrow [\xi_1], \quad \xi \rightarrow \xi_1 \text{ closed}. \]

Pick a harmonic. By Kahler identity, \( \xi \) is also

\[ d_1 \text{ harmonic } \Rightarrow d_1 \text{ closed } \Rightarrow d_1 = 0. \]

\[ d_1: E_1 -> E_0 \]

On \( \xi \in E_1 \), this is as follows:

\[ \xi \in H^{p,q}_X \Rightarrow \\xi \in H^{p,q}_X \]

such that projections to \( E^{2,1} \) \( E^{1,1} \) are all in \( H^{p,q} \)

\[ \ker d_1, \ker d_2, \ldots, \ker d_{p+1} : d_1 \chi_{\xi} = 0 \quad \exists \chi_x \]

\[ 2d_2 = \xi \]

For \( (i) \) notice \( H^{p,q}_X = H^{p,q}(\mathcal{O}_X) = H^{p,q} \).

Case 2: \( X \) proper:

\[ \phi: X \rightarrow X, \quad X \text{ proj scheme, } \phi \text{ birational morphism.} \]

\[ E_1 = H^{p,q}_X(X, \mathcal{O}_X) \Rightarrow H^{p,q}_X(X) \]

\[ E_1 = H^{2,1}(X, \mathcal{O}_X) \Rightarrow H^{2,1}(X) \]

Pullback of forms \( \Rightarrow g^*: E \rightarrow E \), so suffices to show

\[ g^* \text{ is an embedding}. \]
Lemma: Let \( f : X \rightarrow Y \) be a proper birational morphism between smooth varieties. Then \( f^* : \mathcal{N}_Y(Y, \mathcal{N}_Y^n) \rightarrow \mathcal{N}_X^n(X, \mathcal{N}_X^n) \) is injective.

Proof: Use the Gysin map.

Gysin maps (disregard) \( f : X \rightarrow Y \) proper morphism between smooth varieties. Gysin map: \( \text{Tr}_f : R^d f_* \mathcal{O}_X \rightarrow \mathcal{N}_Y^n \mathcal{O}_Y \) \( d = \dim X - \dim Y \), \( m = \deg f \).

It is the dual to natural pullback \( f^! \mathcal{N}_Y^n \rightarrow \mathcal{N}_X^n \).

Special cases:

1. \( f : X \rightarrow Y \) a finite map.
   \( \text{Tr}_f : f^* \mathcal{O}_X \rightarrow \mathcal{O}_Y \). \( f \) is finite, so \( f^* \mathcal{O}_X \) is locally free, so \( \text{Tr}_f \in H^0(Y, \text{Hom}(\mathcal{O}_Y, f_! \mathcal{O}_X)) \).

2. \( \text{Hom}(f^* \mathcal{O}_X, \mathcal{O}_Y) = (f^* \mathcal{O}_X)^* \otimes \mathcal{O}_Y = \) (relative duality)
   \( = f_* (\mathcal{O}_X^* \otimes f_* \mathcal{O}_Y) \otimes \mathcal{O}_Y = \) (projection formula)
   \( = f_* (\mathcal{O}_X^* \otimes \mathcal{O}_X \otimes f_* \mathcal{O}_Y \otimes f^* \mathcal{O}_Y) = \) (contraction)
   \( = f_* (\mathcal{O}_X \otimes f_* (\mathcal{O}_Y)) v = f_* \text{Hom}(f^* \mathcal{O}_Y, \mathcal{O}_X) \)

where we have a preferred section.

Explicitly if \( y \in Y \) is pt where \( Y \) is étale
\[
(f^* \mathcal{O}_X)_y \xrightarrow{\text{Tr}_f} (\mathcal{O}_Y)_y
\]
\[
\oplus_{x \in f^{-1}(y)} (\mathcal{O}_X)_x \xrightarrow{\oplus (df)^{-1}} (\mathcal{O}_Y)_y
\]

Case 2: \( f : X \rightarrow Y \) smooth with connected fibers of dim \( d \).

\( R^d f_* \mathcal{O}_X \) still locally free.

\( \text{Tr}_f : R^d f_* \mathcal{O}_X \rightarrow \mathcal{O}_Y \)

Again \( \text{Tr}_f \) section of \( \text{Hom}(R^d f_* \mathcal{O}_X, \mathcal{O}_Y) \)
\[
= (R^d f_* \mathcal{O}_X)^* \otimes \mathcal{O}_Y = R^d f_* ((f^* \mathcal{O}_Y)^* \otimes \mathcal{O}_Y) \otimes \mathcal{O}_Y
\]
\[
= f_* (\mathcal{O}_X^* \otimes f_* \mathcal{O}_Y \otimes f^* \mathcal{O}_Y \otimes f^* \mathcal{O}_Y)
\]
By induction on \( r \leq s \) set \( \delta^*: E_r \to E_r' \) \( dr = 0 \),
\( E_r = E_r' \) \( \implies \) part (i)

Part ii): Note that \( F^p H^r_{DR}(K) \cap F^{p+r}\ H^r_{DR}(K) = \{0\} \)
by degeneration of \( E_r \). But \( E_r \) is a sum of \( S \)
\( \implies F^p H^r_{DR}(K) \cap F^{p+r}\ H^r_{DR}(K) = \{0\} \)
\( \implies \sum h_i n_i \leq \sum \frac{h_i n_i - 1}{s - p + 1} \)
\( \implies \sum h_i n_i - 1 \geq \sum \frac{h_i n_i - 1}{s - p + 1} \).

But if \( N = \sum h_i n_i \) is some dual \( b_i = (n_i - 1, n_i - 1) \), get

Theorem: S scheme in char 0, \( f:X \to S \) proper generically

i) \( R^s f_* \mathcal{L}^s \) are locally free of finite type

ii) The Hodge de Rham SS degenerates

iii) At every point of \( S \), \( R^s f_* \mathcal{L}^s \) and \( R^s f_* \mathcal{L}^s \) have same rank.

Hodge-de Rham: \( E_r^{s} = R^s f_* \mathcal{L}^s \Rightarrow R^n f_* \mathcal{L}^s / S \times S \)

Proof: f proper \( \Rightarrow R^s f_* \mathcal{L}^s \) are coherent \( \& \) have base change property. In particular it suffices to prove iii at a point \( s = k \).

Reductions: – may assume S affine (statement real in S)
– May assume S noetherian: commutes with inductive limits.
– May assume S local
– Faithful flatness of completion \( \Rightarrow \) assume S complete
– Since every R (local, complete, Noetherian is afft), limit \( \lim R/m^k \)
use comparison theorem in cohomology: same holds in cohomology – E term of H-dR is a projective limit of the \( E_i \) terms of H-dR over R/m^k.
$R/m^n$ are finite length, $k \otimes R$ is projective over them $\implies$ (1) holds automatically. Also $E^p_{q, r}(R/m^n)$ is projective, hence $\varprojlim E^p_{q, r}(R/m^n)$ by reduction and $m^{-k}$.

$\varprojlim E^p_{q, r}(R/m^n)$ is a projective limit which will be a free $R$-module.

Ultimately, may assume $S = \text{spec } A$, $k$-algebra with residue field of char. 0.

Use Lefschetz principle, hence $\implies$ assume $k = \mathbb{C}$.

This $A$ is a local Artinian $\mathbb{C}$-algebra, $S = \text{spec } A$, maximal ideal.

The SS. is $(\pi_1)^h \Rightarrow \Gamma^h \Rightarrow R^\text{proj}$.

Let $X^\alpha$ be the underlying complex analytic manifold of $X$.

$\mathbb{C}^\alpha$ is a resolution of the constant sheaf $A$.

**Proof** - The differential is $f \cdot \mathbb{C}$ linear $\implies A$-linear.

- The monadic filtration on $L^\alpha_{\mathbb{C}/S}$ is a filtration

  by subcomplexes.

  $\vdash \text{Gr}_{L^\alpha_{\mathbb{C}/S}} = \text{Gr} A \otimes \text{Gr} L^\alpha_{\mathbb{C}/S}$.

  But notice that $\text{Gr} L^\alpha_{\mathbb{C}/S}$ is $dR$ complex for $X^\alpha$ reduced manifold.

  $\implies$ resolution $C. \implies \text{Gr}_{L^\alpha_{\mathbb{C}/S}}$ resolves $\text{Gr} A$.

Now the fact that the monadic filtration is filtration

by subcomplexes + claim gives the lemma.

- **Claim** If $A$ is a filtered Noetherian ring (extensively filtered).

and if $M, N$ are two filtered $A$-modules, then

(a) $\text{Gr} M$ is a free $\text{Gr} A$ module iff $M$ is free $A$-module.

(b) If $f: M \to N$ is filtration preserving morphism, then $\text{Gr} f$ is also iff $f$ is so.

Invoke GAGA.

$x : A \to \mathbb{C}$.

$\mathbb{C}^\alpha R^n \Gamma(L^\alpha_{\mathbb{C}/S}) = \mathcal{A}(A) \cdot \lim_{\longrightarrow} R^n \Gamma(L^\alpha_{\mathbb{C}/S})$.

On the other hand we always have

$\text{length}_A \text{Gr} M \otimes \mathbb{C} \leq \text{length}_A \text{Gr} M$.

So equality occurs iff LHS is a free $A$-module.

$H^r (X^\alpha, L^\alpha_{\mathbb{C}/S}) = \text{Gr} A \cdot \lim_{\longrightarrow} R^n \Gamma(L^\alpha_{\mathbb{C}/S})$.

with equality iff $H^r$ degenerates.
Combining these, we get
\[ N(A) \cong \prod_{x_{fr, x_{cr}} \in H^2(X_{fr}, \mathbb{Q}_L)} \text{det} \left( \mathbb{H}^0(x_{fr}, \mathcal{L}^*) \left[ \text{det} \left( \mathbb{H}^2(x_{fr}, \mathcal{L}^*) \right) \right] \right) \]

But this is = by norm for \( X_{fr} \) 
\[ \Rightarrow \text{get all equalities degenerating} \]

Corollary A smooth local algebra \( \mathcal{A} \) over \( X \) satisfies
\[ H^2(X, \mathcal{L}^{\otimes n}) \] is a free \( \mathcal{A} \)-module.

\[ H^2(X, \mathcal{L}^{\otimes n}) = A \otimes H^2(X_{fr}, \mathcal{L}_{cr}^{\otimes n}) \]

**Algebraic Proof of degeneration**: Deligne–Illusie, Invent 1987, p. 91.

Builds on work of Faltings, Fontaine–Messing, Kato, Mazur, M. Raynaud.

**Remark**: Liftings to char 0. Let \( X \) variety over \( k \), char \( k = p > 0 \).

If \( k = \mathbb{F}_{p^2} \), usual candidate is some \( X \rightarrow \text{Spec } \mathbb{Z} \)

s.t. \( X \otimes \mathbb{Z}_p = X \).

Look at the locally at least complete \( \Rightarrow \) another version is a scheme over \( \text{Spec } \mathbb{Z}_p \) specializing to \( X \).

In general, have **uniformization theorem** (Witt): If \( k \) is a perfect field, char \( k = p > 0 \)

\[ \exists \text{ } W_n(k) \text{ ring, uniquely determined by } \begin{align*}
\text{(1) } & W_n(k) \text{ flat over } \mathbb{Z}/p^n \\
\text{(2) } & W_n(k)/p W_n(k) = k
\end{align*} \]

(2) \( \text{Lin } W_n(k) \text{ is a complete DVR uniformized by } p \)

with residue field \( k \).

Let \( X \) scheme over \( k \) perfect \( \Rightarrow \) lift of \( X \) to char 0 \( \tilde{X} \rightarrow \text{Spec } \mathbb{W}(k) \)

specializing to \( X \) at \( \text{Spec } k \).

**Theorem** (Deligne–Illusie...): \( X \) smooth proper \( /k \) perfect char \( p > 0 \).

Assume \( X \) admits a lifting to \( \text{Spec } W_n(k) \). Then the Hodge–Rham ss satisfies \( E^1_j = E^1_{j+1} \) for \( i < p \).

**Corollary**: If \( X \) is proper smooth \( /k \) of char 0 \( \Rightarrow \) HRR degenerates.

Proof: It is standard that for any \( X \) as above \( \mathcal{F} \) an integral domain \( A \) of fin type \( \mathbb{Z} \), equipped with \( A \rightarrow k \).

\( \mathcal{F} \) is a smooth proper \( f : \mathcal{F} \rightarrow \text{Spec } A \) specializing to \( X \).

Since \( R^f_{\mathcal{F}/k} R^f_{\mathcal{F}/k} \) are coherent, one base change property \( \Rightarrow X_{fr} \) is a \( \mathcal{O}_k \) of \( H^0 \) for \( X \).
Suffices to show $\text{Hilb}(\mathbb{C})$ dominates $\mathfrak{S}$. May assume these coherent sheaves are $\mathfrak{S}$-free by specializing further. All we need is $\exists \bar{x} \in R_{\mathfrak{S}}$ such that $\text{spec}(\bar{x})$.

This can be checked at any point $\bar{x}$ of finite rank or $\mathfrak{S}$-coarse point. Therefore, get residue fields of arbitrarily high char.

To simplify things assume $X$ scheme $/\mathbb{Z}/\mathbb{F}_p$ with lift to $\mathbb{A}^0$.

For $X$ Frobenius: identity on points, raises fibres to $p^k$ power. $F^k a = a^{p^k} = a \circ F^k.

Notice that $\mathcal{O}_X$ has $F^k \mathcal{O}_X$-linear differential:

$$d(a^{p^k}) = p^k \cdot \eta a = 0 \ldots$$

so better to look at this complex, $F^k \mathcal{O}_X$.

We haven't done anything to cohomology since $F^k$ is a finite map!

Then (Cartier) there is canonical morphism of graded $\mathcal{O}_X$-algebras

$$C_X: \bigoplus \mathcal{O}_X \rightarrow \bigoplus \mathcal{H}^i F^k \mathcal{O}_X,$$

satisfying $C_X(da) = [a^{p-1} da] \in \mathcal{H}^i F^k \mathcal{O}_X$

$C_X(a) = a$.

If $X$ is smooth this is iso.

Proof: Reduce to affine case, induction by dimension to affine $X = \text{spec } A$. $A' = \mathbb{F}_p [t]$. $\mathbb{F}_p [t] \rightarrow A'$. For $d = k[t^p]$, choose $d = k[t]/(t^p - t).

Then (Deligne-Illusie) de Rham decomposition: $X \rightarrow \text{Spec } k$.

$\mathcal{O}_X \rightarrow \text{Spec } k[t]/(t^p - t)$ smooth lift $\Rightarrow$ canonical

$\varphi_X: \bigoplus \mathcal{H}^i F^k \mathcal{O}_X \rightarrow \bigoplus \mathcal{H}^i \mathcal{O}_X$ (translated complex)

so $\mathcal{H}^i \varphi_X = C_X$, in particular $\varphi_X$ is a quasi-isomorphism.

LHS is complex with differentials $C^k$ (RHS calculated $dR$...)

Sketch of proof: Assume ideal situation when $F$ lifts to $X \in F$.

Then $C_X: \mathcal{O}_X[t] \rightarrow F^k \mathcal{O}_X$, $[ds] \mapsto [e^{\sum (-1)^i \frac{d^i}{i} \circ \mathcal{O}_X} \mod p]$. Independent of $a \in H^i$ lift of $a$. Induces Cartier.

Extend by multiplicativity. Need $\forall W$ has all splitting if irreducible.
Analyze how this depends on choice of \(F\); only depends up to ambiguity, some \(F\) on cohomology.
Lifts always exist locally, consider on level of sheaf complex, which is quasi-direct to sheaf cohomology.

We used the infinitesimal version of smoothness!

If \(A\) or \(H^*(X)\) the \(A\) over \(S\) such that \(F \to X\), \(\text{dim} H^*(X, k) = \text{deg} F\)

"""
iff \(H^*(F)\) is \(A\)-free.

\[\text{etale} \text{ lift \ r to char } r \text{ e.g. } \text{not necessarily dimensional, like for an abelian curve.}\]
Grothendieck constructed example of \(X\) st. \(\text{dim} H^*(X, \mathbb{Q}) > 2 \text{dim } H^*(X)\)

- l-adic cohomology not good - misses the prime \(p\).

Morgan - Watanabe: If we can lift our scheme over \(k\) vectors, can take de Rham there, they prove that if \(X \to \text{Spec } k\) lifts to \(\tilde{X} \to \text{Spec } \mathbb{C} \), then \(H^*(\tilde{X}/\mathbb{C})\) depends only on \(X\) and has right \(\ell\)-adic number.

Grothendieck - use local lifts to \(\mathbb{C}\) & patch together cohomology theory - need for smooth paper.

Properties: i) Rigid \(\ell\)-adic numbers, Parace cohomology (Krause, et al., much more)

ii) de Rham cohomology is \("\text{mod} \ \text{complex}\) of crystalline:

\[0 \to H^*_{\text{cr}}(X, \mathbb{Q}) \to H^*_{\text{dR}}(X/\mathbb{C}) \to \text{Tor}_{\mathbb{C}}(H^*_{\text{cr}}(X), \mathbb{Q}) \to 0\]

\[\Rightarrow b_1 \leq \text{drh}_{1} < b_1 \cdot b_2 \leq b_1 b_2 \cdot b_2 \]

iii) \text{Hodge theory (illusory)} - get complex \(W^2\mathcal{L}_x\) - de Rham \(\ell\)-adic complex.

\[\Rightarrow \mathcal{H}^* (X) = H^i(W^2\mathcal{L}_x), \text{ geometric to do in help.}\]

The spectral sequence (slope 3, s.s.) for this complex

\[E^1 = H^i(W^2\mathcal{L}_x) \Rightarrow H^*_{\text{cr}}(X)\]

degenerates at \(E^2\), mod torsion.
Kodaira Vanishing

Since coherent sheaf ample \( \Rightarrow \) cohomology zero

\[
\dim X \text{ smooth, projective } \Leftrightarrow L \rightarrow X \text{ ample line bundle, }
\]

then \( H^i (X, \mathcal{L} \otimes \mathcal{L}) = 0 \) for \( i > 0 \)

Kodaira (1953) proved that if \( M \) compact complex manifold, \( L \rightarrow M \) a holomorphic line bundle which possesses a Hermitian metric with everywhere \( > 0 \) curvature \( \Rightarrow H^i (M, \mathcal{L} \otimes \mathcal{L}) = 0 \) \( i > 0 \).

This implies the algebraic version, but in fact they're equivalent by Kodaira embeddings.

(0.6.3) Akizuki-Nakano simplified generalized proof, got stronger statement: under some assumptions \( H^i (M, \mathcal{L}) \otimes \mathcal{L} = 0 \)

for \( i > \dim M \) — uses Hodge theorem + existence of a Green operator.

There are algebraic proofs, but not known to what extent it generalizes in char. \( p \).

- Raynaud (uses liftability to char 0) — algebraic
- Kollar (geometric proof) — separates geometry from Hodge theory.

Raynaud proof. \( L \) is smooth, projective over \( k \) perfect of char. \( p \).

lifts to \( W_2 (k) \). Then if \( L \rightarrow X \) is ample, then

\[
H^i (X, N \otimes L) = 0 \quad \text{for } i > \max (\dim X, 2 \dim X - p).
\]

Cor. Kodaira in char. \( 0 \).

Proof. From cohomology decomposition theorem, we know that there

is a quasi-isomorphism \( N \otimes \mathcal{O}_X \rightarrow \bigoplus F^i \mathcal{H}^j \).

Also note that for any line bundle \( M \rightarrow X \), \( F^{\leq} M = \mathcal{M}^P \).

By projection \( \Rightarrow H^j (X, N \otimes M) \otimes \mathcal{O}_X \rightarrow H^j (X, F \otimes M) \).

By decomposition, \( F^i \mathcal{H}^j \cong \dim H^j (N \otimes \mathcal{M}) \).

\[
= \sum \dim H^j (N \otimes \mathcal{M}) \otimes \mathcal{M}^P \]

(1 + (N \otimes L \otimes \mathcal{M}^P) = 0 \quad \text{for } \mathcal{M} \text{ ample, } i, j < d).
Kollár's proof (Sturmfels' mp book). If a coherent cohomology has topological origin, then vanishing occurs.

The fact we need is that the natural map
\[ H^i(X, L) \to H^i(X, O_X) \] is surjective \( i \leq \text{dim } X \) smooth proper.

Discussion on cyclic covers: For a vector bundle \( E \to X \) denote by \( \text{tot } E \) the total space of \( E \).

If \( \mathcal{L} \to X \) any line bundle, \( S \in H^0(X, \mathcal{L}^\otimes n) \) \( n \geq 0 \),

then canonical cyclic cover \( X[\sqrt{n}] \to X \) of degree \( n \) with complete ramification along \( D = \text{div } S \).

It is the fiber product of \( X \times \text{mult } S \)

Equation for \( X[\sqrt{n}] \): Let \( x \in H^0(\text{tot } \mathcal{L}, p^* \mathcal{L}) \) be the tautological section. Then \( X[\sqrt{n}] \) is the divisor of \( x^n - p^* S \in H^0(\text{tot } \mathcal{L}, p^* \mathcal{L}^\otimes n) \).

Proposition: \( p_* O_X[\sqrt{n}] = \bigoplus_{a+b=n} L^a \otimes L^b \), and the \( \mathbb{C} \)-algebra structure is given by \( L_{a+b} \otimes L_{a+b} \to L_{a-b} \otimes L_{a+b} \) if \( a+b \geq n \), \( L_{a-b} \otimes L_{a-b} \to L_{a-b} \).

Proof: The short exact sequence for \( X[\sqrt{n}] \) \( \text{tot } \mathcal{L} \)
\[ 0 \to p^* \mathcal{O}^\otimes n \to \mathcal{O}^\otimes n \to \mathcal{O}^\otimes n \to 0 \]
\[ p^* \mathcal{O}^\otimes n \to \mathcal{O}^\otimes n \to \mathcal{O}^\otimes n \to 0 \]

Push every \( m \)-th to \( \mathcal{O} \): \( p_* \mathcal{O} \otimes \mathcal{L}^{-1} \mathcal{O} \)

Now \( (-1)^m \mathcal{O} \otimes \mathcal{L}^{-m} \otimes \mathcal{L}^m \) is the dual of \( -1 \), \( \mathcal{O} \otimes \mathcal{L}^{-m} \).

Remark: If \( X, D \) smooth \( \Rightarrow X[\sqrt{n}] \) smooth.

fiber product of two maps whose discriminant loc. is empty.

Intersect is smooth.

--- look in Kollár
Proof of Kodaïma. Let $\text{set} \lambda(\mathbb{C}^n)$, $n$ big enough, s.t. $D=\text{div} \lambda$ is smooth & connected. Consider $\mathbb{Z} = \mathbb{N}^{\lambda}$ to $\mathbb{X}$.

Fact: $H^i(\mathbb{Z}, L^{-b}) \to H^i(\mathbb{X}, L^{-b})$ & surjective for all $b \geq 0$ (gives Kodaïma: we know $H^i \lambda = 0$ by setup for $\mathbb{X}$ big enough).

Proof: Consider $\mathbb{Z}$. Let $\mathbb{P}^k \mathbb{Z} \to \mathbb{X}$ be the pushforward of the cotangent sheaf. Decompose this sheaf under the characters of our cyclic group's group $\mathbb{Z}$. $\mathbb{P}^k \mathbb{Z} = \bigoplus_i G_i$. $G_i \to \mathbb{P}^k \mathbb{Z}$.

Arrange $(c_i, i)$ so that $G_i \to \mathbb{Z}^{-1}$.

(Local systems with singularities along each fiber).

Lemma: $-G_0 = G_1 \times \mathbb{Z}$, $-G_1 \mathbb{C} \mathbb{Z}^{-b} \mathbb{D} \mathbb{Z}^{-b}$.

Proof - Later.

Notice that since $\mathbb{Z}$ is smooth, $H^i(\mathbb{Z}, G_2) \to H^i(\mathbb{X}, G_2)$ get some projection on any isomorphic component.

In particular $H^i(\mathbb{Z}, G_1) \to H^i(\mathbb{X}, L^{-b})$.

By the lemma: $H^i(\mathbb{Z}, G_1) \to H^i(\mathbb{X}, L^{-b})$.

$\to$ Kodaïma.

Proof of Lemma: $-G_0 = (\mathbb{P}^k \mathbb{Z})^\mathbb{Z}/\mathbb{Z} = \mathbb{C} \times \mathbb{Z}^{-1}$.

$-G_1 \mathbb{Z}^{-b} \mathbb{D} \mathbb{Z}^{-b} \mathbb{B}^{-b}$.

$\Rightarrow$ local question to show $G_1 \mathbb{C} \mathbb{Z}^{-b} \mathbb{D} \mathbb{Z}^{-b}$.

The sheaves are different only near $D$. . .

Let $U \subset \mathbb{X}$ be a connected open s.t. $U \cap D \neq \emptyset$.

What is $H^0(U, G_1 L^{-b})$? It's zero.

Local system outside $D$, nontrivial nonobvious: $G_1$ has no character, no max.

Local on $U$, $G_0 \times \mathbb{Z} = G_0$ to $G_0$.

$(x_1, \ldots, x_n) \to (x_1^{-1}, x_2, \ldots, x_n)$

$\Rightarrow$ Felce constants of monogeneity degree $>0$.

$\Rightarrow 0$. 

$\square$
General principle: If a geometric property can be expressed in terms of \( H^0(X, M) \) then it's "rigid".

Want to study moduli of \( X \) smooth projective, \( X \) trivial.

\[ \exists \phi \in H^0(X, \omega_X) \text{ nowhere vanishing, gives } \]
\[ \pi: \mathcal{M}_X \to \mathbb{A}^n \text{ (moduli space of } X) \]
\[ H^1(X, \omega_X) = H^1(X, \mathbb{C}) \text{ infinitesimal deformations } \to \]

Then (Bogomolov, Tian, Todorov) The moduli space of such \( X \) is smooth.

**Deformation Theory**

- **Examples:**
  - In differential geometry: moduli of Riemannian metrics - too big & general
  - Einstein metrics: finite dim, maybe simpler
  - Complex structures (in dim. even...)
  - Kähler structures (good but uninteresting)
  - Kodaira proved any deformation of Kähler is complex is Kähler.
  - Extremal Kähler metrics (very interesting)
  - Minimal functional on a Kähler class
  - Subclass of Ricci flat, constant scalar curvature.
    - In general existence not governed by topology - Fintushel, geometric examples.

- In algebra:
  - Associative algebra, Lie alg., reps
  - If an object (algebra) has deformations, then representations don't have deformations & conversely.

- In alg. geometry:
  - Schemes, sheaves, etc.

**Given \( X \), geometric object \( \Rightarrow \) moduli \((X) \), \( M \): If interested in local structure of \((M, \varnothing) \Rightarrow \) reduce by \( \Omega^* \).

Try to describe it or rather \( \Omega^* \)-ideal topological co-algebra.

- Print annotated bibliography of deformation theory:

**Functors on Artin algebras**

- Art - the category of local Artin C-algebras with residue field C.
- Set - the category of sets.

\[ \text{Art} : \text{Art} \to \text{Set} \text{ st. } D(C) \text{ is a one point set.} \]
Basic construction: \( \mathcal{C} \) a class of geometric objects,

- an equivalence relation in \( \mathcal{C} \) \( \sim \)
- \( \forall x \in \mathcal{C} \) distinguished object \( \Rightarrow D(x, y) : \text{Art} \to \text{sch} \)
- \( \text{Art} \to \text{sch} \)
- \( \text{equivalence} \) of \( \text{families} \)
- \( \mathcal{X} \to \text{Spec} A \) of objects in \( \mathcal{C} \)

\[ \mathcal{X} \text{Spec} \mathcal{E} = X \]

Property: existence of formal moduli

\[ \mathcal{X} \times \text{compact analytic space } \to \text{class of all complex manifolds} \to \text{biholomorphism} \]

\[ D : \text{Art} \to \text{Set} \]

\[ A \to \{ \text{set of families } \mathcal{X} \to \text{spec } A \} \]

\[ \text{loc. c. of scheme } \mathcal{O}_X \to X \text{ equipped with } A \text{-algebra structures} \]

Remark: 1) Same construction \( \Rightarrow \) moduli functor

\[ D : \text{Sch} \to \text{set} \]

2) Grothendieck general definition of moduli functor:

In practice, \( (\mathcal{C}, \sim) \) comes from a category \( \mathcal{E} \) of geometric objects, \( \mathcal{O}_E(y) = \mathcal{C} \), \( A, B \in \mathcal{O}_E(y) \)

- all we need are the isomorphisms, so forget the rest of the definitions, and assume \( \mathcal{E} \) is a groupoid.

Given any groupoid, can recover set of isomorphism classes of objects in \( \mathcal{C} \) - get moduli space with Galois group of automorphisms, isotropy group at any point.

Moduli functor in general is a functor

\[ M : \text{Sch} \to \text{Set} \]

that factors through the \( K \)-category of groupoids.

Representability: could come from a scheme, or more generally a category \( M \to \text{Sch} \)

\[ M(S) = \text{Hom}(S, M(\bullet)) \]

Kontsevich's approach to formal deformation theory:

\[ 0 : \text{Art} \to \text{Groupoids} \]

What kind of object can represent \( 0 \)?

Fermi's idea: look for a DGLA representing \( 0 \).

Let a DGLA is a \( Z \)-graded complex vector space

- \( g^n \to g^{n+1} \) equipped with graded bracket
- \( [g^n, g^m] \to g^{n+m} \)
- graded skew symmetry \( [y, x] = -[x, y] \)
- graded Jacobi \( \text{Jac} : [X, [Y, Z]] \to (-1)^{\text{deg}(x) \cdot \text{deg}(y)} [Z, [X, Y]] + (-1)^{\text{deg}(x)} [X, [2, [Z, Y]]] = 0 \)
\[ d \{x, y\} = [dx, y] + (-1)^{\langle x, y \rangle} [x, dy] \]

To any \( g \in \Omega G \), \( D : \mathbb{R} \rightarrow \text{Goroids} \)

Start with \( A = C \circ \text{or} \), Tensor with \( \sigma \)

\( \Omega \text{or} \rightarrow \Omega A \) inclusion of \( \Omega G \).

The objects of the goroid \( D(A) \) will be all elements in \( \Omega \text{or} \) satisfying Maurer-Cartan

\[ Z = \{ X \in \Omega \text{or} \ | \ dX = \frac{1}{2} [X, X] \} \]

Morphisms: look first at \( \Omega \text{or} \), nilpotent Lie algebras. This acts on \( \Omega \text{or} \), preserving \( Z \), by affine vector fields \( \Omega \text{or} \rightarrow \mathbb{R} (\Omega \text{or}, \text{T} \Omega \text{or}) \)

\( V(\mathbf{X}) = dX + [X, X] \) — gauge action

**Lemma (13)** \( V \) is a Lie algebra homomorphism

1. The infinitesimal action of \( \Omega \text{or} \) on \( \Omega \text{or} \) preserves \( Z \).

**Proof:** \( Z \) is zero scheme of \( k : \Omega \text{or} \rightarrow \Omega \text{or} \)

\[ V \rightarrow dV + \frac{1}{2} [V, V] \]

Thus exact that for any \( V \in k, X \in \Omega \text{or} \), we have

\[ C_v(x) (k) (V) = 0 \]

\[ = d(V(x)V) + [V(x), x] \]

\[ = d(dx + [V, x]) + [dx + [V, x], x] \]

\[ = [d(dx + [V, x]), x] - \frac{1}{2} [V, x], x] + \frac{1}{2} [V, x], x] \]

\[ = \frac{1}{2} [V, x], x] + \frac{1}{2} [V, x], x] = 0 \]

Recall \( N \) nilpotent Lie algebra — can exponentiate if always a group \( \exp (N) = \) set of all \( \exp (v), x \in N \), \( x \), \( x \) in BCT multiplication.

Claim \( \exp (\Omega \text{or} \text{or}) \) acts on \( Z \), action given by \( g \in \exp (\Omega \text{or} \text{or}) \)

\[ X \in Z \rightarrow V \rightarrow gXg^{-1} = X^{-1}dX^{-1}X \]

where if \( X = \exp (x) \), \( \exp (-X) = \exp (x)^{-1} = \prod_{n=0}^{\infty} \frac{1}{n!} (\exp (-X)) dX \]

\[ d \exp (X) = \int [\exp (x), dx] = \frac{1}{(2\pi)^{\frac{1}{2}}} \exp (-X) \exp ((-1)X) dX \]

\[ \Rightarrow \text{Goroid: } \mathbb{R}, \mathbb{R}, \mathbb{R} \in \text{Goroids} \Rightarrow Z, \text{Hom}(\mathbb{R}, \mathbb{R}) \} \space \frac{\text{Exp}(\text{or}(g))}{\text{or}(g)} = k \]
Examples: i. $X$ complex manifold, $Def_X$ deform complex structure. 
$\text{deg} \kappa = \Gamma (X, T^{1,0} \otimes \Lambda^k (T^{0,1})^*) = (0, k)$ forms with 
coeffs (so) vector fields.

ii. Lie bracket on $\mathfrak{g}$, acts on a fixed differential $d$.

iii. $X$ complex manifold, $G$, Lie group, $V \to X$ principal
$G$-bundle, $\nabla$ flat connection on $V$.

$d = d^\nabla$, covariant derivative.

$d^\nabla = \frac{1}{2} [\mathcal{E}, \mathcal{E}]$ usual Maurer-Cartan equation, $V_G$.

is gauge action.

Theorem: If $\mathfrak{g}^\circ$ Drinfel'd, $H^2(\mathfrak{g})$ is rationally $\mathbb{Z}$-graded

- Assume $\mathfrak{g}^{\circ} = 0$, $H^2(\mathfrak{g}^{\circ}) = 0$. Then $\exp (\mathfrak{g}^{\circ} \otimes \mathfrak{m})$
acts freely on $\mathbb{Z}!$ for any $\mathfrak{A} = \mathfrak{A} \otimes \mathfrak{m}$

- The deformation functor associated to $\mathfrak{g}$ is representable by
a $\mathbb{C}$-algebra $C$, i.e., $\text{D}(\mathfrak{A}) = \text{Hom}(\mathfrak{A}^\circ, C)$, $\mathfrak{A}$ topological dual.
Moreover $\mathfrak{B} = H^0(\mathfrak{g}^{\circ} \otimes \mathfrak{C})$, Lie algebra homology

- If in addition $H^2(\mathfrak{g}^{\circ}) = 0$, $C$ is (canonically) isomorphic to $\text{Sym}^* (H^0(\mathfrak{g}^{\circ}))$ - smoothness.

Meaning of the theorem: roughly it says, is there are any
infinitesimal automorphisms $\mathfrak{g}$ the formal moduli exists. If
there aren't any obstructions to smoothness the space where
obstructions to smoothness live vanishes, the formal moduli
is smooth. $\mathfrak{g}$ quasi to free Lie algebra
which is a Chevalley complex ... ?

Pro-representability & Smoothness for deformations

$0: \text{Art} \to \mathfrak{A}$ deformation functor - complete local Noetherian $\mathbb{C}$-algebras $\mathfrak{A}$

$s.t. A \to \mathfrak{A} \in \text{Art}$ for

Let $\mathfrak{A}$ counit for a deformation functor $0$ is a pair

$(A, \delta)$, $A \in \text{Art}$, $\delta \in D(A)$, morphism of $\mathfrak{A}$

$\psi: (A, \delta) \to (A', \delta')$ is a morphism $A \to A'$ s.t.

$0(\psi)(\delta) = \delta'$
Report: If we extend $D$ to a functor $D$ on $A$ by

\[ D(A) = \lim D(A/A^n) \Rightarrow \text{pro-representable}. \]

If $R \in A^n$, then we have $h_R: A \rightarrow \text{Set}$, $A \mapsto \text{Hom}(R, A)$.

If $(R, S)$ is a pro-couple, $\text{mor}_R$ is a map of factors $h_R \rightarrow D$:

\[ \exists \tilde{f} \in \tilde{D}(R) \Rightarrow \tilde{f} = \tilde{f}^n \in \tilde{S}_n, \quad S_n \in \text{R}(\mathbb{A}/n) \]

Let $M = \text{Hom}_R(R, S)$. A $A \rightarrow M$ $\Rightarrow$ $F$ st $U: R \rightarrow A$

factor $u$ through $M$. Then:

\[ \Rightarrow \text{take } D(m) (f_0) \in D(A) \]

**Def:** $(R, S)$ pro-couple for $D$ pro-representable $D \Rightarrow h_R \rightarrow D$ is iso.

Geometrically if $D = D(\mathbb{C}, \mathbb{C}_1)$, pro-representability of $D$ by $(R, S)$ says $\tilde{M} = \text{Spec } R$ formal moduli for $(\mathbb{C}, \mathbb{C}_1)$

and a family of objects in $\mathbb{C}$ $\rightarrow \tilde{M}$

s.t.

a) $f$ is complete i.e. for any other family $\eta \rightarrow (S, \tilde{S})$ pointed over here we have

$\eta$ is a pullback via a map $S \rightarrow \tilde{M}$ locally around $S_0$.

b) $f$ is universal i.e. the completion of it at $S_0$ is uniquely determined.

**Universal** - complete family exists but not quite uniquely determined - only its differential is.

**Sufficient condition for pro-representability:**

- Grothendieck general Neron-Severi condition - left-exactness of the functor.

- Schlessinger conditions $\text{H}1, \text{H}2, \text{H}3, \text{H}4, \text{H}5$

  sufficient for hull/universal deformation. $\text{H}1-\text{H}4$

  sufficient for pro-rep. $\text{H}5$ (Mark Helgason) - has admissible

  smoothness. Used this to prove that if $X$ proper

  scheme $/ \mathbb{C} \Rightarrow \text{Def}_X$ has a minimal deformation (Kuranishi's

  theorem) (complex geometry).

- If $X$ has $H^0(X, T_X) = 0 \Rightarrow \text{Def}_X$ is pro-representable.

How about smoothness? Assume $D$ has a hull $(R, S)$

Want to test $\text{Spec } R$ for smoothness.
Reduce to a question that can be handled inductively.

\[ A_n \cong \text{Spec}(A) \times A_n, \quad S_n = \text{Spec}(A_n), \quad \forall : A_n \to A. \]

Def. \( D : A \text{nt} \to \text{Set} \) is called unobstructed if \( \forall n \geq 0 \) the map

\[ D(A_n) \to D(A_{n+1}) \to D(A) \]

is surjective.

Prop. Let \( D \) be an obstruction functor, \( \text{pow}r \) by \( \tilde{M} \). Then \( D \) is unobstructed \( \iff \tilde{M} \) is smooth.

Proof. Let \((R, \mathfrak{m})\) be the pro. couple representing \( R \). \( \tilde{M} = \text{Spec}(R, \mathfrak{m}) \). \( \mathfrak{m} = \text{Der}(R) \). \( \tilde{M} \) is smooth \( \iff \mathfrak{m} = \text{Der}(R) \). \( \mathfrak{m} = \text{Der}(R) \iff \tilde{M} \) is smooth.

Def. unobstructed \& \( \tilde{M} \) not smooth : \( \tilde{M} \in A_{\text{Zarish}} \)

Tangent of \( \text{Spec} \mathcal{O} \) to \( \tilde{M} \) : \( D(A, \mathcal{O}_{\tilde{M}}) \)

\( \tilde{M} \) not smooth \( \iff \) found line \( L \in A_{\text{Zarish}} \) not contained in the tangent cone to \( \tilde{M} \). But \( D \) is unobstructed \( \iff \)

can lift \( L \) to a formal curve \( C \subset \tilde{M} \) ... contradiction.

Rem. Deformation problems in practice always give some objects !

Geometric object \( X \) we want to deform \( \Rightarrow \) get a sheaf or complex \( \mathcal{F} \) on a suitable space, that 'linearizes'
the geometry of \( X \), i.e. \( H^0(\mathcal{F}) \) = inf. automorphisms of \( X \)

\[ H^i(\mathcal{F}) = \text{inf. deformations of } X, \quad H^2(\mathcal{F}) = \text{obstruction} \]

Examples 1) \( X \) proper smooth / \( C \), \( \mathcal{F}_X = T_X \).

2) \( X \) is a vector bundle on scheme \( S \), \( \mathcal{F}_X = \text{End}(\mathcal{F}_X) \)

\( \text{Ker } \text{sec } \text{ch } \mathcal{O} \text{ if } D \text{ comes from a } \mathcal{O}_{\mathcal{L}} \text{ LA, then } H^0(\mathcal{O}^*), \quad i=0,1,2 \text{ give us above objects.} \)

Def. The tangent space at \( x \) of a de Rham \( D : S^+ \) at \( t_0 = 0(6/7/7)^d \)

Example: Hensel tip \( \text{Ex } 4.10. \quad H^0(\mathcal{O}_x) = H^1(\mathcal{H}_x) \text{ for } \mathcal{H} \text{ smooth x cone.} \)

Lemma. If \( D \) is pro-top, to has natural structure of \( C \)-scheme

\[ \text{pf: } \quad \{ \mathcal{O}(A) \to 2 \times \mathcal{O}(C) \} \times \mathcal{O}(A) \]

For any \( D \) there is a map \( \mathcal{O}(A) \times \mathcal{O}(A) \to \mathcal{O}(A) \times \mathcal{O}(A) \) for any \( A \).

If \( D \) is proper \( \Rightarrow H^1 \) is a bijection. Compare with

\[ \text{diagonal map } \to \text{arrows.} \]

Remark. Surjective \( \Rightarrow \text{to be bijection } \Rightarrow \text{essentially Schlessinger's } H^2 (\text{for any } A) \).
Def. A deformation functor $D : \text{Art} \to \text{Set}$ has an obstruction space if $\mathfrak{g}$ a complex vector space $\mathfrak{g}^3$ s.t. for any surjection of Artin algebras $p : A \to A'$ with $\text{ker } p \cdot \mathfrak{g}^3 = 0$, there exists an \textit{external obstruction map} $\hat{f}_p : D(A) \to \text{ker } p \cdot \mathfrak{g}^3$ which fits in an exact sequence

$$0 \to D(A) \xrightarrow{\hat{f}_p} D(A') \xrightarrow{f_p} \text{ker } p \cdot \mathfrak{g}^3$$

in the sense $\hat{f}_p(x) = 0 \Leftrightarrow x \in \text{ker } D(A)$

i.e. map to linear space measuring problem with lifting deformations from $A$ to $A'$.

Prop. Assume $D$ is pro-rep, then $D$ has an obstruction space.

Proof. Relate $D$ to a quotient of polynomial ring over $\mathbb{Z}$.

Let $R$ be an algebra representing $D$, $R = (k[s_1, \ldots, s_n]) / \mathfrak{J}$

where $\mathfrak{J} \subset (s_1, \ldots, s_n)^2$ to make $R$ local.

Set $\mathfrak{g}^3 = (\mathfrak{J} / (s_1, \ldots, s_n) \mathfrak{J})^3$.

To describe $\hat{f}_p : D(A) \to \text{ker } p \cdot \mathfrak{g}^3$, take $f_p(D(A)) = \text{Hom}(R, A)$.

Build $\hat{f}_p(x)$ in stages:

- $x = P(s_i)$, $i = 1, \ldots, n$
- lift $P$ to $P \in B$, $b \in \mathfrak{J}$
- use them to define $g : (k[s_1, \ldots, s_n]) \to \mathfrak{J}$

If $s_i \in \mathfrak{J}$, then $s_i = 0$ since $g$ is a local monomorphism and $\text{ker } p \cdot \mathfrak{g}^3 = 0$.

Moreover, if $s_i \in \mathfrak{J}$, then $\hat{f}_p(s_i) = 0$ since $g$ induces a $\mathfrak{g}$-linear map $\mathfrak{J} / (s_1, \ldots, s_n) \mathfrak{J} \to \text{ker } p$ i.e. an element in $\mathfrak{g}^3$.

\[\text{Exercise:} \quad X \text{ smooth proper } / k, \quad \text{for } \mathfrak{g}(A, T) = 0 \quad \text{then } H^1(A, T_A) = 0\]

\text{is the obstruction space for } D(A).

\textbf{Kawamata-ViehmannCriterion}

Def. (tangent functor) $D : \text{Art} \to \text{Set}$ deformation functor

Then the infinitesimal deformations of $D$ give a functor $D_{\text{inf}}$ to every $D$-scheme $(A, f)$ as set

$$T_D(A, f) = \{ g \in D(A \otimes A) / D(A)(g) = f \}$$

- infinitesimal extensions of a given scheme over dual numbers.
Remarks

i. If $D$ is proper, the same result as before gives us a canonical structure of $A$-module on $T^i_0(A, g)$

iii. Kawamata-Ram achieved this unsmoothed $D$ as a question about $T^i_0$, which is a mean object.

- i.e. this is a characterization procedure

Let $A_n = (C[[x]] / (x^n), S_n = \text{Spec } A_n, \phi_n: A_n \rightarrow A$

Def: we say a finite $D(A) \rightarrow \mathbb{C}$ satisfies the $T^i$ lifting property if for any $n \geq 0$, $S_n \rightarrow D(A)$.

the natural linear morphism $T^i_0(A, g) \rightarrow T^i_0(A, g)$ is surjective.

Geometrically this means $K(Z)$, any family $S_n \rightarrow D(A)$, any

inf. deformation of pre-projection family $S_n \rightarrow S_{n+1}$, can be

lifted to an inf. det. of $S_{n+1}$.

- Linear problem - First order extension of linear object ...

Theorem: If $D$ is proper and satisfies $T^i$ lifting, then $D$ is

unobstructed.

Proof: want to check $\text{Ext}^1_A V, V \rightarrow 0$ the new $D(A_n) \rightarrow D(A_n)$ is surjective.

$C_n = \text{Ann } A_n = C[[x]] / (x^n, y^2)$

Natural maps $u_n: A \rightarrow A_n, v_n: B_n \rightarrow A_n$

$E_n: A_n \rightarrow B_n \rightarrow C_n$

$E_n': A \rightarrow C_n$.

Let $L_0 = \text{obstruction space} \Rightarrow \text{conformal diagram}$

$D(A_n) \xrightarrow{\partial} D(A_n) \xrightarrow{\delta} L_0 \rightarrow (C[[x]])$

$D(B_n) \xrightarrow{\partial} D(C_n) \xrightarrow{\delta} L_0 \rightarrow (C[[x]])$.

But $E_n$ sends $(1)$ to $(y)$ dieromorphically as $(y)$-vector space:

$E_n(1) = (x+y)^{n+1} \text{ and } (x^{n+1}, y^2) = (n+1)(x, y) \text{ mod } (x^n, y^2)$

only place where we need $\text{char } = 0$.

$\Rightarrow \delta_n = 0 \Leftrightarrow \delta_n = 0$

To prove $\delta_n = 0$ use $T^i$ lifting property: consider

$\rho_n: C_n \rightarrow A_n, x \mapsto x, y \mapsto 0$

$\varphi_n: C_n \rightarrow B_n, x \mapsto x, y \mapsto y$.
Let $\xi \in D(\mathcal{A})$, $D(\mathcal{A}) \xi \in D(\mathcal{A})$, $D(\mathcal{A}) \xi \in D(\mathcal{A})$

By the core of $D$, $\xi = O(\mathcal{A}) \xi \times D(\mathcal{A}) \xi$

But $D(\mathcal{A}) \xi \in T' (\mathcal{A}_n, D(\mathcal{A}) \xi)$

By $T'$ lifting, for $\eta$ s.t., $\eta$ maps to $D(\mathcal{B}) \xi \in D(\mathcal{B})$

By pro-rep $D(\mathcal{A}) (\eta) = (\text{image of } \eta \in D(\mathcal{B})) \times D(\mathcal{B}) (\text{image of } \eta)$

Remarks
1. Don't need $D$ pro-rep: need a hull-linear structure of target functor. First comes from $H_1 - H_3$, latter either by $H_4$ or weaker $H_5$.
2. Condition $H_5$ Recall that $\forall \mathcal{A} \rightarrow \mathcal{A}' \rightarrow \mathcal{A}$, the canonical map $D(\mathcal{A}) \times D(\mathcal{A}') \rightarrow D(\mathcal{A}')$ is an isomorphism.

Every $D$ has a hull $D$ satisfies $H_5$ if and only if $H_5$.

Fact. (Core) If $X$ is any scheme, $\text{def}_X$ satisfies $H_5$.

Applications

Theorem. (Bogomolov-Tian-Todorov) $X$ is a smooth projective variety with torsion $\mathcal{E} \Rightarrow \text{def}_X$ is (un)obstructed.

Proof. (Core) The same calculation as for $H_5$, gives $\forall \mathcal{A} \in \text{Art}$, $\forall \mathcal{E} \in \text{def}_X (\mathcal{A}) \Rightarrow T_0 (\mathcal{A}, \mathcal{E}) = H^1 (Y, T_Y / \mathcal{E})$.

Now assume $\text{def}_X$ is trivial.

Fact 1. For any family $X_\mathcal{E}$, $\text{def}_X (\mathcal{E})$, when $\text{def}_X$ is also trivial.

By the main corollary from descent of $H^0 \rightarrow \text{def}_X$, $H^0 (X_{\mathcal{E}}, \mathcal{E} \otimes \mathcal{O}(1))$ is a free $\text{Art}_\mathcal{E}$ module.
Fact 2. \( V(X, x, \xi(x, x')) \) by relative duality & fact 1 we have
\[
\omega_{x, x'}^{(n-1)} = \omega_{x, x'}^{(n-1)} \times_{\omega_{x, x'}^{(n-1)}} T_x \times_{\omega_{x, x'}^{(n-1)}} T_{x'} \ni 1
\]

T' lifting says \( H'(\omega_{x, x'}^{(n-1)}, X) \rightarrow H'(\omega_{x, x'}^{(n-1)}, X) \) is onto.

Use Hodge theory \( \Rightarrow H'(\omega_{x, x'}^{(n-1)}, X) \cong H'(\omega_{x, x'}^{(n-1)}, \Omega_{x, x'}^{(n-1)}) \cong A_{n-1}
\)

same over \( S_n \Rightarrow \text{surjection} \)

Torsion case \( \Rightarrow \text{get} \ \pi_1(x, X) \rightarrow \text{yclic def. comes}
\)

sit. \( \pi_1(x, x') = 0 \) is trivial

\( T_{x, x'} \rightarrow T_{x', x} \Rightarrow T' \) lifting for \( x \)

---

**Deformations of normal subvarieties**

Set-up: \( X, Y \) varieties, \( f: X \rightarrow Y \) morphism. Study some def. nots.
- \( \mathcal{D}(X, y) \) - local not. of the triple \( X \rightarrow Y \)
- \( \mathcal{D}(x, y) \) := \( \mathcal{D}(X, y) \) with \( y \) fixed
- \( \mathcal{D}(x, y) \) := \( \mathcal{D}(X, y) \) a pair \( X \rightarrow Y \), bottom

\( \mathcal{D}(x, y) \) := local embedded deformations of \( X \) in \( \mathbf{V}(k) \)

\( \mathcal{D}(x/y) \) := deformations of \( Y \)

Remark: all these are specialization of \( \mathcal{D}(x/y) \)

we can use fibers to study \( \mathcal{D}(x/y) \) by reducing it to others

In particular, can try to infer the obstruction complex for \( \mathcal{D}(x/y) \) from those of others

Sufficient condition for pro-rep:
- If \( Y \) is smooth & proper, then \( \mathcal{D}(x/y) \) has a hull, and if \( H^{1}(\Omega_{y}) = 0 \Rightarrow \mathcal{D}(x/y) \) is pro-rep (Kuranishi)
- If \( Y \) proper \( \Rightarrow \mathcal{D}(x/y) \) has a hull. If \( Y \) - variation of \( \Omega_{y} \)

and if \( H^{1}(\Omega_{y}) = 0 \Rightarrow \mathcal{D}(x/y) \) is pro-rep (Schlessinger)
- If \( Y \) proper/even, \( X \) arbitrary \( \Rightarrow \mathcal{D}(x/y) \) is pro-rep

(Grothendieck-Hilbert scheme).
- If \( X, Y \) smooth, proper then \( \mathcal{D}(x/y) \) has a hull, and if

\( f: X \rightarrow Y \) has a lift, automorphism \( \Rightarrow \) pro-rep. (Hartshorne)
- \( X, Y \) proper same is true. (Grothendieck)
A list of linearizing complexes for $X/Y$ smooth (up to quasi-isom.)

- Exact
  - $\mathcal{E}_{\mathcal{O}_Y}$
  - $D(x)$
  - $D(\mathcal{O}_X)$
  - $D(x+y)$
  - $D(x + c)$
  - $D(x, y)$
  - $D(x, c)$
  - $D(x+v)$
  - $D(x, c)$

- Works for $H^1, H^2$, don't always get exact in cohomology.

Remarks:
1. $T_X \to T_Y \to N_{X/Y}$ exact $\Rightarrow$ so $N_{X/Y}$ is quasi-isomorphic to $T_X \to f^* T_Y$ when $X \subset Y$.
2. Another check is look at $D(x+c) \to D(x+y)$

- comes from natural short exact sequence $0 \to D(x+y) \to D(x+c)

\[ 0 \to \left[ \begin{array}{c}
\mathcal{O} \\
\mathcal{O}^* \end{array} \right] \to \left[ \begin{array}{c}
T_Y \\
T_{x+Y} \end{array} \right] \to \left[ \begin{array}{c}
T_Y \\
0 \end{array} \right] \to 0 \]

Ziv Kan's approach to degeneration of general $f: X \to Y$

Take $A = \mathcal{O}_X$ and $f = \mathcal{O}_X$ for $R^f$

Hom$(B, A) = \text{Hom}_{\mathcal{O}_X} (f^* B, A) = \text{Hom}_Y (B, f^* A)$

$f$-linear maps

Goal: given two $f$-linear morphisms, $\phi, \psi$ $(\text{Hom}_{\mathcal{O}_X} (B, f^* A))$, $f \neq 0$,

define functorial groups $\text{Ext}^i(f^* B, A)$ satisfying:

1. $\text{Ext}^0(f^* B, A)$ is pairs $0 \to A \to B \to 0$ fitting in commutative diagram $f^* B \to f^* B \to f^* A$

2. $\text{Ext}^1(f^* B, A)$ is the set of $\lambda_2, B, \beta_2$

lifting in short exact $0 \to [\mathcal{O}] \to [\mathcal{O}] \to [\mathcal{O}] \to 0$

3. There's a long exact

$0 \to \text{Hom}(A, B) \to \text{Hom}(A, f^* B) \to \text{Hom}(f^* A, B) \to \text{Ext}^1(f^* B, A) \to \text{Ext}^1(A, f^* B) \to \text{Ext}^1(A, f^* A) \to \text{Ext}^2(A, f^* B) \to \text{Ext}^2(A, f^* A) \to \cdots$

where $\text{Ext}^i(B, A)$ is the derived factor of $\text{Hom}_{\mathcal{O}_X}$

in either variable. $s, s \in \text{Ext}^i(f^* B, A) \Rightarrow \text{Ext}^{i+2}(B, A)$

$E_2^{i, j} = \text{Ext}^{i+j}(B, f^* A) \Rightarrow \text{Ext}^{i+j}(A, f^* B)$

4. If inclusion $i: \mathcal{O}_X \to \mathcal{O}_Y$ with kernel $K$, $\Rightarrow$ exact sequence

$0 \to \text{Hom}(\mathcal{O}_X, B) \to \text{Hom}(\mathcal{O}_Y, B) \to \text{Hom}(K, B) \to \text{Ext}^1(i^!, B) \to \cdots$
To construct $\text{Ext}^1_0 (\mathcal{O}, \mathcal{O})$ first define a Grothendieck topology $\mathcal{T}$ associated with $\mathcal{O}$:

Open sets in $\mathcal{T}$ will be pairs $(U, V)$, $U \subseteq X$ Zariski open, $V \subset Y$ open, $\mathcal{O}_X(U) \subset \mathcal{O}_Y(V)$. The coverings of $(U, V)$ will be collections $\{ (U_i, V_i) \}$ s.t. $U_i$ cover $U$, $V_i$ cover $V$.

Define $\mathcal{O}_X$ - structure sheaf on noncommutative rings:

$$\mathcal{O}_X((U, V)) = \{ (a, b, c), a \in \mathcal{O}_X(U), b \in \mathcal{O}_X(V), c \in \mathcal{O}_X(U) \}$$

with multiplication:

$$\left( (a, b, c), (a', b', c') \right) = (a + b + c, ab, ac')$$

Lemma: The category of $\mathcal{O}$-linear maps is equivalent to the category of left $\mathcal{O}_X$ modules.

Define two mutually inverse functors

$$\mathcal{O}_X \longrightarrow \text{left } \mathcal{O}_X \text{-modules} \, f \, \mapsto \, \mathcal{O}_X \text{ module } f$$

$$\mathcal{O}_X \longrightarrow \text{right } \mathcal{O}_X \text{-modules} \, g \, \mapsto \, \mathcal{O}_X \text{ module } g$$

Inverse functor $\mathcal{T} :$

$$\mathcal{T} \text{ module } E \mapsto \text{ triple } A = (a, b, c), E, \mathcal{O}_X \text{-module, } b = (a, b, c), E \mathcal{O}_X \text{-module, } f = (a, b, c)$$

Define $\text{Ext} : \mathcal{T} \times \mathcal{T} \mapsto \text{Hom}_0 (\mathcal{O}, \mathcal{O})$ Yoneda $\text{Ext}$.

If $D = D(x, 0)$ and $\mathcal{T} : \mathcal{T} \mapsto \mathcal{T}_x$.

$$\mathcal{T} : \mathcal{O}_X \mapsto \mathcal{T}_x \mathcal{O}_X$$

then $\mathcal{T}_x = \text{Ext} (\mathcal{O}, \mathcal{O}) = 0, 1, 2, ..$.

More: In general deformation theory need closed model categories derived categories of additive categories.

**Bloch semi-regular map** want to study $\mathcal{O}_X$'s

Smooth, projective of dim $n$, $X$ local complete

Intersection of codimension $p - \text{ dim } X \leq \mathbb{H}^{1/6}(X)$

is a smooth point.

Bloch introduced a special linear map whose inactivity $\Rightarrow$ smoothness in $\mathcal{X} \leq \mathbb{H}^{1/6}(Y)$.

Notation: $\mathcal{I}$ ideal of $\mathcal{X}$ in $\mathcal{Y}$.

$$\mathcal{N}_{\mathcal{X}/\mathcal{Y}} = \text{Hom}_{\mathcal{X}} (\mathcal{I}, \mathcal{O}_{\mathcal{X}})$$

$$\mathcal{L} : \mathcal{X} \mapsto \mathcal{N}_{\mathcal{X}/\mathcal{Y}}$$

Standard exact sequence:

$$0 \rightarrow \mathcal{N}_{\mathcal{X}/\mathcal{Y}} \rightarrow \mathcal{L}^\text{tr} \rightarrow \mathcal{L}$$

$\omega_{\mathcal{X}/\mathcal{Y}}$ canonical bundle of $\mathcal{Y}$, $\mathcal{D}$ dualizing sheaf of $\mathcal{X}$ (line bundle).

$$\mathcal{O}_{\mathcal{X}/\mathcal{Y}} \otimes \mathcal{D}_{\mathcal{X}/\mathcal{Y}} = \mathcal{L}^\text{tr}$$

Relative dualizing sheaf.
Def. $X \subset Y$ as before is called semi-regular if $T_X$ is injective.

Example: Hilbert scheme of a curve in its Jacobian.
- $C$ smooth genus $g \geq 2$; $JC = \text{Jacobian}$ (degree 1)
- In general, $\text{Hilb}(J)$ is very bad; $\text{Hilb}(C)$ is a set theoretically and identity $\text{Hilb}(J)$ with $J$ (parametrizes schemes).
- More precise: let $C$ be a very smooth curve in $M_g$ of rank $\text{NS}(C) = 1$, then $\text{Hilb}(J)_{\text{red}} = J_0$.
- Indeed $J_0$ acts on $J$ and on $\text{Hilb}(C)$, suffices to check this action is simply transitive on closed points.
- If $X \subset J$ a curve, $[X] \in \text{Hilb}(J)$, then normalization $\tilde{X} \to J$, but arithmetic genus $\cdot X \subset J \Rightarrow$ smooth curve $\tilde{X}$ of genus $\leq g$ new to $J$.
- If $g(\tilde{X}) < g$, then get map $\tilde{X} \to J / C \to J / C$ contradiction since rank $(\text{NS}) = 1 \Rightarrow J$ is indecomposable.
- Moreover $J(C) \subset \text{NS}(C)$; only one polarization is isomorphic as principally polarized Ab. varieties by Torelli: $C \simeq X$.
- The isomorphism $C \simeq \tilde{X}$ induces an automorphism of $J \Rightarrow J_0$ acts transitively on $\text{Hilb}(C, J)$.
- If $T_X$ (translation by 1) stabilizes $C \Rightarrow C$ has $1$'s, $C / \phi \subset C / \phi$ curve of lower genus sitting in isogenous $A$, $C$ contradiction with irreducibility of $J$.

Lema: Let $C$ smooth of genus $3$, non-hyperelliptic.

$\Rightarrow C \subset J$ is semi-regular.

Proof
- $H^1(M, C) \to H^3(J, \mathbb{Z})$
- By def $H^i$ is dual to $\ast : H^0(J, \mathbb{Z}^2) \to H^0(J, \mathbb{Z}^2)$
- $\Rightarrow H^0(C, N^0 \otimes \omega_C)$
- By $\text{Tor}$ it surjective for $\omega_C$.
- $\Rightarrow$ defined as follows
- $0 \to N^0 \to \mathbb{Z}^2 \to N_v \otimes \omega_C \to 0$ (part of Keresi complex)

Take $N^2$:
- $0 \to N^2 \to \mathbb{Z}^2 \to N_v \otimes \omega_C \to 0$ (pair of Keresi complex)

In conclusion:
- $0 \to H^0(N^2 \otimes \omega_C) \to H^0(N^2 \otimes \omega_C) \to H^0(N \otimes \omega_C) \to 0$.
- $\Rightarrow (N^2 \otimes \omega_C) \to H^1(J, \mathbb{Z})$. 
- $H^1(J, \mathbb{Z})$ is finite.
\[
\mathcal{N}_{X/Y} = \mathcal{O}_X \otimes \mathcal{O}_{X/Y}
\]

We have contraction maps \((\mathcal{L}^{n-k})^* \otimes \mathcal{O}_X \to \mathcal{L}_X^k\
\mathcal{N}_{X/Y} \otimes \mathcal{O}_{X/Y} \to \mathcal{L}_X^k \otimes \mathcal{N}_{X/Y}
\]

\[\Lambda^m \cdot e : \Lambda^m \mathcal{N}_{X/Y} \to \mathcal{L}_X^k\]

\[\Rightarrow \mathcal{L}_X^k \to \mathcal{N}_{X/Y} \otimes \mathcal{O}_X\text{ inducing a map on cohomology}\]

\[(*) \quad \cap \quad H^{m-p} (\mathcal{N}_{X/Y}^k) \to H^{m-p} \left( \mathcal{L}_X^k \otimes \mathcal{O}_X \right)\]

**Def**. The semi-regularity map \(\Pi\) of \(X\) is the dual to *:\n
\[\Pi : H^1 (X, \mathcal{N}_{X/Y}) \to H^p - p \mathcal{O}_X (\mathcal{L}_X^k)\]

Q: Is the Hilbert point \([X]_{Hilb} \mathcal{O}_X (\mathcal{L}_X^k)\) smooth? 3/17

\(\mathcal{O}_X \text{ smooth at } X = Y (c.i.)\)

\(\Rightarrow \) is \(\mathcal{O}_X \) unobstructed? Bloch: this implied by injectivity of semi-regularity map.

\[\mathcal{N}_E \in H^0 (\Lambda^{m-p} \mathcal{N}_{X/Y} \otimes \mathcal{L}_X^k)\]

\[\Lambda^{m-p} \mathcal{N}_{X/Y} \otimes \mathcal{L}_X^k = \mathcal{N}_E \otimes \mathcal{O}_X \otimes \mathcal{L}_X^k\]

\[\Rightarrow \mathcal{N}_E : \mathcal{L}_X^k \to \mathcal{N}_E \otimes \mathcal{O}_X\text{ induces } \Pi \text{ above.}\]

\[\Pi : H^1 (X, \mathcal{N}) \to H^p - p \mathcal{O}_X (\mathcal{L}_X^k) \text{ map defined on the obstruction space!}\]

**Example** If \(X \subset Y\) is a divisor, \(\Pi : H^1 (X, \mathcal{N}) \to H^2 (Y, \mathcal{O})\)

- comes from edge hom of \(\mathcal{O} \to \mathcal{O}_X \to \mathcal{O}_X (k) \to \mathcal{N} \to \mathcal{O}\)
- discovered by Kodaira–Spencer, who proved \(\Pi\) injective \(\Rightarrow X\) smooth pt of \(H_{16} (X)\)

**Def**. Let \(X \subset Y\) be as before, then the inf. Abel–Jacobi map for \(X\) is the morphism

\[\mathcal{A}^0 : H^0 (X, \mathcal{N}) \to H^0 (Y, \mathcal{L}_Y)\]

which is dual to

\[\Lambda^0 : H^{n-p} (\mathcal{L}_X^k) \to H^{n-p} (\mathcal{N}_E \otimes \mathcal{O}_X)\]

- introduced by Clemens, differential of A–J map to Griffiths intermediate Jacobian.
Theorem (S. Bloch) If $X \to Y$ is before and the semi-regularity map $\tau$ for $X$ is injective $\Rightarrow D_X(x)$ is unbounded.

Proof (Kawamata)
As before have $A_n, S = \text{ Spec } A_n, \gamma_n (S_n \to S_n)$

say $X_n = X \times S_n$. We'll check $\tau$ lifting holds $\Rightarrow$ the recursion.

Suppose we are given a flat deformation $X_n \subseteq Y_n$

$T' (C_n, S_n) = H^0 (X_n, N_n(x))$

Need to check that $H^0 (N_n(x)) \to H^0 (N_n/(x-1))$ is surjective.

Done by $\text{ Ext }_Y^1 (H^0 (N_n(x)), 0)$

The relative inf. A-E map standard exact sequence:

Restriction $0 \to N_n(x) \otimes (t^{n-1}) \to N_n(x) \otimes (t^{n-1}) \to N_n(x) \otimes (t^{n-1}) \to 0$

Inflation $N_n(x) \otimes (t^{n-1})$ can be induced to $N_n(x)$ as an $O_{x_n}$ module.

$0 \to N_n(x) \otimes (t^{n-1}) \to N_n(x) \to N_n/(x-1) \to 0$

Remark: If we're looking at usual deformations of a variety $X$

we know that a ring extension $0 \to Q_n \to Q_n \to Q_n \to 0$

$\iff$ Ext of $Q_n$ module $0 \to Q_n \to T_n(X) \to T_n(X) \to 0$

So if $X$ is smooth, e.g., then $0 \to Q_n \to T_n(X) \to T_n(X)$

Using these sequences we get a commutative diagram:

$H^1 (N_n(x), t^{n-1}) \to H^1 (N_n/(x-1), t^{n-1})$

and $H^1 (N_n/(x-1), t^{n-1}) \to H^1 (Q_n(x), t^{n-1})$

We want to show $x$ surjective $\Rightarrow$ show $\gamma = 0$.

$Y$ doesn't deform, $X_n = X \times S_n, Y_n = Y \times S_n$

$\Rightarrow$ $x$ surjective $\Rightarrow \phi \circ x = 0$.

But it is injective $\Rightarrow y = 0$.
$\varphi$ is surjective $\iff h'(N^2\nu) + h'(N^2\otimes\omega) = h'(L_2)$

$$h'(N^2\nu) : L^2\nu = \omega \otimes = \omega^{-1}$$

$h'(N^2\nu) = 0$ and $h'(N^\nu\otimes\omega) = 0$.

To calculate $h'(N^\nu\otimes\omega)$, consider $0 \rightarrow N^\nu\otimes\omega \rightarrow L_1\otimes\omega \rightarrow C_2 \otimes \rightarrow 0$.

The map

$$\mu : H^6(\mathcal{O}_{L_1}/\mathcal{O}_{C_2}) \rightarrow H^6(\mathcal{O}_{C_2})$$

$$\mu(\mathcal{O}_{L_1}/\mathcal{O}_{C_2}) \otimes \mu(\mathcal{O}_{L_1}/\mathcal{O}_{C_2}) = H^6(\mathcal{O}_{C_2})$$

by the third isomorphism theorem.

$\Rightarrow h^0(\mathcal{O}_{C_2}/\mathcal{O}_{C_2}) = 2$ and $h^1(\mathcal{O}_{C_2}/\mathcal{O}_{C_2}) = 0$.

$\Rightarrow h'(N^\nu\otimes\omega) = 2$.

But note $h'(N) \neq 0$.

3. Theorem (2v Run): Let $\mathcal{X}$ be a smooth curve, $X \rightarrow X$ a small birational modification.

$\Rightarrow O(c,x,y)$ smooth and $O(c,x,y) \rightarrow O(c)$ is smooth.

Then $\mathcal{X}$ is a variety, $T'$ lifting holds for $\mathcal{X}$.

$\Rightarrow O(c,x,y)$ is unobstructed.

Corollary: $\mathcal{X}$ is smooth, $\mathcal{X}$ rigid at $x$, (in embedded sense) $\Rightarrow 0(c,x,y)$ unobstructed.

Proof of Theorem (2v Run):

Definition: Given a deformation functor $D : \text{Art} \rightarrow \text{Set}$, a functor $f : D\text{-couples} \rightarrow \text{modules}$/Artin algebras, is called deformation invariant if $f$ has the base change property $\mathcal{L}$ if $f(\mathcal{A}, \mathcal{f})$ is a free $\mathcal{A}$-module (like cohomology of a relative sheaf of forms ...).

Note: Given $D \rightarrow T_0$. If $T'_0$ is deformation invariant then $T'_0$ lifting holds (Nakayama).

Example: If $X_0 \rightarrow X_0$, $D = A : X_0$ is deformation invariant if $X_0$ proper, smooth.
A - artin local algebra.

\[ X_A \xrightarrow{f_A} X \xrightarrow{\delta_A} \mathcal{D}(X, x) \]

\[ T^1(X_A, \mathfrak{m}, X) = H^1(T_{X_A}/\mathfrak{m}, \mathcal{F}_\mathfrak{m} \rightarrow \mathcal{F}_\mathfrak{m} \otimes N_{X_A}/\mathfrak{m} X) \]

\[ \begin{array}{ccc}
0 \rightarrow & \frac{\mathfrak{m}}{\mathfrak{m} \cdot N_{X_A}/\mathfrak{m} X} & \rightarrow \frac{T_{X_A}/\mathfrak{m}}{\mathfrak{m} \cdot N_{X_A}/\mathfrak{m} X} & \rightarrow \mathcal{F}_\mathfrak{m} \otimes N_{X_A}/\mathfrak{m} X \rightarrow 0
\end{array} \]

We saw that \( Y \rightarrow C_Y \rightarrow \mathcal{O}_{Y_A} = \mathcal{O}_Y \Rightarrow T_{Y/A} = T_{X/A} \otimes \mathcal{O}_{Y/A} = T_{Y/A} \).
\( N_{Y/A} = \mathcal{O}_{Y/A} \) Also \( h^0(CT_{X/A}) = 0 \) for \( C_Y \).

In cohomology:
\[ 0 \rightarrow H^0(\mathcal{O}_Y, \mathcal{O}_{Y_A}) \rightarrow H^0(T_{Y/A} \rightarrow \mathcal{O}_Y, \mathcal{O}_{Y_A}) \rightarrow \]
\[ \rightarrow H^1(\mathcal{O}_Y, \mathcal{O}_{Y_A}) \rightarrow H^1(Y, \mathcal{O}_{Y_A}) \]

Now \( H^0(\mathcal{O}_Y, \mathcal{O}_{Y_A}) \) is deformation invariant \( \Rightarrow \) map smooth
\( \ker (H^1(\mathcal{O}_Y, \mathcal{O}_{Y_A}) \rightarrow H^1(Y, \mathcal{O}_{Y_A})) \) is deformation invariant
as well - both sides free, look at closed point map of vector spaces
\( T^1 \) also invariant: total space smooth.

Proposition: \( X \) a variety s.t. \( T^1 \) lifting holds for \( \text{det} x, c \rightarrow X \)
\( \text{det} \) of dim one, \( \mathbb{C} \) assume \( H^1(N_{X_A}) = 0 \)
\( \Rightarrow H^1(c = h) \) is weakly fixed.

Proof: Start with \( G \rightarrow X \), flat deformation of \( G \times X \)

Let \( G_{X_A} \xrightarrow{\psi} X_{A-1} \) be the restriction on \( X_{A-1} \).
\( T^1(\text{det}) \rightarrow (A_n, G \times X) = H^1(T_{X_A} \rightarrow \mathcal{O}_X, N_{X_A}/X_A) \)
\( H^0(G, N_{X_A}/X_A) \rightarrow H^1(\mathcal{O}_{X_A} \rightarrow \mathcal{O}_X, N_{X_A}/X_A) \)
\( H^0(G, N_{X_A}/X_A) \rightarrow H^1(\mathcal{O}_{X_A} \rightarrow \mathcal{O}_X, N_{X_A}/X_A) \)
\( T^1 \) lifting for \( X \rightarrow \mathbb{A} \) is surjective, we want to show \( \psi \) surjective.
\( H^1(N_{X_A}) = 0 \Rightarrow T^1 \) lifting holds
for \( D(c) \Rightarrow \psi \) surjective.

If we can show \( H^1(N_{X_A}) = 0 \), \( \psi \) will be surjective.
(Diagram chase . . .)
By Leray's theorem, it suffices to show that 

$$\text{H}^0(S, R^{1+} N(x))$$

and 

$$\text{H}^1(S, \Omega^{1+} N(x))$$

are 0 (⇒ Ext term 0).

Second is automatic since S affine, and first
follows from an infinitesimal version of semi-continuity +

$$\text{H}^1(N(x)) = 0$$

Corollary: A smooth 3d C→C(x rigid) is: curve ⇒

$$\text{V}(x)$$ unobstructed.

Proof: 

$$\text{Ext}^1(N(x), \Omega^{1+} N(x))$$

so rigidity gives vanishing of the obstruction space.

Mumford's example: Given a smooth space curve of degree

14, genus 24, such that nearby Pd C is in the

Hilbert scheme, V(x) is unobstructed.


"Further pathological in algebraic geometry."

C CP^3, curve, H = X(1, 24, \text{CP}^3).

Thus, CP^3 surface H = O(1) \times \text{C}, \text{h} = O(1) \times \text{C}.

Fact: Any nonsingular space curve C \subset H is

contained in a pencil of quartics.

Proof: 

$$\text{H}^0(O(C, 4 \times \text{h})) \to \text{H}^0(C, 6 \times \text{h})$$

First, finite, kernel =

$$\text{h}^0(O(C, 4 \times \text{h})) = 4^{4+4-1} = 35$$

Next, by 4 \times 24 = 46

$$\Rightarrow (R-1, h^0(O(C, 4 \times \text{h})) = 56 - 23 = 33, \dim \text{ker} \geq 2$$

Let now C be a space curve, P pencil of quartics through C. Assume P has no fixed components ⇒

F', F" span P ⇒ F \times F" = C \times q, q conic

⇒ C \times q has at most double points (smooth, 2 \times \text{not in examples})

⇒ F' \times F" share no double points. Hence F' \times F"

will be smooth either in F' or in F".

Also, C \times q is smooth transverse intersection of F' \times F", both

are smooth along it; \Rightarrow \text{generic F' \times F"

is smooth along all of C.}

Fact 2: Any algebraic family of smooth curves

in \text{H} \times \text{C} that are contained in pencils of quartics whose fixed components

has dimension \leq 56.
If $P$ has a fixed component, must be a cubic surface (deg = 1, 2 or 3) $\Rightarrow$ CCF cubic if $F$ has 10 be unique (deg $c = 14$ $\Rightarrow$ deg $F = 10$ only the cubics).  

Fact 3: Any maximal algebraic family of curves $C$ contained in cubics is of dimension exactly 10.

Proof: If CCF, $F \subseteq C$ curve $\Rightarrow$ $F = -H$.  
By RR: dim $1c_F = (C \cdot 2H + H^2) = 1 + 2 \cdot (O_F(c)) - h^1(O_F(c))$

$\omega_c = C \cdot (O_F(c) = C \cdot (H)_{F} = 46$.  $\deg(C)_{H} \Rightarrow CC \cdot H = 60$

$h^1(O_F(c)) = h^2(\mathcal{O}_F(-H - c))$ duality

$\mathcal{O} \rightarrow \mathcal{O}_F(-H - c) \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_{H+c} \rightarrow 0$  
$H+c$ reduced, connected $\Rightarrow h^0(\mathcal{O}_{H+c}) = 1$

$\Rightarrow h^1(\mathcal{O}_F(c)) = 0$ $i = 1, 2$. $\Rightarrow$ dim $1c_e = 3$

If CCF is generic in a maximal/family, declare CCF (generic $F$) specializing to $C$.

Picard doesn't move for cubic, by upper semi-continuity, get the count we want: $17 + 37 ?$