Existence of connections

**Example 1:** If $L \to X$ is a line bundle, then $L$ has an 
alg. connection iif its Atiyah class $\alpha(L) = c_1(L) = 0$.

**Example 2:** $E \to X$ v.b. on a curve $(smooth, projective)$

$E$ has connection iif $\alpha(E) = 0$. By construction, 
a is additive under direct sum. (Split the Euler sequence...)

If $E = \mathcal{O}(1) \oplus \mathcal{O}(-1) \to p^*$, $c_1(E) = 0$

$a(E) \in H^1(End \mathcal{O}(1) \otimes \mathcal{O}(1)) \to H^1(End \mathcal{O}(1) \otimes \mathcal{O}(1)) = (c_1(\mathcal{O}(1)))$

$H^0(End \mathcal{O}(1) \otimes \mathcal{O}(1)) \Rightarrow a(E) \in H^0(\mathcal{O}(1)) \otimes H^0(\mathcal{O}(-1))$ 

$\Rightarrow \alpha(E) = \alpha(\mathcal{O}(1)) \otimes \alpha(\mathcal{O}(-1)) \neq 0$.

Def: $E \to X$ vector bundle is decomposable (split) if it doesn't split into a direct sum of vector subbundles.

If $X$ is compact connected, every $E \to X$ has 
Decomposition: $E = E_1 \oplus \cdots \oplus E_r$ into indecomposables.

unique up to isom. (Remark: Decomposition)

Thm. (Well): $E \to X$ v.b. on smooth projective curve, 
$L = E, \theta, \theta_\ast$.

$E$ has an algebraic connection iif $\deg c_1(E) = 0$ v.b.

**Proof:** Since Atiyah class is additive, enough to check 
that if $E$ is indecomposable, then $\alpha(E) = 0$ iif $\deg c_1(E) = 0$.

Proposition. $X$ compact complex manifold, $E \to X$ v.b.

$\Rightarrow E$ is indecomposable iif $A = H^0(X, End E)$ satisfies

1. The nilpotent elements form a subalgebra $N \subset A$.
2. As a vector space $A = C^\infty \oplus N$.

**Proof:** ($\Rightarrow$) $E$ indecompos. $\Rightarrow \forall e \in A \to \det (\chi_2) \neq 0$.

$= f_{\chi_1} f_{\chi_2} \cdots + \psi \text{ clearly s'ly. } \Rightarrow 0 \in H^0(X, \mathcal{O}_X) = \text{constants}$

$\Rightarrow$ eigenvalues constant. $\Rightarrow$ take generalised eigenspaces

for any $\lambda$, $E_\lambda = \{s \in (\mathbf{C}, \mathbf{C})^n \text{ s'ly for } \lambda \}$

subbundle, $\Rightarrow \prod \chi_i$ can have only one eigenvalue

$\Rightarrow \det (\chi_2 - \psi) = 0$. $\Rightarrow \lambda \cdot \mathbf{1}$ is nilpotent

(Any endo is either nilpotent or an isomorphism.)

($\Leftarrow$) If (1), (2) hold and $E = E_1 \oplus E_2$ can look at

projections $g_1, g_2: \mathbf{1}$.

$E_1 = \mathbf{1} \oplus 0$, $g_2 = 0 \oplus \mathbf{1}$.
Now if $E \rightarrow \mathcal{X}$ is a curve, then $H^{1}(\mathcal{X}, \mathcal{E} \otimes \mathcal{L})^{\vee} = H^{0}(\mathcal{X}, \mathcal{E})^{\vee}$.

To calculate $a(E)$ as function on $H^{0}(\mathcal{End} \mathcal{E})$.

Proposition. Let $\mathcal{E} \in H^{0}(\mathcal{End} \mathcal{E})$ be nilpotent.

Then $\langle a(E), \mathcal{E} \rangle = 0$.

Proof. For any $\mathcal{F} \in H^{0}(\mathcal{End} \mathcal{E})$, we have $\langle \mathcal{F} \mathcal{E}, \mathcal{E} \rangle = 0$.

Let $E \in \mathcal{E}$ be the class of $\mathcal{E}$, but $\mathcal{T} \mathcal{E} = -2 \mathcal{T} \mathcal{E}$.

 Identify part. Nilpotent part: Look at $E$, $\mathcal{E}$ nilpotent, $\mathcal{E} = 0$.

$\mathcal{E}$ is a subsheaf of $\mathcal{E}$, since we're on a curve this generates a proper subsheaf $\pi$.

Set a flag of subsheaves preserved by $\mathcal{E}$.

By the Fact, $a(E) \in H^{0}(\mathcal{End} \mathcal{E})^{\vee}$.

$\langle \mathcal{E}, \mathcal{E} \rangle = \mathcal{E} \mathcal{T} \mathcal{E} = 0$.

Prop. Whitt. Theorem.

Why is the Atiyah class compatible with filtrations?

Differential operators revisited:

$a(E)$ will be compatible with filtration if $a(E)$ is an exact functor.

$0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}^{\vee} \rightarrow T \mathcal{E} \rightarrow 0$.

Not exact $(\operatorname{re}) (\operatorname{curve}) a(E)$.

$x_{n}$ smooth $\mathcal{A}: x_{n} \rightarrow x_{n-1}$:

$x_{n}$ is a neighborhood.

The $n$th jets of functions on $x_{n}$ is $J_{x_{n}} = \mathcal{O}_{x_{n}} / \mathcal{I}_{x_{n}}^{n}$.

As an $\mathcal{O}_{x_{n}}$-module in the following sense:

$J_{x_{n}}$ is an $\mathcal{O}_{x_{n}}$-module, $J_{x_{n}} \rightarrow J_{x_{n}}$.

$J_{x_{n}}$ induces via pullback $\mathcal{O}_{x_{n}}$.

Two ways $\mathcal{O}_{x_{n}} \rightarrow J_{x_{n}}$ in order to order generation.

$\partial^{n}_{x}: \mathcal{O}_{x} \rightarrow J_{x_{n}}$, induced by $\mathcal{O}_{x_{n}}$. 

\[ \]
\[
J^1_x \text{ is the sheaf of analytized } \mathcal{O}_X \text{ over } X.
\]
\[
J_x = \mathcal{H}^1(J^1_x) \rightarrow \mathcal{H}^0(J^0_x) = \mathcal{O}_X
\]

There is always \( S \cdot J^1_x \rightarrow J^0_x \), i.e., \( X \) smooth.

We have short exacts \( 0 \rightarrow S \cdot J^1_x \rightarrow J^0_x \rightarrow \mathcal{H}^1(J^1_x) \rightarrow 0 \).

Note \( \mathcal{H} \circ \mathcal{O}_X : \mathcal{H}^1; f \mapsto \mathcal{H}^1 \circ f \) usual de Rham cl.

A, B quasi ish sheaf, \( \mathcal{O}^k(A, B) = \text{Hom}_X(J^k(A), B) \).

In particular symbol sequences come from jet sequences.

\[
F \text{ flat } \Rightarrow 0 \rightarrow \mathcal{O}^{k-1}_x(F) \rightarrow \mathcal{O}^k_x(F) \rightarrow S^k T_x \otimes \mathcal{O}_F \rightarrow 0
\]

\[
0 \rightarrow \mathcal{O}^{k-1}_x(F) \otimes \mathcal{O}_F \rightarrow \mathcal{O}^k_x(F) \otimes \mathcal{O}_F \rightarrow \mathcal{O}^k_x(S^k L_x \otimes \mathcal{O}_F) \otimes \mathcal{O}_F \rightarrow 0
\]

Note if \( F \) flat, \( J^k_x(F) = J^k_x \otimes \mathcal{O}_F \rightarrow P_{\mathcal{O}_X}^k \mathcal{O}_F \rightarrow 0 \)

Claim: \( a(F) = -j^!(\mathcal{F}) \)

where \( j^!(\mathcal{F}) \) is now \( 0 \rightarrow \mathcal{O}^k_x \otimes \mathcal{F} \rightarrow \mathcal{O}^k_x(F) \rightarrow 0 \)

\( \Rightarrow j^!(\mathcal{F}) \) is exact.

**Higher Dimensions**

Example 3 (Atiyah–Viel's theorem doesn't generalize to dim > 1)

\( X = Y \times Z \), \( Y = \mathcal{P}^1, Z = \text{elliptic curve} \)

Choose base points \( y_0 + Y, z_0 \in Z \) : \( Y = Y \times \{z_0\} \times Z \),

\( Z = \{y_0\} \times Z \subset X \).

Look at the exact seq on \( X \),

\( \mathcal{O}_X^1(\mathcal{O}_Z) \rightarrow \mathcal{O}_X(\mathcal{O}_Z) \rightarrow \mathcal{O}_Z(\mathcal{O}_Z) \rightarrow 0 \)

\( \Rightarrow \) in cohomology

\[
H^1(\mathcal{O}_X(\mathcal{O}_Z)) \rightarrow H^1(\mathcal{O}_Z) \rightarrow H^2(\mathcal{O}_Z)
\]

\[
= \mathcal{H}^0(\mathcal{O}_Z(-2) \otimes \mathcal{O}_X)
\]

\(
= \mathcal{H}^0(\mathcal{O}_Z(-2) \otimes K_X)
\]

\( \Rightarrow x \in H^1(\mathcal{O}_Z) \) mapping to the generator of

\( H^2(\mathcal{O}_Z) \), \( \Rightarrow x \) \( \text{ a rank two vector bundle} \)

\( \text{Claim: } (1) \ a(E) = 0 \quad (2) \ E \text{ is irreducible} \)

\( \Rightarrow (3) \ a(E) \neq 0 \).
In particular, affine bundles modeled on $E$ are given by surjective $\mathbb{G}_a$-equivariant maps $\tilde{E} \to E$. They are called affine vector bundles.

Example 1: Let $\mathbb{G}_a$-equivariant maps $\tilde{E} \to E$.

Example 2: Let $\mathbb{G}_a$-equivariant maps $\tilde{E} \to E$.

Example 3: Let $\mathbb{G}_a$-equivariant maps $\tilde{E} \to E$.

In particular, affine bundles modeled on $E$ are given by surjective $\mathbb{G}_a$-equivariant maps $\tilde{E} \to E$. They are called affine vector bundles.
If $E \to S$ a vector bundle, replace $S$ with an affine bundle

$\pi : \mathcal{E} \to S$ so that $\pi^* \mathcal{E}$ has a canonical connection:

$\alpha(\mathcal{E}) \in H^1(S, \text{End } \mathcal{E} \otimes L)$

But take $\mathcal{E}$ to be affine bundle modeled on $\text{End } E \otimes L$'s corresp to $\alpha(E)$.

Check: there is a canonical connection on $\pi^* \mathcal{E}$.

Q: Can we do this for all vector bundles simultaneously?

Theorem (Jendeleau's trick) If $X$ quasi-projective $\Rightarrow$

3 affine bundle $Y \to X$, Zariski loc. trivial so met $Y$ is an affine variety

Proof 1. May assume $X$ projective: indeed $X \subset \mathbb{P}^N$ projective
can blow up $X \times X$ to set $R \to X$ st. $R$ is projective,
complement is Cartier divisor.

If now $Y \to X$ an affine bundle with $Y$ affine $\Rightarrow$

$\pi^{-1}(x)'s$ an affine bundle $X'$, $Y \times \pi^{-1}(X%)$ is Cartier.

But the complement of a Cartier divisor in an affine variety is affine.

2. May assume $X$ is a $\mathbb{P}^N$'s $\Rightarrow$ induct $X \subset \mathbb{P}^N$ closed

subvariety. If $Y \to \mathbb{P}^N$ affine affine bundle,
$Y \times \mathbb{P}^N$ affine & closed subvariety in $Y \to \mathbb{P}^N$.

3. For $\mathbb{P}^N$ take $\text{GL}(N+1)/\text{GL}(N) \times \text{GL}(N) \to \mathbb{P}^N = \text{GL}(N)/P$

lies form. Fibers are $\text{unipotent}$ radical $R\times P$

This is the affine bundle endomorphism on $\mathbb{P}^N$

Corollary $X$ quasi-projective $\Rightarrow \exists \mathcal{E}\to X$ affine bundle

$\pi^* \mathcal{E}$ admits a connection $\nabla E \to X$.

$\text{iii)}$ Connections as forms on principal bundles

(= Endo with fibers, $ii$ = infinit. section of $\mathcal{E}$)

We saw that a vector bundle $E$ gives a principal $GL(N)$ bundle
for $n$-ranks $E$. Conversely a map $P \to GL$ gives

principal bundle into fib. $P \times V / G$
Look for a connection on $P$, inducing connections on all associated vector bundles.

A connection on $P$ is a splitting of
\[ 0 \to \mathfrak{tp} \to T_P \xrightarrow{dH} \mathfrak{ts} \to 0 \] (1)

It has to be compatible with $G$ action — ask for the splitting to be equivariant (all terms have $G$ action).

It is easy to check that $T_P P_S = g * \mathfrak{g}$.

Lift of connection on $T_P P_S$ is adjoint action of $G$ on $g$:
\[
(T_P P_S)_x \xrightarrow{J} (T_P P_S)_x g
\]
\[
\downarrow
\]
\[
\begin{array}{c}
g \mapsto Ad(g) \mapsto g
\end{array}
\]

Def. a connection on $P$ is a $G$-equivariant 1-form $\omega \in H^0(P, \mathfrak{g} \otimes \mathfrak{g})$ such that it is horizontal, i.e., splits $\mathfrak{g} \otimes \mathfrak{g} \to T_P$.

For existence, notice that if we push $\omega$ forward to $S$ and make $G$-equivariant again get short exact:
\[
0 \to (T_{\mathfrak{p}} T_{\mathfrak{p}_S}) G \to (T_{\mathfrak{p}} T_{\mathfrak{p}_S}) G \to (T_{\mathfrak{p}_S} \times \mathfrak{g}) G \to 0
\]
\[
\downarrow
\]
\[
0 \to P \times \mathfrak{g} \xrightarrow{\omega} \text{Ad}(P) \xrightarrow{\omega} T_S \to 0
\]

by definition

A $G$-equivariant splitting of $\omega$ is same as splitting of $\mathfrak{g}$,

so obstruction to existence of a connection on $P$ is $\omega(P) \in H^1(S, \mathfrak{p}_S \otimes \mathfrak{g})$. (extension class of $\omega$).

If $P$ was the frame bundle of $E \to \mathfrak{g}(P) = \mathfrak{g}(E)$.

Check Exact:

For general $P$ and a representation $\rho: G \to GL(V)$

take $E = P \times E$, Atiyah sequence for $E$ is

push-forward of $\omega(P)$ via map $T_P \to \text{End} E$.

-- a splitting of $\omega(P)$ gives a splitting.

A connection $d\omega$ on a $G$-equivariant is integrable

if $d^\omega \circ d\omega = 0$, i.e., $F \xrightarrow{d\omega} F \otimes \mathfrak{g} \xrightarrow{d\omega} F \otimes \mathfrak{g}^2$ etc.

is complex.

In version (1), a connection was a splitting
\[ 0 \to \text{End} E \to \mathfrak{A}(E) \xrightarrow{\omega} T_S \to 0 \]

But $\mathfrak{A}_E$ is a sheaf of $G$-linear Lie algebras (like $T_S$), End $E$ $G$-linear bundle.
Integrability & the fundamental form

Integrability of a connection \( \nabla \) in the sense of a connection \( \nabla \) \( \Rightarrow \) a Lie algebra homomorphism

\[ \text{Integrability } \Rightarrow \nabla : T_S \to \mathfrak{g} \text{ a Lie algebra homomorphism} \]

In interpretation iii, \( \nabla \) is integrable iff it's closed.

1. On a curve any function is integrable \( (d\xi = 0) \)

2. \( S \) smooth, connected, projective \( L \rightarrow S \) in bundle,
   \( d\xi \Rightarrow \xi \) integrable ...

   Indeed let \( \xi \) be the complex str Each structure operator on \( L \)

\[ D = \nabla + d\nabla : F \rightarrow F \otimes A \]

is an \( \infty \) connection on \( F \)

\[ \text{curvature } (\nabla) \in H^2 \Omega(\mathcal{C}) \text{ - closed by Bianchi identity} \]

\[ \text{curvature } (\nabla) \in H^2(\mathcal{C}) \]

Let \( U \in S \) be open set where \( F \) is trivialized, \( \text{ and } \)

\[ \text{let } \theta = F(\xi U) \text{ non vanishing holomorphic section} \]

\[ \Rightarrow d\xi = 0 \Rightarrow \xi \in \mathfrak{g} \]

\[ \Rightarrow \xi \in \mathfrak{g} \]

So the curvatures \( \text{curvature}(\nabla) \Rightarrow \text{curvature}(\xi) \)

\[ \text{by Chern-Weil } \quad c(F) = 2\pi i \text{ curvature}(\xi) \]

but \( c(F) = \text{tr}(\text{Atiyah class}) = 0 \) \( \Rightarrow \) since \( F \) has a connection

3. \( S \) smooth, projective \( \Rightarrow \) if \( F \rightarrow S \) has connection \( d\xi \Rightarrow \)
   all invariant polynomials of \( \text{curvature}(\xi) \) vanish
   - by some argument.
Especially, \( \text{curv}(\theta) = 0 \).

2. Connection as infinitesimal action: splitting of
\( \alpha(F) : 0 \to \text{End } F \to \mathfrak{X}_c(F) \xrightarrow{\theta} \mathfrak{g}_c \to 0 \)

Note: \( \text{End } F \) is a sheaf with \( \mathfrak{g}_c \)-linear Lie bracket.
\( \mathfrak{g}_c \) is equipped with a \( \mathfrak{g}_c \)-linear Lie bracket.
\( \mathfrak{g}_c \) also has a \( \mathfrak{g}_c \)-linear Lie bracket.

A subspace of \( \mathfrak{g}_c \) which has natural multiplication:
\( \to \text{End } c \mathfrak{g}_c \to \mathfrak{g}_c \).

Lemma: \( \mathfrak{g}_c \) is integrable iff \( \mathfrak{g}_c \) is a morphism of Lie algebras.

Proof: \( \gamma \in \mathfrak{g}_c \) is a morphism of Lie algebras.

Different interpretation: \( \mathfrak{g}_c \) is naturally a subalgebra of \( \mathfrak{g}_c \) such that \( \mathfrak{g}_c(\mathfrak{g}_c) = \mathfrak{g}_c \).

Because \( \mathfrak{g}_c \) has canonical integrable connection \( \theta \).

\( 0 \to \mathfrak{g}_c \to \mathfrak{g}_c \subseteq \mathfrak{g}_c \).

5. Lemma \( \implies \) the action of infinitesimal symmetries \( \mathfrak{g}_c \) extends to an action of \( \mathfrak{g}_c \), i.e., a homomorphism \( \mathfrak{g}_c \to \mathfrak{g}_c \).

Interpret by one-forms on principal bundles:
\( G \)-equivariant 1-form \( \omega \) on a principal \( G \)-bundle \( \implies \) integrable \( \implies \) \( \mathfrak{g}_c \mathfrak{g}_c \).

\( \omega \) is the curvature (after pushing down & taking invariant).

Integrable connections carry topological info...

Assume connected, universal cover \( \tilde{\mathcal{S}} \to \mathcal{S} \) is a principal \( \tilde{\Pi}_1(\mathcal{S}) \) bundle \( \to \) can take associated vector bundles.

Given \( \rho : \tilde{\Pi}_1(\mathcal{S}) \to \text{End } \mathcal{V} \).

\( \chi(x, y) = (xy, \rho(y)g) \).

To describe all \( \rho \)'s coming in this way, need notation:

6. Complex algebraic group \( G \) acts on \( \alpha \).

Def: \( G \)-complex algebraic, \( P \to \mathcal{S} \) principal \( G \)-bundle \( \implies \)

say that \( P \) has a discrete form if exists a principal \( G \)-
bundle \( P' \to \mathcal{S} \) and a continuous bijective \( G \to G \)-equivariant

map \( P' \to P \).

An \( \alpha \)-principal \( G \)-bundle \( P \) is called a \( G \)-local system if it

can be expressed by constant coordinate transitions.
transitions in $H'(S, G) < H'(S, G/C)$

Proposition: $P \rightarrow S$ principal $G$-bundle, $T$ distinguished

1. $P$ arises from a rep of $G$.
2. $P$ is a $G$-local system.
3. $P$ has an integrable connection.
4. $P$ has a fixed grad.

Proof: $G$ acts on $N$, $b \rightarrow c$ since cosets of $N$ are constant, thus choosing $g$ to be our connection on every patch $\gamma$, the trace of $G_{\gamma}g$ term vanishes.

Let $\mathcal{G}$ be the sheaf of germs of analytic sections of $P$. Then $\mathcal{G}$ is a sheaf of sets with natural topology on total space with fiber discrete, fibers continuous.

$\mathcal{G} : P \rightarrow \mathcal{G}(x)$.

There is a continuous $G$-action on $P$, let $\rho : P \rightarrow P/G$ the topological quotient. Integrable connection gives a continuous action $\rho \in \Gamma(S, P/G)$.

$\rho = \rho^{-1}(\rho(S))$.

$G$ is discrete $\Rightarrow \rho_d \rightarrow G$ has contractible components where

are covering spaces $\Rightarrow \rho_d : \pi(S) \rightarrow G \Rightarrow G$.

Corollary. Smooth proj. $L \rightarrow S$ line bundle $\Rightarrow L$ comes from a character of $G$, iff $L$ alg. equivalent to $O$.

(Ch. 1) If $S$ is a proj curve $\Rightarrow$ a subspace comes from a rep of $G$, iff all its irreducible summands have degree zero.

$\Rightarrow$ Topological interplay of integrability: $F$ a vb with connection $\nabla$

Can look at $D = \delta_F + dP$, s.t. $\text{curv}(D) = \text{curv}(\nabla)$.

Given a path $Y : [0,1] \rightarrow S$ with endpoints $x,y$, piecewise smooth,

Can use 0 to lift $Y$ to path in $F$ with any horiz $\text{vec}x$.

$\Rightarrow$ attach to any path $Y$ an iso $\xi : F_x \rightarrow F_y$ parallel transport with $D$ along $Y$.

If $\text{Th}$ is the signature of all loops in $S$ based at $x$,

get hom $\mathcal{C} : \text{Th} \rightarrow \text{Aut} F_x$, holonomy action of $D$, graph generated by image is holonomy group.

$\Rightarrow$ $D$ integrable $\iff$ $C$ factors through $\text{Th}(Sx)$ $\Rightarrow$ monodromy.
Crystalline interpretation of an integrable connection

Let $F \to S$ be a vector bundle. Then an integrable
correction $d^0$ on $F$ is the same as an isomorphism

$p: p^* F \to p_2^* F$ on $(S \times S)^+$ s.t. $p$ satisfies cocycle condition

$(p_1^* \phi)(p_2^* \phi) = (p_3^* \phi)$ on $(S \times S)^+$, + $\phi$ is an isomorphism restricted on $S$.

**Proof:** Given $p$ we can construct $d^0: F \to F \otimes L_S$ as follows:

Let $\Delta^{(2)}: S \to S \times S$, $\Delta^{(3)}: S \to S \times S \times S$

For any $a \in F$, we look at $p_2^* a - q(p_1^* a) \mod I^2$.

$I$ is ideal of $\Delta^{(2)}/(S)$. This belongs to $p_2^* F \otimes O_S/I^2$

but is actually in $p_2^* F \otimes I/ I^2$.

$= d^{0}(a) := \Delta^{(2)} \times (p_2^* a - q(p_1^* a) \mod I^2) \in F \otimes O_S/I^2$.

So we've constructed a functor from pairs $(F, \phi)$ to pairs $(F, d^0)$,

$d^0 \in \text{Hom}(F, F \otimes L_S)$.

We want to show that this gives an equivalence with
the full subcategory $(F, d^0$ integrable $) \to \text{local in } S$.

- Reduce to case $S$ affine: over $S$ by open $V$.
- State results on affine, take direct image.

So assume $S = U$ affine open and $F|U$ is trivial.

Want $U$ affine open in $V$ and $d = 0$ $\to \text{can be done }$ in $\text{etale}$.

Let $\phi: C^0 = C^1$ be given by a function $g: U \times U \to O(U)$.

$g(x, y) = 1 \to A(x)(x-y) + O((x-y)^2)$

where is a $M_n(C)$ valued $1$-form on $U$.

$\phi(x, y) = a(y) - g(x, y) a(x)$

$= a(y) - a(x) - A(x)(x-y) \to d^0 \phi = \frac{a(y) - a(x)}{y-x} = A(x)(x-y)$

i.e., $d^0 \phi = d \phi$. So $d^0 \phi$ is a morphism...
The cycle condition on $g$ gives a formal DE on $A$:

\[ g(y, z) = g(y, x) = (1 + A(y)(x-z)) g(x, y) \]

\[ g(x, z) - g(x, y) = A(y) g(x, y) \]

\[ \frac{\partial g}{\partial y} = A(y) g(x, y) \]

Projection onto part of the differential.

Get initial value problem

\[ 2 g(x, y) = A(x) g(x, y), \quad g(0) = I \]

\[ (*) \]

- $g$ is unique if it exists, for given $A$.

**Remark**: If $g$ is a soln then cycle condition is automatically satisfied since $g(x, z), g(x, y), g(x, y)$ are solutions & they coincide for $y=z$.

Need to show $g$ has solution iff $A$ is integrable.

Change variables $t = (y-x) = (t_1, \ldots, t_k)$

\[ A(x, t) = \frac{1}{2} A; \quad A(x, 0) = A(0, 0) \]

The system $(*),$ in these variables, reduces to

\[ \frac{\partial g(x, t)}{\partial x} = A(x, t) g(x, t), \quad g(x, 0) = I \]

Standard form: Integrability $\iff$ commuting of characteristics (second order) $\iff$ $A, A^0, \frac{\partial A}{\partial x_i} - \frac{\partial A_i}{\partial x_i} = [A_i, A_j] = 0$ (i.e., $A^2 = 0$).

**Def**: A stratification of scheme $S$ is a scheme $X \rightarrow S$ together with $g: X \times S \rightarrow \text{Spec} S$ on $S^{(s)}$ s.t. $g_{/S}$ is identity & satisfies the cycle condition.

Replace $(S^{(s)})^\times$ by any thickening $\text{crystalline}$ & any two retracts $\rightarrow$ crystalline. Two sections agree for smooth bases.

**Gauss-Manin**: $f: X \rightarrow S$ smooth, proper between smooth varieties, Relative de Rham: $H^k_{dR}(X/S, \mathcal{O}) = H^k_{dR}(O_X, f^*S)$

Want to exploit topological nature of

$H^k_{dR}(X/S) \rightarrow \cdots \rightarrow$ connection on $H^k_{dR}(X/S, \mathcal{O})$. 
1. **Topological construction**: By Grothendieck comparison theorems, the coherent analytic sheaf corresponding to $\mathcal{H}^n(X/S)$ is

$$R^k f_* \mathcal{L}^{\text{an}}_{X/S} = (R^k f_* \mathcal{C}^{\text{an}}_{X/S}) \otimes_{\mathcal{O}_S} \mathcal{O}_S$$

by universal coefficient theorem in analytic context, in particular $\mathcal{H}^n(X/S)$ acts as a discrete form, i.e., it is associated with the frame bundle of $R^k f_* \mathcal{L}^{\text{an}}_{X/S}$, so has an integrable connection.

Explicitly $\mathcal{G}_n$ is given by among the horizontal sections to $\mathcal{L}^{\text{an}}_{X/S}$ the sections of $R^k f_* \mathcal{C}^{\text{an}}_{X/S}$.

Topologically $R^k f_* \mathcal{L}^{\text{an}}_{X/S}$ is a coming space of $S \to \text{canonical connection}$, & this spans.

2. **Algebraic construction**: Given a smooth morphism of schemes $f: X \to S$, Grothendieck constructs a connection on $R^k f_* \mathcal{L}^{\text{an}}_{X/S} \in \mathcal{D}^b(C(S))$. Moreover if $R^k f_* \mathcal{L}^{\text{an}}_{X/S}$ are locally free (in particular, have the base change property) it shows the connection induces connections on any $R^k f_* \mathcal{F}$.

\[ \mathcal{D}(S, \mathcal{F}, \mathcal{D}(S)) \] S scheme, \( \mathcal{F} \in \mathcal{D}(S) \). An integrable connection on \( \mathcal{F} \) is a consistent way of associating to every diagram $S' \to S \to S$ $S \to S'$ is a square, \( \mathcal{D}(S') \to \mathcal{D}(S) \), restrictions of \( \mathcal{F} \) to \( \mathcal{D}(S') \) give an isomorphism \( \mathcal{F} \mid_{S'} \to \mathcal{F} \mid_{S} \).

That restricts to \( \text{Id} \) is associative for any given third restriction, i.e., cocycle condition.

**Special cases**:

(c.) \( f: X \to S \) smooth & affine

Preliminaries: \[ f^* (\mathcal{O}_S) \text{ is a } f^{-1}(\mathcal{O}_S) \text{-linear deriviation of } \mathcal{O}_X \text{ (vertical vector field)} \] i.e., \( \mathcal{O}_X \otimes_{\mathcal{O}_X} f^* \mathcal{O}_S \to \mathcal{O}_X \) induces a $f^{-1}(\mathcal{O}_S)$-linear homomorphism

\[ \Theta(\mathcal{O}_S): \mathcal{L}^{\text{an}}_{X/S} \to \mathcal{L}^{\text{an}}_{\mathcal{O}_S} \] (Lie derivative along flows)

Characterized by: \( \Theta \) acts on \( \mathcal{L}^{\text{an}}_{X/S} \), \( \Theta^\circ d \) Leibniz (Hahn of DGA's)
Special cases of the algebraic definition of Courant algebroids:

a. \( f : X \to S \) smooth, affine morphism.

Last time: to any \( \mathcal{O}_f \) an \( \mathcal{O}_f \)-module, \( \mathcal{O}(\mathcal{O}_f) : \text{Hom}_{\mathcal{O}_f}(\mathcal{O}_f, \mathcal{O}_f) \to \mathcal{O}_f \) associated a Lie derivative along the fibers \( \mathcal{O}(\mathcal{O}_f) \)
taking \( \mathcal{O}_f \) to \( \mathcal{O}_f \), endomorphism as differential graded algebra.

Main Observation: certain homotopy property:
\( \Theta(\mathcal{O}(\mathcal{O}_f)) = \text{d} \circ \text{Id} + \text{Id} \circ \text{d} \) contraction

so \( \Theta(\mathcal{O}(\mathcal{O}_f)) \) homotopic to the identity.

[ LHS & RHS agree on \( \mathcal{O}_f \) & commute with \( \text{d} \), using \( \text{d} \).

\( \Theta(\mathcal{O}(\mathcal{O}_f)) \) acts as \( \text{Id} \) on hypercohomology sheaves.

Went: morphism \( \mathcal{G} \cdot \mathcal{M} : \text{Der}_c(\mathcal{O}_f) \to \text{End}_c(\mathcal{H}^0_c(X; \mathcal{O}_f)) \)
connection.

First given \( \mathcal{V} \in \text{Der}_c(\mathcal{O}_f) \) try to lift to an infinitesimal automorphism of \( X \), i.e., derivation of \( \mathcal{O}_f \).

Local in \( S \), so may assume \( S \cdot X \) affine
\( S = \text{Spec} \mathcal{A}, \quad X = \text{Spec} \mathcal{B}, \quad f : A \to B \).

Every element of \( \text{Der}_c(\mathcal{O}_f) \) will lift to an automorphism of \( \mathcal{O}_f \)
\( \text{iff} \) the natural map \( \text{Der}_c(\mathcal{O}_f, \mathcal{B}) \to \text{Der}_c(A; B) \)
has image containing \( \text{Der}_c(AA) \).

Recall for any commutative \( (\text{Rokhlin}) \) rings \( f : A \to B \)
and any \( B \)-module \( \mathcal{E} \) there is an exact sequence
\[ 0 \to \text{Der}_c(\mathcal{O}_f, \mathcal{E}) \to \text{Der}_c(\mathcal{B}, \mathcal{E}) \to \text{Der}_c(A; \mathcal{E}) \]
\[ \to \text{Ext}^1(\mathcal{B}^\bullet, \mathcal{E}) \to \cdots \]
\( \text{Ext}^1(\mathcal{B}^\bullet, \mathcal{E}) \) = equivalence classes of \( A \)-algebras that are square 0 extensions of \( B \) by \( F \), i.e., \( \mathcal{E} \)-classes of \( \mathcal{O} \to \mathcal{I} \to \mathcal{E} \to B \to \mathcal{O} \) \( I^2 = 0 \), \( I = \mathcal{F} \) as an \( \mathcal{A} \)-module.

This sequence terminates for rings... but not in general for operads.

The map \( \mathcal{I} \) can be described explicitly:
\[ \mathbf{f} \in \text{Der}_c(A; \mathcal{E}) \]
has underlying vector space \( B \otimes \mathcal{E} \) as central term, multiplication is...
\((b, c) : (b', c') = (bb', b'e + b'e)\) and \(A\)-module

structure is given by \(a : (c, e) = (f(a)b, f(c) + fe)\).

Since our \(f\) is smooth \(\Rightarrow f\) satisfies the inf. lifting property, i.e.
\[ \text{Ex}^f (B, f) = 0 \]

in nonrelative case.

\[ 0 \rightarrow \text{Der} (B, B) \rightarrow \text{Der} (B, B) \rightarrow \text{Der} (B, B) \rightarrow 0 \]

so any derivation from \(\text{Der} (A, A) \rightarrow \text{Der} (A, B)\) can be lifted to \(\text{Der} (B, B)\) with ambiguity exactly \(\text{Der} (A, B)\) namely vertical vector fields, which act homotopically to 0

\[ \rightarrow \text{G-M connection} \]

Integrability - above are morphisms of \(L\)-Lie algebras. \((\text{Der} (A, B)\) is Lie module, \(\text{Der} (A, A)\) Lie algebra in \(X\).)

**B.** \(f : X \rightarrow S\) smooth morphism of smooth varieties.

Again we want to construct \(G_M : TS \rightarrow A_S (\mathcal{H}^1 (X/S))\) integrable.

By hypothesis we have a tangent exact sequence of vector bundles,
\[ 0 \rightarrow f^* N' \rightarrow N' \rightarrow \mathcal{L}' \rightarrow 0 \]

\(\mathcal{L}'\) has a filtration \(\mathcal{L}' = I^0 \supset I^1 \supset I^2 \supset \ldots\)

where \(I^k = \text{Im} f^* N'_S \otimes \mathcal{L}^{* - k} \rightarrow \mathcal{L}^{* - k}\)

Since \(\mathcal{L}'_x, N'_x\) are locally free \(\Rightarrow \text{the associated graded of } I^*\) are

\[ \mathcal{G}^* := \mathcal{G}^* \mathcal{L}'_x = f^* N'_S \otimes \mathcal{L}^{* - k} \]

by (1),
\[ 0 \rightarrow I^0/I^1 \rightarrow I^1/I^2 \rightarrow I^2/I^3 \rightarrow 0 \]

by (4),
\[ 0 \rightarrow f^* N'_S \otimes \mathcal{L}^{* - k-1} \rightarrow \mathcal{L}^{* - k} \rightarrow \mathcal{L}^{* - k} \rightarrow 0 \]
Take the long exact sequence of hyperderived images.

The $k^{th}$ edge homomorphism of this sequence is

$\mathbb{R}^k f_! L^1_{X/S} \to \mathbb{R}^{k+1} f_! (f^* L^1_{Y/S} \otimes L^0_{X/S})$

(projection formula + fact that differenctial

$\mathbb{L}^1_{X/S} \otimes \mathbb{R}^k f_! L^1_{X/S}$

in $\mathcal{L}^1_{X/S}$ is $f_! f^* L^1_{X/S}$)

$\mathbb{L}^1_{X/S} \otimes \mathbb{R}^k f_! L^1_{X/S}$

$\Rightarrow$ map $d^m_{\mathbb{L}} : H^k_{\mathbb{L}}(X/S) \to H^k_{\mathbb{L}}(X/S) \otimes \mathbb{L}^1_{X/S}$

- Gauss-Manin as differential in ( Emmy ? ) spectral sequence.

$E^{p,q}_2 \Rightarrow gr_{\mathbb{L}} (R^{\bullet,k} f_* L^1_X)$

$\Rightarrow d^{p-2} (gr \mathbb{L}) = \mathbb{L}^1_{X/S} \otimes \mathbb{L}^1_{X/S} : H^q_{\mathbb{L}}(X/S)$

This spectral sequence is multiplicative : $I^*$ is compatible

with $\wedge$ ( D.G.A. structure ) i.e. $I^* \wedge I^* = I^{*+*}$

get multiplicative structure on the $E_{p,q}^2$

$E_r^{p,q} \otimes E_r^{p',q'} \Rightarrow E_r^{p+p',q+q'}$

superrammandule

$e \cdot e' = (-1)^{pq}(2r+q) e \cdot e'$

$d_\mathbb{L}^p (ae) = (d_\mathbb{L}^p a) e + (-1)^{p+q} e \cdot d_\mathbb{L}^q a$

The differential on $E_1$ is given as follows : set $k = pq$

$d_\mathbb{L}^p : E^{p,q}_1 \to E^{p+1,q}_1$

is a homomorphism

$\mathbb{R}^k f_* (gr \mathbb{L}) \to \mathbb{R}^{k+1} f_* (gr \mathbb{L})$

that is the $k^{th}$ edge homomorphism for $0 \to gr_{\mathbb{L}} \to I^{\bullet,k} f_* L^1_X \to gr_{\mathbb{L}} \to 0$.

In particular $\forall k$ the Gauss-Manin operator $d_{\mathbb{L}}^k$

operator $d_{\mathbb{L}}^k$ is just $d^k$

If we have a complex

$\cdots \to H^k_{\mathbb{L}} \otimes \mathbb{L}^1_{X/S} \to H^{k+1}_{\mathbb{L}} \otimes \mathbb{L}^1_{X/S} \to \cdots$

diagram complex of the spectral sequence.

If $k = 0$ this complex is $H^0_{\mathbb{L}}(X/S) \otimes \mathbb{L}^1_{X/S}$

d_0 is just usual exterior differential

$- H^0_{\mathbb{L}}(X/S)$ is trivial bundle - smooth map

is (very) locally a product...

push forward of constant sheaf

(étale) - locally constant - know how to differentiate functions upstairs...
If \( x \in S \subseteq E \downarrow_0 \) is a local section, \( \mathcal{E} \in E \downarrow_0 \), then \( \mathcal{E} E \downarrow_0 = \mathcal{E} H_q(X/S) \)

\( d_{0,k}^{1,h}(x, \mathcal{E}) = d_{0,k}(x, \mathcal{E}) + (-)^k \mathcal{E} \rightarrow d_{1,k}^{1,h}(x, \mathcal{E}) \)

\( d_{1,k}^{1,h} \) is a connection + \( d_{0,k}^{1,h} \) are recovered from \( d_{0,k} \)

by imposing the Leibniz rule.

\[ \text{But } E_{-1} \text{ is a complex } \Rightarrow \text{ curvature vanishes} \]

Remark: \( H_{\mathcal{E}}(X/S) \) is a sheaf of algebras, +

by multiplicativity of \( E \rightarrow E \rightarrow H_{\mathcal{E}}(X/S) \)

\( d_{0,k}^{1,h}(x, \mathcal{E}) \) is a function \( E \rightarrow E \mathcal{E} \)

which rewritten for \( G \)-in \( g \)

\[ \forall \mathcal{E} \in T_G \quad GM_{\mathcal{E}}(x, \mathcal{E}) = GM_{\mathcal{E}}(x, \mathcal{E}) \cdot \mathcal{E} + \mathcal{E} \cdot GM_{\mathcal{E}}(x, \mathcal{E}) \]

multiplicative

- **Each calculation of** \( d_{0,k}^{1,h} \) **Assume S is affine (everything local in S).** Choose a finite cover \( U \) of \( X \) by affine opens that are étale covers of open sets in \( A^d \), \( d = d_{0,k}^{1,h}(X/S) \), and such that

\( \mathcal{D}_\mathcal{E} \) is a free \( \Omega^d \) module for any \( \mathcal{E} \in U 

with basis \( dx^1, \ldots, dx^d \).

\( H_{\mathcal{E}}(X/S) \) is an \( \Omega^d \) module that is the \( k \)-cohomology of the total complex of the double complex

\( C^{p,q} = C^p(U, \mathcal{E}_{k,q}(X/S)) \)

We will describe a non-canoncial, non-integrable connection on \( tot(C^{p,q}) \) which

gives Gauss-Manin on passing to cohomology.

For a derivation \( \xi \in \mathcal{E}R \), \( \mathcal{E} \in U 

define \( \xi \mathcal{E} \in \mathcal{E} \text{End}_G(C^{p,q}) \) \( \text{via} \)

\[ \xi \mathcal{E} (dx^1, \ldots, dx^d) = \xi \mathcal{E} (dx^1, \ldots, dx^d) \]

\( \xi \mathcal{E} \) is an endo. of bidegree \((0,0)\).
Each calculation of Gauss-Manin
\[ f : X \to S \text{ smooth between smooth varieties. May assume affine.} \]
\[ U \text{- finite cover of } X \text{ by affine opensets, } \mathcal{C} \text{ takes over affine opensets in } A^n_S \text{. } \]
Choose a trivialization of \( \mathcal{C}_U \) over every \( U \in \mathcal{U} : dx_1^U, \ldots, dx_n^U \).
We have \( \mathcal{C}^p \equiv C^q(U, N_x \chi S) \) - realize \( C \cdot M \) on total complex.
Want correction on \( \text{tot}(C^{**}) \) inducing \( C \cdot M \) on cohomology.
Problem of ambiguity - can lift derivations on opensets, but which lift to choose over intersections?
For \( \xi \in \text{Der}(C) \) denote \( \xi_U \in \text{Der}(C_U) \), the unique lifting of \( \xi \) that kills \( dx^U_1, \ldots, dx^U_n \).
Choose a total order \( \prec \) on \( U \). For \( \xi \in \text{Der}(C) \), choose \( \xi_U \in \text{End}_C(C^{**}) \) by setting
\[ \xi_U|_{(U_0, \ldots, U_p, x^U)} = \xi_U|_{(U_0, \ldots, U_p, x^U,)} \]
where \( U_0 \) is the minimal among \( U_0, U_1, \ldots, U_p \).
\( \xi \) has bidegree \((0, 0)\) - don’t change order of diff form or order of intersections.

For each pair \( U, V \in \mathcal{U} \), have an \( O_x \)-linear morphism
\[ \lambda(\xi^{U V}) : N_{x^U_{1 V}} \to N_{x^V_{1 U}} \text{, contract by } \xi_U - \xi_V. \]
This gives an \( O_x \)-linear endomorphism \( \lambda(\xi) \) of \( C^{**} \) of bidegree \((1, -1)\) by \( \lambda(\xi)(\sigma_{U_0, \ldots, U_0}) = \]
\[ (-1)^{n_{U_0}^U} \xi_U|_{(U_0, \ldots, U_0, x^U)} \sigma_{U_0, \ldots, U_0}, \quad \sigma \in C^p(U, N_x \chi S) \]
where \( U_0 < U_1 < \ldots < U_0 \).

Define \( C \cdot M : \xi \to \text{End}_C(\text{tot } C^{**}), \xi \mapsto \partial + \lambda(\xi) \)
\( \lambda(\xi) \) also has total degree 0 & is zero part.
Diff operator part + diff 0 linear part (endomorphism)
Depends on all the choices, not integrable in general.
Everything else comes from $C^\infty$, & we have the two $S$ seqs. $\Rightarrow H^*(X, C^\infty)$ from the two filtrations.

The horizontal filtration of $\text{tot}(C^\infty)$ is called the Zariski filtration

$$E_i = \bigoplus \mathbb{C}^a_i \quad \text{vertical filtration is the Hodge filtration}$$

By examining how $G_M$ shifts the filtrations, we obtain:

1) $G_M$ is compatible with the Zariski filtration of $\text{tot}(C^\infty)$

$\Rightarrow$ acts on the assoc. $S$ $S$

$Zar \quad E_i \quad = \quad \mathbb{C}^a_i \quad \text{hor. det.} \quad \Rightarrow \quad H_{\text{hor}}^*(\mathbb{C}^a_i)$

$\Rightarrow$ $G_M$ induces a morphism on $H_{\text{hor}}^*(\mathbb{C})$

- filtered by horizontal degrees. check that

This is Gauss-Manin as before.

2) $G_M$ is not compatible with Hodge $\Rightarrow$ doesn't act on $H_{\text{hor}}^*(\mathbb{C})$

$E_i \quad = \quad R^k f_* \Omega^a_{X/S} \quad \Rightarrow \quad H_{\text{hor}}^*(X/S)$

but it only shifts Hodge at most by one $\Rightarrow$

$G_M : F_{\text{hove}} H_{\text{hor}}^*(\mathbb{C}) \rightarrow F_{\text{hove}} H_{\text{hor}}^*(\mathbb{C})$

$\Rightarrow$ Griffiths Transversality

(doesn't depend on degeneration of Hodge-deRham!

(filtration is always there, as is $G_M$, as is

transversality!)

Consider the Kodaira-Spencer map for the family $f^* X \rightarrow S$.

By definition this is the map $p_{X/S} : T_s \rightarrow R^1 f_{X/S}^*$

first edge homomorphism of the direct image of

the tangent sequence $0 \rightarrow T_{X/S} \rightarrow T_X \rightarrow f^* T_S \rightarrow 0$

$p_{X/S} \in$ the composition $T_s \rightarrow (f^* T_S) \rightarrow R^1 f_*(T_S) .

Over dual numbers this is our usual deformation map.

Let $2_{X/S} \in H^0(S, \mathcal{J}_S \otimes R^1 f_{X/S}^*)$ correspond to $p_{X/S}$.

Note that for any $s \in T_s$, the image

$p_{X/S}(s) \in H^0(R^1 f_{X/S}^*)$ into $H^*(X, T_{X/S})$ by Leray
is represented by the Čech cocycle $\xi_v - \xi_u$.

$\implies$ so degree 0 part of $\xi_v$ is just contraction with the Kodaira-Spencer class (cup product)

iii) Griffiths infinitesimal period relations:
Assume $f$ is such that $1 + dR$ is s.s. degenerate at $E$.
The associated graded of $\mathrm{CM} : \mathcal{M} \to \mathcal{M} \circ \mathcal{S}$
is an $\Omega^1$-linear homomorphism
\[ \mathrm{KS} : H^k_{\text{dR}}(X/\mathcal{S}) \to H^k_{\text{dR}}(X/\mathcal{S}) \circ \mathcal{S} \]
given by a cup product with $\mathcal{S}(\mathcal{S})$, i.e.
\[ \mathrm{KS} : R^k_{\text{dR}} \mathcal{S} \to R^{k+1}_{\text{dR}} \mathcal{S} \circ \mathcal{S} \]
$\mathcal{S}$ gives

KS map comes the same way as CM: take relative $dR$ with $0$ differential, inflate to full $dR$ with $\Omega$ differential, push downstairs, (filter first) $\implies$ s.s. degenerate, $\implies$ Dolbeault, corresponding $\mathcal{O}_X$-linear edge homomorphism is KS class

**Remark** A pair $(E, \Theta)$ of a vector bundle $E \rightarrow S$ and an $\mathcal{O}_X$-linear homomorphism $\Theta : E \rightarrow E \circ \mathcal{S}$ is called a Higgs bundle.

iii) says the local system $(H^k_{\text{dR}}, d_m)$ has a canonically associated Higgs bundle $(H^k_{\text{dR}}, \mathrm{KS})$.
Equivalently, the pushforward of the trivial local system $(\mathcal{O}_X, 0)$ is a trivial Higgs bundle $(\mathcal{O}_X, 0)$ corresponding to each other as $\mathrm{CM}$. high assoc graded!

Let $f : X \longrightarrow S$ be a smooth map between arbitrary schemes.
We define a connection in the derived category on
\[ R^\ell_X \mathcal{O}_{X/S} \otimes D^b(S) \]
This means that given a diagram
\[ S \xrightarrow{h} S' \xrightarrow{f} S \]
Let an inclusion given by a square zero $x_{i,j}$, $g_i$ retraction of $x_{i,j}$ functional isomorphism

$\mathbb{L}^2(x, \mathbb{R}^*_x) \cong \mathbb{L}^2(\mathbb{R}^*_x x_x)$

identity on $S$, with natural cocycle condition.

Since $f$ is smooth, it commutes with base change in the derived category $=>$ suffices to construct a functional isomorphism $\phi \mathbb{R}^*_x x_x \cong \mathbb{R}^*_x x_x$ where $x_i = x_i x_i$, $S \rightarrow S$, i.e. $x_i$ smooth liftings of $x_i$ to $S$.

We'll find $(*)$ in general for any two smooth liftings $x_i, x_i$, difference between crystal & stratification: crystal need isos for any lifting, stratification for liftings given by retractions (smooth case) any lifting is (locally) given by retractions ...

=> in fact we'll be constructing a crystal: build $(\ast)$ for any two liftings $x_i, x_i$.

Consider $G = \{ g \in \text{Aut}_x x_i \mid g |_{x_i} = 1 \}$

$P = \{ p \in \text{Isom}_x x_i, x_i \mid p |_{x_i} = 1 \}$

$G$ is a commutative group scheme, $P$ a right torsor over $G$.

SGA I III Prop 5.33 : $G \simeq \text{Hom}_x x_i (x_i, \mathbb{L}^1)$

$I O_{x_i}$ - ideal sheaf of the will of $x_i$.

Geometrically this says given $g : x_i \rightarrow x_i$ inducing $x_i \rightarrow x_i$ can lift uniquely to $\tilde{g} : x_i \rightarrow x_i, x_i$.

Also set natural of $\mathbb{L}^1$ over $\mathbb{L}^0$.

$\mathbb{L}^0 x_{i,j} = x_{i,j} x_{i,j}^\ast \mathbb{L}^0 x_{i,j} x_{i,j}$, $I O_{x_i} \simeq \mathbb{L}^0 x_{i,j}$

+ there are natural maps.
**Transport of structure** \( U : R \to \text{Hom}_{A}^{(n)} (\mathfrak{g}, \mathfrak{g}^\ast, \mathfrak{g}^\ast) \)

- Sheaf of algebras, these are deg \( A \) homomorphism between \( A \)-modules.

**Interior product** \( i : C \to \text{Hom}^{(n)} (\mathfrak{g}, \mathfrak{g}^\ast, \mathfrak{g}^\ast) \)

\[
i (f \wedge x') = \rho f \in \mathfrak{g}^\ast \subset (\mathfrak{g}^\ast)^\ast, \quad \Theta \subset \text{Hom} \mathfrak{g}, \mathfrak{g}^\ast
\]

where \( \Theta \in \Gamma (U, \Theta) \)

**Lie derivative along fibers**

\[
\Theta : C \to \text{Hom}^{(n)} (\mathfrak{g}, \mathfrak{g}^\ast, \mathfrak{g}^\ast)
\]

- Defined on \( \mathfrak{g}^\ast \mathfrak{g}^\ast = \mathfrak{g}^\ast \), by \( \Theta (f) = \langle \Theta, d(f^\ast) \rangle \)
- Extended by the usual rules (Leibniz)

\[
\wedge x' = \mathfrak{g}^\ast \mathfrak{g}^\ast \otimes \mathfrak{g}^\ast \mathfrak{g}^\ast
\]

- Everything

5/2

**Cartan homotopy formula**

\[
\Theta (f) = \partial f + d f \otimes i
\]

**Relations**

\[
\Theta (\partial f) = \Theta (\partial f) = \Theta (\partial f) + \Theta (\partial f)
\]

**L(U \circ \partial)**

**Since \( G \) is commutative**

**This generates a cyclic extension**

\[
0 \to G \to \mathbb{C} \to \mathbb{Z} \to 0
\]

**We can add \( G \)-torsors**

**Cyclic group generated by \( i \), gives extension \( \mathbb{Z} \) pull this back to \( \mathbb{Z} \).**

**Let**

\[
\tilde{H} = \text{Hom}_{\mathfrak{g}^\ast} (\mathfrak{g}^\ast, \mathfrak{g}^\ast)
\]

**and define a complex**

\[
0 \to L^{-1} \to L^0 \to 0 \ldots
\]
This complex is quasi-isomorphic to $\mathbb{Z}$.

There's a morphism $\varphi : \mathbb{Z} \to H^*$

\[
\varphi^{(-1)} = 1, \quad \varphi^{(0)} \rho = u, \quad \varphi^{(0)} \delta = 0 \quad \Rightarrow \text{determinacy}
\]

In the derived category of abelian sheaves on the topological space underlying $X$, we have a morphism

\[ \mathbb{Z} \xrightarrow{\varphi} L^* \]

\[ \Rightarrow \text{get} \quad \mathbb{Z} \xrightarrow{\varphi} L^* \xrightarrow{\eta} H^* \quad \text{or equivalently an element} \quad \psi \in \mathcal{R}^0 \Gamma_X(H^*) \]

Composing with the canonical morphism $H^* \to \mathbb{R} \text{Hom}^g_{\text{Galois}}(L^{\sigma}, L^{\sigma'}_{\text{Galois}})$

\[ \Rightarrow \text{get element in} \quad \mathcal{R}^0 \text{Hom}_{\text{Galois}}(L^{\sigma}, L^{\sigma'}_{\text{Galois}}) \quad \Rightarrow \mathcal{E} \]

$D(F^r(q))$, derived category of sheaves of $F^r(q)$ algebra

The element $\mathcal{E}$ is an isomorphism because it gives us a transitive system of morphisms between different liftings $L^\sigma, L^{\sigma'}$ of $L^\sigma$. It also specializes to the identity if $X_1 = X_2$ (degeneracy datum...)

\[ \Rightarrow \text{applying } \mathcal{R} F^r \text{ get an isomorphism } \mathcal{R} F^r \mathcal{E} \mapsto \mathcal{R} F^r(L^{\sigma}, L^{\sigma'}_{\text{Galois}}) \]

Application: Theorem (Deligne) Let $f : X \to S$ be a smooth projective morphism between smooth varieties. Then the topological Leray spectral sequence

\[ E^{pq}_2 = \mathbb{H}^p(S, R^q f_\ast \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}) \]

degenerates at $E_2$.

Proof (Sketch) $f$ is projective $\Rightarrow f$ a globally defined bundle $H \to X$ that is relatively ample (f-analytic)

Consider the first relative Chern class of $H$

\[ y \in \Gamma(S, R^1 f_\ast (\mathcal{F} \otimes H^{\otimes S})) \]

\[ y(s) = c, (H^{\otimes S}) \]

Notice that, since $\delta(S)$ is an integral class for any $s$

\[ \Rightarrow y \in \Gamma(S, R^2 f_\ast \mathcal{F} \otimes \mathbb{Z}) \subset \Gamma(S, R^0 f_\ast \mathcal{F}(df_s, \mathcal{F})) \]

So $\varphi$ is flat w.r.t. $G_M$, $\delta_M(\varphi) = 0$
Algebraically this can be seen by looking at \( \mathcal{C}(W) \in H^2(X, \mathcal{E}) \).

Since \( \mathcal{L}_x^* \) is a resolution of \( \mathcal{E} \), \( H^2(x, \mathcal{E}) \cong H^2(\mathcal{L}^*_x) \).

There is an isomorphism \( \mathcal{L}^*_x \to \mathcal{L}^*_x \mathcal{E} \) induced by the canonical quotient map \( \mathcal{L}^*_x \to \mathcal{L}^*_x \mathcal{E} \).

The Leray s.s. gives a map \( H^0(X, \mathcal{L}^*_x \mathcal{E}) \to H^0(\mathbb{C}^2, \mathcal{L}^*_x \mathcal{E}) \).

This is identified with the image of \( \mathcal{E}, \mathcal{E} \), under \( H^2(X, \mathcal{E}) \cong H^2(\mathcal{L}^*_x) \to H^2(\mathcal{L}^*_x \mathcal{E}) \to H^2(\mathbb{C}^2, \mathcal{L}^*_x \mathcal{E}) \).

which is the relative de Rham.

From the definition of \( d_{\text{can}} \) as edge homomorphism for the pushforward of \( 0 \to \mathcal{L}^*_x \rightarrow \mathcal{L}^*_x \mathcal{E} \rightarrow \mathcal{L}^*_x \mathcal{E} \rightarrow 0 \), we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}(W) & \to & H^2(\mathcal{L}^*_x) \\
\downarrow & & \downarrow \\
H^1(\mathbb{C}^2, \mathcal{L}^*_x) & \to & H^2(\mathcal{L}^*_x \mathcal{E})
\end{array}
\]

But \( \mathcal{L}^*_x \to \mathcal{L}^*_x \mathcal{E} \) factors through \( \mathcal{L}^*_x / \mathcal{E} \) so we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}(W) & \to & H^2(\mathcal{L}^*_x) \\
\downarrow & & \downarrow \\
H^1(\mathbb{C}^2, \mathcal{L}^*_x) & \to & H^2(\mathcal{L}^*_x \mathcal{E})
\end{array}
\]

So the variation of Hodge structure is polarized: we have flat Kähler form \( \nabla \) moving with the variety ...

\( \Rightarrow \) do Lefschetz: get the triangle:

\(
\begin{array}{ccc}
\& & \\mathfrak{h}_2 \\
\downarrow & \searrow \downarrow & \SEarrow \\
\& & \mathfrak{h}_2 \\
\end{array}
\)

Lefschetz decomposition of the fibers of \( H^2(W, \mathcal{E}) \) which is horizontal w.r.t \( \mathfrak{h} \).

\( \Rightarrow \) the Lefschetz operators \( L^k \cdot = \gamma^k \cdot : \mathbb{R}^2 \to \mathbb{R}^{2+k} \) are compatible with the differentials of the Leray spectral sequence - (see Froh Weyl)

\( \Rightarrow \) de Rham \( \Rightarrow \) Bott

\( \Rightarrow \)

1) Stiffness to check the vanishing of the differentials of Leray on the primitive cohomology (incld the fibers.)
2) If \( d = \dim X / S \Rightarrow \text{we have } H^p(\mathbb{R}^d, \mathfrak{c}) \cong H^p(\mathbb{R}^{d-2}, \mathfrak{c}) \),
\[ H^p(\mathbb{R}^d, \mathfrak{c}) \xrightarrow{d_2} H^{p+2}(\mathbb{R}^{d-2}, \mathfrak{c}) \]

By Hard Lefschetz, top \( L^{d-2} = 0 \) and the bottom is an isomorphism \( \Rightarrow d_2 \) on the right must be zero. Combined with (i) \( \Rightarrow c_2 = 0 \),
+ induct for higher differentials, same argument.

---

**Cycle maps**

\( \mathfrak{c} \): smooth, irreducible, codimensional \( p \)

\( Z^p(x) \): group of cycles of codimension \( p \)

Free abelian gen. by all irreducible subvarieties.

(i) The classical/topological cycle map \( \gamma^p \):

There's a natural map \( \gamma^p : Z^p(x) \to H_{Betti}^{2p}(x, \mathfrak{c}) \)

by Poincaré duality - consider \( Z \to X \) smooth, codimension \( p \) subvariety. defines functional on \( H^{2p-d}(x, \mathfrak{c}) \to H_{Betti}^{2p}(x, \mathfrak{c}) \).

For general, irreducible \( Z \), take a resolution of singularities \( \mathring{Z} \to Z \to X \), and integrate the pull back of forms to \( \mathring{Z} \). Any two resolutions will differ by a set of measure zero, ... affected by linearity.

Betti natural place to compare topological/analytic properties.

Consider \( H^p_{\mathbb{R}}(x, \mathfrak{c}) \subset H^{2p}_{\mathbb{R}}(x, \mathfrak{c}) \)

\( \forall Z \) codim \( p \Rightarrow \partial \) is of dimension \( d-p \)

\( \Rightarrow \) a \( C^\infty \) form \( \omega \) of degree \( d-2p \) will be non-zero restricted on \( Z \) if it has type \( (d-p, d-p) \) after restriction.
\[ Y_{top}(Z) \leq H_{Z_p}^p(X) \]

\( Y_{top} \) can be generalized to a map with values in a Tate twist of \( H_{Betti} \). Tate twist is a device to keep track of the way multiplicative structure on \( H^* \) interacts with possible embeddings of the abelian group of coefficients in \( \mathbb{C} \).

Let \( A \subset C \) subring \( (\mathbb{Z}, \mathbb{Q}, \mathbb{R} \text{ usually}) \).

If introduce objects \( A(n) \in C \), abelian groups abstractly isomorphic to the additive group of \( A \), \& when used as coeff they remember weight of a cohomology class - in terms of primitive generators - the ns add up.

Intrinsic way to write it is put \( \mathbb{Z}(1) \approx H_2(\text{IP}^1) \) (noncanonically isomorphic to \( H_0(\text{IP}^1) \)). If we know what this is in our cohomology theory. Set \( A(n) = \mathbb{Z} \otimes A \n A(n) = A(1)^{\otimes n} = \mathbb{Z}(1) \otimes^\oplus A \).

Explicitly, define \( A(n) = (2\pi i)^n A \in C \).

The coeff \( (2\pi i) \) in \( \mathbb{Z}(1) \) comes from the fact that the fundamental class \( [\text{IP}^1] \) sets identified \( \mathbb{Z}(1) \).

Under the cycle class map \( C(\mathbb{C}(1)) \).

Def: The \( n^{th} \) Tate twist \( \wedge^n \) of the Betti cohomology with coeff in \( A \) is \( H^*(X, A) \otimes A(n) = H_{Betti}^*(X, A(n)) \).

\[ Y: H_{Betti}^p(X) \rightarrow H_{Z_p}^p(X, \mathbb{Z}) \]

\( Y \approx (2\pi i)^p Y_{top} \).

Here \( H_{Z_p}^p(X) \) is the intersection of \( H_{Z_p}^p \) \& \( H_{Z_p}^p(X, \mathbb{Z}(p)) \).

\( Y \) takes into account how deeply a cohomology class runs with twists coming from primes in cohomology.

\( H_{Z_p}^p(X) \subset \mathrm{subgroup} \) of deRham \& integral cohomology.

I would like to lift our map \( Y \) to more refined maps to \( H_{Betti}^*, H_{Z_p}^*(X, \mathbb{Z}) \).

The natural place where the fundamental class of \( \mathbb{Z} \) lives is in local cohomology with support in \( \mathbb{Z} \).
Let $X$ be a smooth complex variety, $Z \subset X$ irreducible, and $\text{codim } Z = p$. For a complex $K$ of sheaves of abelian groups on $X$, denote $H^i_Z(X, K^*) = H^i(\Gamma_Z(X, K^*))$ local hypercoho of $X$ supported in $Z$. Let $\Gamma_Z(X, \_)$ be the category of sheaves. Remark: $\Gamma_Z(X, F) \to \Gamma(X, F)$ represents the functor $\Gamma \circ \Gamma_Z$. For $i \geq 0$, $H^i_Z(X, K^*)$ is computed via a injective resolution of $\mathbb{Z}$-complex $K^* \to I^{\bullet,*}$. Important properties:

1. **Gysin sequence** for $U = X \setminus Z$: $\cdots \to H^i_Z(X, K^*) \to H^i(U, K^\bullet_U) \to H^i(U, K^\bullet_U) \to \cdots$
2. **Grothendieck vanishing**: If $F \to X$ is a free sheaf, $j < p \implies H^j_Z(X, F) = 0$.
3. **Excision**: If $V \subset X$ open s.t. $Z \subset V \implies H^i_Z(X, K^*) \to H^i_Z(U, K^\bullet_U)$. (ii) **Splitting**: $K^*$ is a complex of locally free sheaves with $K^i = 0$ for $i < p \implies H^i_Z(X, K^*) = 0$ for $j < 2p$, and $\cdots \to H^{2p}_Z(X, K^*) \to H^p_Z(X, K^p)$. Remark: Gysin follows from observation that for a flasque sheaf $F$, $U \to \Gamma_Z(X, F) \to \Gamma(X, F) \to \Gamma(U, F) \to 0$ is exact. Vanishing is proven by Čech covering, counting the face. Excision comes easily, e.g. from Gysin. Splitting: Filter $K^*$ by good filtration $K^* = K^{\geq 0} \to K^{\geq 1}$. Look at associated s.s.s. $E^{i,j}_k = H^j_Z(X, k^i) \to H^j_Z(X, k^{i+1})$. Now $H^j_Z(X, k^i) = 0$ for $i < p$ by Grothendieck vanishing. $H^j_Z(X, k^i) = 0$ for $i > p$ by assumption. $\Rightarrow E^{i,j}_k = 0$ for $i + j < 2p$. Also $E^{i,j}_k$ relates into $E^{i,j+1}_k$ for $i + j = 2p$.
By Lefschetz duality, $H^{2p}(X, \mathbb{Z}) \cong H_{2d-2p}(X, \mathbb{Z})$

$\Rightarrow H^{2p}(X, \mathbb{Z}(p)) \cong \mathbb{Z}(p)$, generator is fundamental class $g(Z) \in H^{2p}(X, \mathbb{Z}(p))$, and $g(Z)$

maps to $g(Z) \in H^{2p}(X, \mathbb{C})$

Define $Y_g(Z) < H^{2p}(X, \mathbb{Z}(p))$ as the image of $g(Z)$

$Y_g(Z)$ projects on $Y(Z)$.

(iii) The cycle map $Y_{10}$

$Z \subset X$ irreducible at generic at $Z$ is smooth $\Rightarrow$ fix an affine open $X_0 \subset X$ $\mathbb{C}$ divisors $D_1, \ldots, D_p$ in $X$ s.t. $D_0 = D_1 \cap \ldots \cap D_p = 0$ $\cap X_0$ smooth, intersect transversely and $Z^0 = \cap D_0$ (Theorem of the rank).

Look at $X_0 - Z^0$ and the affine cover

$(U_i, \mathbb{D})$,

Define $C(Z_0) \in H^{p-1}(X^0 - Z^0, \mathbb{L}^{p-2})$

as the element given by the Chech cocycle

$\tilde{h}_1, \ldots, \tilde{h}_p$ on $U_1 \cap U_2 \cap \ldots \cap U_p$ (Poincaré residue).

By Gysin, $H^{p-1}(\mathbb{L}^{p-2}) \rightarrow H^p(X_0, \mathbb{L}^{p-2})$

Let $C(X^0 - Z^0)$ be the image of $C(Z_0)$ in $H^{p-1}(\mathbb{L}^{p-2})$

We have a map (inclusion) $H^{p-2}(X, \mathbb{F} \cap \mathbb{L}^{p-2}) \rightarrow H^{p-2}(X, \mathbb{L}^{p-2})$

by splitting.

It can be checked that $C(X^0, Z^0) < H^{p-2}(X, \mathbb{F} \cap \mathbb{L}^{p-2})$

comes from a class $C_{GR}(Z) \in H^{p-2}(X, \mathbb{L}^{p-2})$

To show this project on constant, get cohomology with coefficients logarithmic forms, use Poincaré summation of residues to see this vanishes.

Fact: $C_{GR}(Z), C_g(Z)$ project to the same element in $H^{p-2}(X, \mathbb{L}^{p-2})$

This definition is motivated by splitting principle in the case of $d$-logs around a divisor $\rightarrow \frac{dz_1}{z_1} \cap \ldots \cap \frac{dz_n}{z_n}$ around intersection of divisors.
\[
\delta_{DR} : \mathbb{Z}^p(x) \rightarrow H^{2p}(x, F^p L^\infty_x) \quad \text{lifting } \delta_B
\]

\[\delta_{DR}, \delta_B \text{ motivate introduction of Deligne cohomology :} \]
\[
\begin{array}{ccc}
\mathbb{Z}^p(x) & \xrightarrow{\gamma} & H^2_{DR}(x, \mathbb{C}^\infty) \\
\delta_B & \xrightarrow{\gamma} & H^2_{DR}(x, \mathbb{C}) \\
\end{array}
\]

Natural to look at fiber product of \(H^2_{DR}(x, \mathbb{C}) \times H^1_{DR}(x, F^p L^\infty_x)\) over \(H^2_{DR}(x, \mathbb{C})\) as target of most refined cycle map, includes all \(\delta_B, \delta_D\), Deligne introduction ...

Fiber products of cohomology theories not well behaved...

e.g. will long exacts of all three \(\Rightarrow\) do it on level of complexes.

**Def:** The Deligne cohomology groups of \(X\) are the groups
\[
H^i_{DR}(x, F^p L^\infty_x \otimes_{\mathbb{Q}} \mathbb{Z}(p)) \quad (\mathbb{Z}(p) \text{ in deg } 0)
\]

Category of complexes not abelian: initially not clear...

Fiber product exists \(\Rightarrow\) also is there a cycle map to \(H^{2p}_{DR}\)

Denote \(\pi : F^p L^\infty_x \rightarrow L^\infty_x, \quad j_p : \mathbb{Z}(p) \rightarrow L^\infty_x\)

Existence of fiber product \(\Rightarrow\) take \(F^p L^\infty_x \otimes \mathbb{Z}(p)\)

and take kernel of difference map \(j_p \otimes 1\). Kernels do exist in category of complexes.

Rather let's try to find a simpler quasi-isomorphic complex ...

To understand kernel \(\rightarrow\) core construction:

\[f : A^* \rightarrow B^* \quad \text{morphism of complexes} \quad \text{(in an abelian category)} \quad \Rightarrow \quad \text{canonical complex } \quad \text{the core of }\]
\[C^*_f = A^*[1] \otimes B^* \quad \text{as a graded object, with differential}\]
\[d^*_f = \begin{pmatrix}
A & 0 \\
\text{shift} & B
\end{pmatrix},
\]

\[d = \begin{pmatrix}
0 & 0 \\
\text{shift} & B
\end{pmatrix}
\]

\[\Rightarrow \text{short exact } 
0 \rightarrow B^* \rightarrow C^*_f \rightarrow A^*[1] \rightarrow 0
\]
If \( f: A \to B \) surjective (on cohomology eva.)

\[ K = \ker (f^*: B^* \to A^*) = C_f \cdot \mathcal{L}_{-1} \]

Now define \( F^0 \mathcal{L}_x^* \times \mathcal{L}_x^* \mathbb{Z}(p) := \ker (F^0 \mathcal{L}_x^* \mathbb{Z}(p)) \to \mathcal{L}_x^* \mathbb{Z}(p) \cdot C_f \cdot \mathcal{L}_{-1} \]

definition of \( H^{*0}(x, \mathbb{Z}(p)) := H^*(C_{\mathcal{L}_{-1}}) \)

\( F^{*0} \mathcal{L}_x^* = 0 \to \ldots \to 0 \to \mathcal{L}_x^* \to \mathcal{L}_x'^* \to \ldots \)

\( Z(p) := \mathcal{L}(p) \to 0 \to 0 \to \ldots \)

one \( [1] \) : \( \mathcal{L}(p) \to C_x \to \mathcal{L}_x^* \to \ldots \to \mathcal{L}_x^{n-1} \to \mathcal{L}_x^n \to \mathcal{L}_x^n \mathcal{L}_x^{n-1} \to \ldots \)

Consider the complex \( \mathcal{L}(p) : \mathcal{L}(p) \to C_x \to \ldots \to \mathcal{L}_x^{n-1} \to 0 \to 0 \ldots \)

there's an inclusion \( \mathcal{L}(p) \to C_{\mathcal{L}_{-1}} \)

which is quasi-isomorphism \( \text{(check k)} \)

\[ H^k_d(x, \mathcal{L}(p)) = H^k(x, \mathcal{L}(p)) \]

**Examples**

1. \( n=0 \) : \( \mathcal{L}(0) = \mathbb{Z} \), \( H^k_d(x, \mathbb{Z}(0)) = H^k_B(x, \mathbb{Z}) \)

2. \( n=1 \) \( \mathcal{L}(1) = (\mathcal{L}(i) \to C_x) = \text{kernel of exponential map :} \)

\[ 0 \to \mathcal{L}(i) \to C_x \to \mathcal{L}_x^* \to 0 \]

\( \mathcal{L}(1) \to C_x \to \mathcal{L}_x^* \cdot [-1] \)

quasi-isomorphism \( \text{ quasi-isomorphism.} \)

\[ H^k_d(x, \mathcal{L}(1)) = H^{k-1}(x, \mathcal{L}_x^*) \]

\[ H^0_d(x, \mathcal{L}(1)) = \text{Pic } C_x \]

3. \( n=2 \) \( \mathcal{L}(2) = [\mathcal{L}(2) \to C_x \to \mathcal{L}_x^*] \)

\[ L \left( \exp(\mathcal{L}(2)) \right) \]

\[ [0 \to \mathcal{L}_x^* \to \mathcal{L}_x^* \to \mathcal{L}_x^* \to 0] \]

\[ L(2): \mathcal{L}_x^* \otimes \mathbb{Z}(2) \]

Replace \( \mathcal{L}_x^* \) by \( \mathcal{L}_x(1) \) since \( \exp \) kills \( \mathcal{L}(1) \) not \( \mathcal{L}(2) \).
$\mathbb{Z}(2) \rightarrow 0$

So $\mathbb{Z}(2) \otimes [C^\infty_x \rightarrow \mathbb{Q}_x] \rightarrow [-1]$

$C^\infty_x \rightarrow C^\infty_x$

$\Omega^1 \rightarrow \Omega^1$

$\text{dlog} \rightarrow \text{dlog}$

$H^k_D(x, \mathbb{Z}(2)) = H^{k-1}(x, C^\infty_x \rightarrow \log N^x)$

$H^2_c(x, \mathbb{Z}(2))$ is represented by Cech cocycles

$[g_{\mu\nu}, \{x_\delta\}] \in Z^2(U, C^\infty_x) \otimes \Omega^1(U, \Omega^1_x) \rightarrow \text{chain}$

satisfying $\text{dlog} g_{\mu\nu} = x_\nu - x_\mu$.

$\Rightarrow \{g_{\mu\nu}\}$ is a hol line bundle $L \rightarrow \{x_\mu\}$ holomorphic connection on $L \rightarrow \text{hol}$.

i.e. $H^2_D(x, \mathbb{Z}(2))$ line bundles with hol connections

$X$ smooth projective, $i_p : F^pL \rightarrow L$ induces

$l_p : H^{2p}(x, F^pL^*) \rightarrow H^{2p}_D(x)$ with image $F^pH^{2p}(x)$

(by degeneration of Hodge-de Rham).

The short exact sequence

$0 \rightarrow H^p_\mathbb{Z}(x, L^*) \rightarrow C^p_{i_p, p} \rightarrow F^pL^* \otimes \mathbb{Z}(p) \otimes \mathbb{Z} \rightarrow 0$

induces long exact in cohomology

$\ldots \rightarrow H^*_D (x, \mathbb{Z}(p)) \rightarrow H^*_D (x, \mathbb{Z}(p)) \otimes H^*_t(1 - L^*) \rightarrow H^*_D (x) \rightarrow \ldots$

$k = 2p$

$0 \rightarrow \frac{F^pH^p\otimes \mathbb{Z}(p)}{H^p_\mathbb{Z}(x, L^*)} \rightarrow H^{2p}_D(x, \mathbb{Z}(p)) \rightarrow H^{2p}_D(x, \mathbb{Z}(p)) \otimes \frac{F^pH^p}{H^p_\mathbb{Z}(x, L^*)} \rightarrow 0$

$\Rightarrow \text{def}$ $F^p(x)$ is first term $\mathcal{S}$ above is the $p^\text{th}$ Griffiths intermediate Jacobian.

The cohomol: $H^{2p}(\mathbb{Z}(p)) \cong j^{-1}F^pH^p_{\mathbb{Z}(p)}$

$H^2_\mathbb{Z}(Z) \otimes j^{-1}F^pH^p_{\mathbb{Z}(p)} \otimes j^{-1}F^pH^p_{\mathbb{Z}(p)}$

since $H^2_\mathbb{Z}(Z)$ is real

$= H^{2p}_\mathbb{Z}(Z) \otimes j^{-1}H^p, (p)$ integral $(p, p)$ classes $H^{2p}_\mathbb{Z}$.

So $0 \rightarrow J^p(x) \rightarrow H^{2p}_D(\mathbb{Z}(p)) \rightarrow H^{2p}_D(\mathbb{Z} \otimes (p)) \rightarrow 0$

Disconnected group, connected component is tors $J^p(x)$.
The Deligne cycle map \( \delta_D \)

Let \( X \) be a smooth irreducible variety of dimension \( d \), and \( Z \subset X \) a smooth irreducible subvariety of codimension \( p \).

The short exact sequence of the core \( G_{d-p} \) induces the exact sequence of local intersection cohomology with supports on \( Z \):

\[
0 \rightarrow H^{d-p}_Z(X, \mathbb{C}) \rightarrow H^{2p}_Z(X, \mathbb{Z}(p)) \rightarrow H^{2p}_Z(Z, \mathbb{Z}(p)) \rightarrow 0.
\]

We have \( \delta_G(Z) \subset H^{2p}_Z(Z, \mathbb{Z}(p)) \), \( \delta_G(Z) \subset H^{2p}_Z(F^p T^*_X) \).

We want to find an element in \( H^{2p}_Z(X, \mathbb{C}) \) mapping to \( \delta_G(Z) \).

By Grothendieck's vanishing, \( H^{2p-1}_Z(X, \mathbb{C}) = 0 \).

2p \leq d \), \( d \) is the real codimension of \( Z \).

But we know that \( \delta_G(Z), \delta_G(Z) \) both map (under \( j^*, i^* \)) to some topological class.

\( \exists \gamma \) such that \( \delta_G(Z) \subset H^{2p}(X, \mathbb{Z}(p)) \) mapping to \( \delta_G(Z) \). 

Set \( \delta_D(Z) \) is the image of \( \delta_G(Z) \) in \( H^{2p}_D(X, \mathbb{Z}(p)) \) via the natural map in the Gysin sequence.

**Multiplicative structure on Deligne cohomology**

- generalizes cohomology with two indices, like Chow group,
- motivic cohomology, Voevodsky motives & K-theories...
There's a multiplication of Deligne complexes \( \mathbb{Z}(L) \times \mathbb{Z}(M) \rightarrow \mathbb{Z}(L \otimes M) \) given by
\[
x \cdot y = \begin{cases} x \cdot y & \text{deg } x = 0, \text{deg } y > 0 \\ 0 & \text{otherwise} \end{cases}
\]

Remark: Beilinson gave construction (Regulators paper) of \( \mathcal{E} \) by taking at graded complex, direct sum of all Deligne complexes \( \mathcal{E} \)
complex, \( \mathbb{Z}(a) \) both have natural products \((a, b) \)
try to construct a product on \( \otimes \mathbb{Z}(a) \) compatible
with these - one number of ambiguity: fiber
product of the two bilinear forms has rescaling at each
ambiguity \( \delta \) for \( d \mathcal{E} \).
He proved they're all homotopic, & each is homotopy
associative commutative & shows it agrees with \( \mathcal{E}(L) \rightarrow
\)
\[ \mathcal{E}(L) \rightarrow \mathcal{E}(L) \Rightarrow H^k(\mathbb{Z}(n)) \otimes H^k(\mathbb{Z}(n)) \rightarrow H^{k+k}(\mathbb{Z}(n+n)) \]

The cycle map is functorial wrt morphisms of
varieties, it's multiplicative on intersections,
& descends to (hanging complexes on difference
of rational equivalence), & restricts to
A-J map on things homologically equiv to zero.
Integrate over chain where this base!

Characteristic Classes

(Regulators) (unpublished)

Weil algebras \( X \) smooth (algebraic or analytic) variety
\( \Omega^c_x \) - Kähler differentials
\( P_x \) - category of all \( \Omega^c_x \)-extensions.
Ob \( P_x \) are short exacts of vector bundles on \( X \), object \( P \) is
\( 0 \rightarrow \Omega^c_x \rightarrow \widehat{\Omega}^c(P) \rightarrow M(P) \rightarrow 0 \)
\( \widetilde{\Omega}^c(P) \), \( M(P) \) bundle.

Observe that if \( \pi : X \rightarrow Y \) is a morphism of smooth
varieties \( \rightarrow \) can pull back \( \Omega^c \)-extensions \( \mathcal{E} \)
\( \pi^* P : 0 \rightarrow \Omega^c_x \rightarrow \Omega^c_x (\pi^* P) \rightarrow \pi^* M(P) \rightarrow 0 \)
where \( \Omega^c_x (\pi^* P) \) is push-out of
\( 0 \rightarrow \Omega^c_x \rightarrow \Omega^c_x (\pi^* P) \rightarrow \pi^* M(P) \rightarrow 0 \)

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via the differential map \( d_{\Omega^*} : \Omega^* \Omega^f \rightarrow \Omega^1 \).

Equivalently, we can say \( P \rightarrow \Omega^f \) is a fibered category.

Given \( P \in \text{Ob} (\Omega^f) \), can construct \( \Omega^* (P) \), the sheaf of commutative graded derivations generated by:

- a subalgebra \( \Omega^1 \) in degree 0
- the \( \Omega^1 \) module \( \Omega^* (P) \) in degree 1
- the \( \Omega^1 \) module \( \Omega^* (P) \) in degree 2

with the only relation: \( \forall f \in \Omega^1 \), the differential of \( f \) in \( \Omega^* (P) \) should coincide with the usual exterior derivative \( df \in \Omega^1 \).

This forces the differential \( \Omega^* (P) \rightarrow \Omega^1 (P) \) to be the map in the exact sequence \( 0 \rightarrow \Omega^1 (P) \rightarrow \Omega^* (P) \rightarrow M \rightarrow 0 \) since \( \Omega^2 (P) = 0 \).

Examples 1. Let \( P_0 \) be the trivial \( \Omega^* \) extension:

\[
P_0 : 0 \rightarrow \Omega^0 \rightarrow \Omega^1 \rightarrow P_0 \rightarrow 0
\]

\[
\Rightarrow \Omega^* (P_0) = \Omega^0 \quad - \quad \text{the de Rham algebra}
\]

(Exterior algebra with single relation \( d(f) = df \).

\[
F^i \Omega^1 = \Omega^{i+1} \quad - \quad \text{the good filtration}
\]

\( \Omega^* (P_0) \) is called contractible...

Notice that \( P \in \text{Ob} (\Omega^f) \) has a unique map \( P \rightarrow P_0 \)

\( \Rightarrow P_0 \) is the universal initial object in \( \Omega^f \),

\( \Rightarrow \) there is a morphism of filtered derivations \( \Omega^* \rightarrow \Omega^* (P) \).

Equivalently \( \Omega^* (P) \) is an algebra over the de Rham algebra \( \Omega^* \).
2. \( X \) is a point: \( L_X^0 > 0 \), any extension \( \mathcal{P}_0 \mathcal{O}(\mathcal{P}_1) \)
reduces to a vector space \( \mathcal{M} = \mathcal{M}^0 \).
\( 0 \rightarrow \mathcal{O} \rightarrow \mathcal{M} \rightarrow M \rightarrow 0 \)

By definition \( \hat{\mathcal{L}}^0(M) = \mathcal{O} \), \( \hat{\mathcal{L}}^1(M) = M \),
\( d : \mathcal{O} \rightarrow \mathcal{M} \) is 0.

As a graded commutative algebra \( \hat{\mathcal{L}}^*(M) \)
will be freely generated by two copies of \( M \)
\( M^{(0)} = M \), \( M^{(1)} = M \) in degs 1, 2.

**Theorem**

\[
\hat{\mathcal{L}}^1(P) = \bigoplus_{a+b = 1} \wedge^a M \otimes \wedge^b M
\]

The differential on \( \hat{\mathcal{L}}^*(M) \) is determined by the rule
\[
\begin{align*}
\hat{\mathcal{L}}^1(M) & \rightarrow \hat{\mathcal{L}}^0(M) \\
\mathcal{M}^{(1)} & \rightarrow \mathcal{M}^{(0)} \otimes \mathcal{M}^{(2)} \\
0 & \rightarrow (0, 0)
\end{align*}
\]

Explicitly, get the usual Koszul differential
\[
d(m_1, \ldots, m_a \otimes n_1, \ldots, m_b) :=
= \sum_{k=1}^{a+b} (-1)^k m_1, \ldots, \hat{m}_k, \ldots, m_a \otimes m_k n_1, \ldots, n_b
\]

**Definition** The Weil algebra of \( \mathcal{P} \) is the differential algebra
\( \hat{\mathcal{L}}^*(\mathcal{P}) \).

**Properties**

(i) If \( \pi : X \rightarrow Y \) morphism of smooth varieties,
\( \forall P \in \mathcal{O}(\pi(Y)) \Rightarrow \hat{\mathcal{L}}^*(\pi^*P) = \pi^* \mathcal{L}^* \hat{\mathcal{L}}^*(P) \)
where \( \pi^* \) is sheaf theoretic
inverse images (differentials aren't \( \mathcal{O} \)-linear).

(ii) Let \( P \in \mathcal{O}_X \), then the complex
\( F^1 \hat{\mathcal{L}}^*(P) / F^2 \hat{\mathcal{L}}^*(P) \) coincides with \( \hat{\mathcal{L}}^1(P) \rightarrow \mathcal{M}(P) \)

- \( P \) is an exact sequence of vector bundles
  - locally split \( \Rightarrow \) locally \( P \) is pullback from a point
  so we're done by (i) and example 2.
Denote by $S^*(F_1/F_2)$ the graded commutative algebra generated by the complex $F_1/F_2$, i.e.,

$$S^*(F_1/F_2) = \bigoplus S^i(F_1/F_2), \quad S^i(F_1/F_2)$$

is the Koszul complex

$$\ldots \rightarrow \mathcal{I}^i(r) \otimes S^{i-1} M(p) \rightarrow S^i M(p)$$

in degree $i, i+1, \ldots, 2i$.

Thus $S^*(F_1/F_2)$ looks like:

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots$$

$$S^0 \rightarrow \mathcal{I}(r) \rightarrow M(p)$$

$$S^1 \rightarrow \mathcal{I}^0(r) \otimes M(p) \rightarrow S^2 M(p)$$

etc.

Property (iii)  The natural map $S^*(F_1/F_2) \rightarrow Gr_F(\mathcal{I}^\infty(x))$ is an isomorphism of graded commutative algebras.

(Trivialize locally and calculate for a point...)

$S^*(F_1/F_2)$ much simpler - e.g. no differentials $\mathcal{I}^i \rightarrow \mathcal{I}^{i+1}$ etc.

(iv) The canonical morphism $\mathcal{I}^0 \rightarrow \mathcal{I}^0(r)$ is a quasi-isomorphism - follows from (iii) and the fact that the sequence $0 \rightarrow \mathcal{I}^i(r) \rightarrow S^i M(p) \rightarrow \ldots$ is exact (symmetric power of short exact).

Let $G$ be a complex algebraic group, $\mathfrak{g} = lie(G)$.


The dual of the Atiyah sequence of $E$ is an $\mathfrak{g}X$-extension

$$p^\vee : E^\vee \rightarrow E \rightarrow \mathcal{I}^1_X, \quad \otimes E \rightarrow 0$$

where $\mathcal{I}^1_X = \mathfrak{g}X(\mathfrak{x})$.

$\mathfrak{g}^\vee E = (p^\vee \mathcal{I}^1_X) = \text{coadjoint bundle associated to } E$. 
\[ \widetilde{\mathfrak{L}}_{\mathfrak{X}}, \mathfrak{E} \] - the Weil algebra of \( \mathfrak{E} \), is just \( \mathfrak{L}^\bullet(\mathfrak{E}) \),
- filtered \( \mathfrak{L}^\bullet \) algebra, quasi-isomorphic to \( \mathfrak{L}^\bullet \mathfrak{X} \)
as complex.

\[ \mathfrak{E} \to Y \] principal \( G \)-bundle \[ \Rightarrow \mathfrak{L}^\bullet_{\mathfrak{X}, \mathfrak{E}} = \mathfrak{L}^\bullet \mathfrak{X} \mathfrak{E} \mathfrak{Y} \]

**Key point:** \( \mathfrak{L}^\bullet_{\mathfrak{X}, \mathfrak{Y}} \) has a natural grading.

Notice that \( \Lambda^* \widetilde{\mathfrak{L}}_{\mathfrak{X}, \mathfrak{E}} \) has a natural differential \( d' \).
One way to see this is to notice that \( \Lambda^* \widetilde{\mathfrak{L}}_{\mathfrak{X}, \mathfrak{E}} = (\mathfrak{L}^\bullet \mathfrak{X} \mathfrak{E} \mathfrak{Y}) \) which has natural differential, exterior differential along fibers
This is de Rham complex along fibers.

Equivalently recall that Atiyah algebra \( \mathfrak{A}_X(\mathfrak{E}) \) had a natural \( \mathbb{C} \)-linear Lie bracket \( \mathbb{C}J : \Lambda^2 \mathfrak{A}_X(\mathfrak{E}) \to \mathfrak{A}_X(\mathfrak{E}) \), which after dualizing gives \( d' : \widetilde{\mathfrak{L}}_{\mathfrak{X}, \mathfrak{E}} \to \Lambda^2 \widetilde{\mathfrak{L}}_{\mathfrak{X}, \mathfrak{E}} \), extend by Leibnitz : \( \langle d'^2 = 0 \iff \text{Jacobi} \rangle \mid \)

\[ \Rightarrow \quad d' : \widetilde{\mathfrak{L}}_{\mathfrak{X}, \mathfrak{E}} \to \Lambda^2 \widetilde{\mathfrak{L}}_{\mathfrak{X}, \mathfrak{E}} = \mathfrak{L}^2 \mathfrak{L}_{\mathfrak{X}, \mathfrak{E}} \]
- define \( d'' = d - d' : \Lambda^2 \widetilde{\mathfrak{L}}_{\mathfrak{X}, \mathfrak{E}} \to \Lambda^2 \widetilde{\mathfrak{L}}_{\mathfrak{X}, \mathfrak{E}} \)
- But by construction, \( d, d' \) coincide on \( \mathfrak{L}^\bullet \mathfrak{X} \)
\[ \Rightarrow \quad d'' \text{ descends to a map} \ x : \mathfrak{E}_0 = \mathfrak{L}^\bullet \mathfrak{X} / \mathfrak{L}^1 \mathfrak{X} \to \mathfrak{L}^2 \mathfrak{L}_{\mathfrak{X}, \mathfrak{E}} \]

By property \((i')\) we know that \( \mathfrak{E}_0 / \mathfrak{L}^{1} \mathfrak{X} = [\mathfrak{L}^2 \mathfrak{X}, \mathfrak{E} \to \mathfrak{E}_0] \)
so \( \mathfrak{L}^2 \mathfrak{X} / \mathfrak{L}^{1} \mathfrak{X} = \mathfrak{E}_0 = \mathfrak{L}^2 \mathfrak{L}_{\mathfrak{X}, \mathfrak{E}} / \mathfrak{L}^1 \mathfrak{X} \)

\[ \Rightarrow \quad d''(v) \text{ mod } \mathfrak{L}^{1} \mathfrak{X} = v \text{ mod } \mathfrak{L}^{1} \mathfrak{X} \] and
\[ \mathfrak{L}^{2} \mathfrak{X} \mathfrak{E} \to \mathfrak{L}^{2} \mathfrak{L}_{\mathfrak{X}, \mathfrak{E}} \text{ is an inclusion, and it splits} \]
\( \mathfrak{L}^2 \mathfrak{X}, \mathfrak{E} \to \mathfrak{L}^2 \mathfrak{L}_{\mathfrak{X}, \mathfrak{E}} \oplus (\mathfrak{E}_0 \mathfrak{E}) \)
Consider the free commutative algebra with generators \( \tilde{\Lambda}_x, \tilde{\pi} \) in \( \deg 1 \), \( \tilde{\omega} \) in \( \deg 2 \), i.e. \( \tilde{\Lambda}_x = \tilde{\Lambda}_x \tilde{\omega} \tilde{\pi} \).

The algebra \( \Lambda_\ast \tilde{\Lambda}_x, \tilde{\omega} \otimes \text{Sym} \tilde{\omega} \).

We have a canonical map \( \tilde{\alpha} : \Lambda_\ast \otimes \tilde{\omega} \rightarrow \tilde{\Lambda}_x, \tilde{\pi} \),
\( \tilde{\alpha} |_{\tilde{\Lambda}_x, \tilde{\omega}} = \text{id}, \tilde{\alpha} |_{\text{Sym} \tilde{\omega}} = \otimes \).

Lemma: \( \tilde{\alpha} \) is an isomorphism of graded commutative algebras.

Proof: Let \( F' \) be the filtration by powers of the augmentation ideal in \( \Lambda_\ast \otimes \tilde{\omega} \), then \( \tilde{\alpha} \) is a filtered homomorphism, and \( \text{gr} \tilde{\alpha} : \text{gr} \Lambda_\ast \otimes \tilde{\omega} \rightarrow \text{gr} \tilde{\Lambda}_x, \tilde{\pi} \) is an isomorphism \( \square \).

[Note: \( \Lambda_\ast \otimes \tilde{\omega} \) has natural differential from \( d' \).

\* Set \( \tilde{\Lambda}_{a+b} = \otimes \left( \Lambda^{a-b} \tilde{\Lambda}_x \tilde{\omega} \otimes \text{Sym} \tilde{\omega} \right) \subset \tilde{\Lambda}^{a+b} \).

\( \Rightarrow \tilde{\Lambda}_x = \bigoplus_{a+b} \tilde{\Lambda}_{a+b} \tilde{\omega} \), \( F' \tilde{\Lambda}_x \tilde{\omega} = \bigoplus \tilde{\Lambda}_{a+b} \tilde{\omega} \).

\( d = d' + d'' \), \( d' : \tilde{\Lambda}_{a+b} \rightarrow \tilde{\Lambda}_{a+1} \tilde{\omega} \). (Exterior deriv. along \( \tilde{\omega} \).

\( d'' : \tilde{\Lambda}_{a+b} \rightarrow \tilde{\Lambda}_{a+b+1} \).

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Example: \( \tilde{\pi} = \tilde{\pi}^1 \), \( \tilde{\pi} \) trivial.
\( \tilde{\Lambda}_{a+b} = \Lambda_a \tilde{\omega} \otimes \text{Sym} \tilde{\omega} \) classical Weil algebra.

The horizontal differential \( d' : \Lambda^a \tilde{\omega} \otimes \text{Sym} \tilde{\omega} \rightarrow \Lambda^{a+1} \tilde{\omega} \).

- Chevalley complex (Lie algebra cohomology) with coefs in \( \text{Sym} \tilde{\omega} \).
- The vertical differential \( d'' : \Lambda^a \tilde{\omega} \otimes \text{Sym} \tilde{\omega} \rightarrow \Lambda^a \tilde{\omega} \otimes \text{Sym} \tilde{\omega} \).

Every \( Ad \)-invariant polynomial on \( \tilde{\omega} \) gives us a polynomial on any vector bundle associated to a principal \( \tilde{\omega} \)-bundle via \( Ad \)....
Vi. So we have a canonical map \( \tilde{\varphi}: (S^0 \mathcal{O}^\vee)^G \to S^0 \mathcal{O}^\vee \).

- Chern-Weil homomorphism.

Lemma. The image of \( \tilde{\varphi} \) consists of cycles wrt the differential in \( \tilde{\Omega}^\bullet \), i.e.
\[\tilde{\varphi}: (S^0 \mathcal{O}^\vee)^G \to \tilde{\Omega}^\bullet \] is a morphism of \( \text{d}g \).

Proof statement is local in \( X \). May assume \( X = \text{pt}. \) follows from previous example. \( \square \)

1) Recall the Topological Chern Classes. A topological group \( G \) has classifying space \( BG \), classifying principal \( G \)-bundles topologically: \( BG \) topological space, characterized uniquely up to homotopy by properties:
   - \( \exists \) a contractible topological space \( \Delta G \), and monomorphism \( \Delta G \hookrightarrow BG \) principal \( G \)-bundle.
   - For every \( \varepsilon \to X \) principal \( G \)-bundle, \( \exists f_\varepsilon: X \to \Delta G \) st.
     \( f_\varepsilon^* \Delta G \cong \varepsilon \).
     (\( \Delta G \) simplicial scheme, \( E \) \( G \)'s geometric realization)

Example: \( G = GL(n, \mathbb{C}) \), \( BG = G(n, \mathbb{C}^n) \), \( \Delta G \) universal bundle.\footnote{Fact: \( G \) reductive, \( H^\bullet(BG, \mathbb{C}) = (S^0 \mathcal{O}^\vee)^G \).

The Chern-Weil homomorphism corresponding to \( \varepsilon \) is the map
\[ \tilde{\varphi}_\text{top}: \quad (S^0 \mathcal{O}^\vee)^G \to H^2i(X, \mathcal{O}^\vee) \to H^2i(X, \mathcal{O}^\vee). \]

Chern-Weil Theory: if \( X \) is \( \text{C}^\infty \Rightarrow \tilde{\varphi}_\text{top} \) can be taken with values in \( H^\bullet(X, \mathcal{O}^\vee) \) and can be expressed in terms of any connection \( D \) on \( \varepsilon \) as
\[ \tilde{\varphi}_\text{top}(D) = \left[ \varphi(D) \right] \]
2) Chern classes $\chi$ algebraic, $G$ principal
$G$-bundle, we have a quasi-ism $\mathcal{L}_x \rightarrow \mathcal{L}_x$, e
$\Rightarrow \mathcal{H}^*(\mathcal{L}_x, e) \simeq \mathcal{H}^*(\mathcal{L}_x)$
and even on level of filtration $\mathcal{H}^*(F_i \mathcal{L}_x) \simeq \mathcal{H}^*(F_i \mathcal{L}_x)$ (**)

$\text{We have } (\Sigma' \text{cyc})^* \rightarrow H^3(F^* \mathcal{L}_x)$
is composition of $\psi^*$ and $\chi$.

Again, $\psi^*$ can be interpreted as pullback of forms by a classifying map $\Delta G$.

Given $G$ an algebraic group, can ask if there is a simplicial classifying space for $G$, i.e., a simplicial variety $BG$, s.t.
- $\exists \Delta G$, simplicial & contractible variety
- $\forall X$, simplicial variety, $\forall F^* \rightarrow X$, s.t.
  $H^*(X, \Delta G, p^* F^*) = H^*(X, F^*)$

$\Delta G$ a map $\Delta G \rightarrow BG$, principal $G$-bundle.
- $\forall \Delta G \rightarrow X$ principal $G$-bundle $\Rightarrow \exists \Psi : X \rightarrow \Delta G$
  s.t. $\Psi^*(\Delta G) = \Delta G$

$\Delta G$ exists: $se^d \Delta G = G^{n+1}$, $BG = G^{n+1}$ diagonal action
Also $BG \simeq BG$, $\Delta G \simeq \Delta G$

Denote $\Delta G : \Delta G \rightarrow BG$.

**Lemma** $G$ reflective, then $\Delta G \rightarrow BG$ is isomorphism
and for $j \neq 1$, $H^j(BG, \Omega^1 BG_e) = 0$

Proof: Use Koszul complex for
$\Delta G$, $BG$ affine & $\Delta G$ contractible, most cohomologies outside degree 0: $j + 1$ tim don't
forms compare with top piece = invariant pairs $\square$

**Corollary** $(\Sigma' \text{cyc})^* : H^2(BG, \mathcal{L}^* BG_e) \rightarrow H^3(\mathcal{L}^* BG_e)$
is an isomorphism, $\chi$ odd dimensional cohomologies vanish.
We discuss the selection via a connection:

\[ \text{if } D \text{ is a hol. connection on } E \to X \Rightarrow \]

\[ D : \tilde{L}^\ast x \otimes \tilde{L}^\ast x \to \tilde{L}^\ast x \text{ splits the } D^\ast \text{extension } P^\ast x \text{ or} \]

\[ \text{equivalently gives a monomorphism } P^\ast x \to P_0 \]

\[ \Rightarrow \tilde{X} \to \tilde{L}^\ast x, x \to \tilde{L}^\ast x \text{ left inverse to} \]

\[ \tilde{L}^\ast x \otimes \tilde{L}^\ast x. \]

So we have \( \tilde{D} \circ W^\ast(x) = \varphi (\tilde{X} \otimes x) = [\tilde{L}^\ast x] \text{ closed} \)

\[ \tilde{D}^{1} = \tilde{D} / \tilde{L}^\ast x, x = \varphi x \tilde{D} \to \tilde{L}^\ast x : \text{this is just curv}(17) \]

\[ \text{or rather connection with curvature } \]

3. **Rethink Chern Classes** - in cycle case we construct:

it first in local cohomology which depended on our cycle, where we had universal cycle class... so now we need

"universal" groups depending on \( x \)

\[ \{ W^\ast(x) : (\text{sing})^6 \to L^\ast x \}

\[ \tilde{L}^\ast x, x \in \text{degree zero} \]

\[ \text{take fiber product:} \]

\[ \tilde{L}^\ast x, x \in \text{degree zero} \]

\[ \text{set } U^\ast(x) = \text{cone } \tilde{L}^\ast x, x \in \text{degree zero} \]

\[ \text{Define universal } \xi \text{-choy} \text{by } H^1_u (x, x) : = H^1 (x, U^\ast(x)) \]

We have canonical maps \( \xi^\ast : U^\ast(x) \to \tilde{L}^\ast x \)

\[ \text{Also have long exact sequence} \]

\[ \text{from the sequence of the cone} \]

\[ C [-1] \to U^\ast(x) \to \tilde{L}^\ast x, x \]

\[ \text{(} C \text{ is quasiomorphic to } \tilde{L}^\ast x, x \text{ ...}) \]
and its quotient (quotient) by $G \to G \times \mathbb{Z}_p \to C^*(X)$.

$C^*(X) \to U(E) \to (\mathbb{Z} \otimes \mathbb{Q})^n \to \mathbb{G}_m \to 1$

Kill the $\mathbb{Z}_p$ on left & right so middle term doesn't change!

$\Rightarrow$ Properties of $H^j(X, \mathbb{Z}_p)$:

i) $H^j(U(E))$ is a torsion module in $X / \mathbb{Z}_p$

ii) $H^{j+1}(X, U(E)) \to H^j(X, U(E))$ is isomorphic

$U \to H^{j+1}(X, U(E)) \to H^j(X, U(E)) \to \mathbb{Z}$

G Griffith's theorem:

Griffith's torsion here sits inside torsion in Griffith's

extra integrality structure comes from Chevalley's theorem

$(\mathbb{Z} \otimes \mathbb{Q})^n \mathbb{Q} = \text{inv. polynomials } p, q \text{ s.t. } \sum_{i} w_i \otimes \alpha_i \in \mathbb{Z}$

for $\forall \alpha \in H^2(X, \mathbb{Z})$

-integrality assumption related to $X$.

iii) If $\Pi : Y \to X$ st.

$H^\ast(Y, \mathbb{Z}) \simeq H^\ast(X, \mathbb{Z})$

then $\Pi^\ast : H^\ast(U(E)) \to H^\ast(Y, U(E))$

iv) The same formulas as in the Deligne cohomology define (homotopy associative, commutative) product $\otimes U(E) \otimes U(E) \to U(E)$.

Take again $E_n = \Delta G \to B G$.

Lemma $\mathbb{E} \otimes : H^{2i}(BG, U(E)) \to H^i(BG, \mathbb{Z}_p)$

(analog of cohomology with supports where fundamental class lives...

Now consider $\hat{X}_E = \Delta G \times E / G$
Now \( \pi^*_X E_{\text{un}} = \pi^*_X E \), also \( \pi^*_X : H^\bullet(\mathbb{B}E, \mathbb{Z}) \to H^\bullet(\mathbb{X}E, \mathbb{Z}) \) is an isomorphism because fibers of \( \pi_X \) are \( \mathbb{B}E \), contractible.

\[ \Rightarrow \text{by iii) } \pi^*_X : H^1(\mathbb{C}, \mathbb{Z}) \cong H^1(\mathbb{X}E, U_{\mathbb{B}E}, E_{\text{un}}(\mathbb{C})) \]

Define \( V_{\mathbb{E}, \mathbb{C}} : H^2(\mathbb{B}E, \mathbb{Z}) \to H^2(\mathbb{X}E, \mathbb{Z}) \) as the composition \( H^2(\mathbb{B}E, \mathbb{Z}) \overset{\cong}{\to} H^2(\mathbb{B}E, U_{\mathbb{B}E}, E_{\text{un}}(\mathbb{C})) \overset{\pi^*_X}{\to} H^2(\mathbb{X}E, U_{\mathbb{B}E}, E_{\text{un}}(\mathbb{C})) \).

Notice that \( E_{\mathbb{Z}} \circ V_{\mathbb{E}, \mathbb{C}} = V_{\mathbb{E}, \mathbb{C}, \mathbb{Z}}(\mathbb{C}) \) specializes well.

3) Deligne-Chern classes: \( H^0(\mathbb{X}E, \mathbb{Z}) \to H^0(\mathbb{X}E, \mathbb{Z}) \) induces

\[ \mathbb{D}(\mathbb{C})_X \to \mathbb{D}(\mathbb{C})_{\mathbb{X}E} = \text{core}(\mathbb{Z}(\mathbb{C}) \otimes F^i_{\mathbb{X}E} \to \tilde{\mathbb{R}}_{\mathbb{X}E}) \]

But \( V_{\mathbb{E}} : (S^1)^{[2]} \to F^i_{\mathbb{X}E} \Rightarrow \exists! \text{ map } U_{\mathbb{E}}(\mathbb{C}) \to \mathbb{D}(\mathbb{C})_{\mathbb{X}E} \text{ which is identity on } \mathbb{Z}(\mathbb{C}) \text{ and on } \mathbb{R}^i_{\mathbb{X}E} \text{ and is } V_{\mathbb{E}} \text{ on } (S^1)^{[2]} \].

\[ \Rightarrow \text{Chern classes in Deligne cohomology.} \]