1. Review of spectral construction: basic objects of mathematics, duality: e.g. Fourier transform on locally compact abelian groups.

Spectral construction: particular matrix model of a spectral operator on vector space described using its spectrum.

V fid. vector space \( V \) \( \phi: V \to V \) endomorphism

\( \phi \) generic [diagonalizable] \( \Rightarrow \) can specify \( \phi \) by giving:
- \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) eigenvalues
- \( L \subseteq V \) eigenspaces
- matching of \( \lambda_i \) with \( L_i \).

Spectral covers: let \( \phi \) vary in families: \( S \) parameter space, \( \phi_s \) \( E \to \mathbb{C} \) family of endomorphism, depending on \( s \in S \).

Repeating the construction for each \( s \) get covering space \( \tilde{S} = S \times C \), \( \tilde{S} = \{(s, \lambda) : \lambda \) eigenvalue of \( \phi_s \} \)

If all \( \phi_s \) have distinct eigenvalues \( \Rightarrow \)
- \( L((s, \lambda)) \subseteq V \) eigenspace with eigenvalue \( \lambda \) for \( \phi_s \).
- \( C \) complex line bundle \( L \to \tilde{S} \)

The data of \( \phi: S \to E \) \( \mathbb{C} \) completely encoded in data: \( (\tilde{S}, C, L \to \tilde{S}) \) line bundle.

If have \( \phi \) with repeated eigenvalues \( \Rightarrow \) \( \tilde{S} \) branched covering space ... but \( L \) does not always make sense as a spectral object (except ...)

Important special case: \( \phi_s \) regular: can have repeated eigenvalues, but only one Jordan block per eigenvalue. 

... in this case go again the bundle \( L \to \tilde{S} \times \mathbb{C} \)
More invariant: have the clear. pullback map
\[ h : \text{End} \ V \rightarrow \mathbb{C}^n \quad \phi \mapsto (a_1(\phi), \ldots, a_n(\phi)) \]
\[ \text{det} (I - \phi) = f \quad \text{a, } f = 1, -1, \ldots, -1 \]
All spectral maps (correspond to maps \( \phi : S \rightarrow \text{End} \ V \))
come as pullbacks (fiber products) of a universal spectral\(^\ast\)
cover \[ \overline{C}^n \subset \mathbb{C}^n \times \mathbb{C}^n \]
\[ \overline{C}^n = \left\{ (a_1, \ldots, a_n, t) \mid a_1^2 = a_2^2, \ldots, a_n^2 \right\} \]
\[ \overline{S} : S \rightarrow \text{End} \ V \rightarrow \mathbb{C}^n \quad \overline{S} = S \times \overline{C}^n \]

Remark: Fibers of \( h : \text{End} \ V \rightarrow \mathbb{C}^n \) are \( GL(V) \)-invariant.
\( C^n = \text{End} \ V \sslash GL(V) \) GIT quotient:
points \( \overset{\sim}{\rightarrow} \) closures of orbits....
Orbit of regular elements (generic) are closed
in each orbit, there are two preferred orbits: an open and a closed orbit: maximal
and minimal dimensional orbits. Closed \( \Leftrightarrow \) semisimplicity of orbit, remember only \( \mathbb{C} \)-values.

Open \( \Leftrightarrow \) regular orbit
Regular elements/open orbits are ones that vary continuously
in families... good representatives for the moduli problem
here always have a preferred limit of many objects which are regular ends - on that of
our space as moduli of regular elements.

Extensions: i) Replace \( \text{End} \ V, GL(V) \) by \( \text{}\) \( G \) lie \( \text{\&} = \text{Ad} \) groups, reductive
ii) Allow twisted versions of \( \phi \):

- \( \phi \) can very \( \text{\&} \) vector space \( V \rightarrow \phi \in \Gamma(S, \text{End} E) \)
- \( E \rightarrow S \) vector bundles
- \( \phi \) can very coefficients of \( \phi \): replace \( C \) (ex. use \( \mathbb{C} \)-values like ) by \( K \) coefficient object,
  \( \phi \in \Gamma(S, \text{End} E \otimes K) \)
  \( K \) abelian group
1.5 \( K = \text{vector bundle}, \quad K = \text{torus} \) 
\( K = \text{affine torus}, \quad K = \text{commutative group stack} \) 
\( K = \text{affine bundle} \) (Huyghs 
\( \to B \cdot \text{fields} \) 
\( \) as expansion of group by \( C \times \mathbb{Z} \).

2. Higgs bundles \( K = \text{vector bundle} \)

Start with vector bundles \( E \to S, \quad K \to S \) such \( \phi : E \to E \otimes K \) \( C^* \)-linear map

- replace \( \phi \) by spectral data...

Problem: spectrum of \( \phi \) may not be well defined for \( \mathbb{C} \) scalar \( \phi \).

Indeed: if we trivialize \( K \) on open set \( U \subset S \)
\( K|_U \cong C^*_U \), on \( U \phi \) is given by

\( \chi \) representation \( \Rightarrow \phi_1, \ldots, \phi_k \in \Pi(U, \text{End}(E)) \)

\( K \) spectral covers \( U \) associated to \( \phi_i \)’s...

have to take this union, but this depends on trivializability of \( K \)...... need to put conditions on \( \phi \) to make sense of this!

Most brutal & useful condition: require

that \( \phi_i \) span a commutative subalgebra of \( \Pi(U, \text{End}(E)) \)

\( \iff \phi \wedge \phi = 0 \) as solution of \( \Pi(S, \text{End}(E) \otimes K) \)

"integrability"

Def. A Higgs bundle on \( S \) is a pair \( E \to S, \quad \phi : E \to E \otimes K \) s.t. \( \phi \wedge \phi = 0 \).

Similarly Higgs sheaf.

Note \( \phi \) gives a map \( K^* \otimes E \to E \) from free associative algebra generated by \( K^* \) \( \text{Ann}(K) \otimes E \to E \)

Integrability \( \phi \wedge \phi \iff \text{Ann} \) retracts through torsion subalgebra symmetric algebra \( \text{Sym} \cdot K^* \otimes E \to E \).

\( \text{Sym} \cdot K^* = \text{bundle of algebras on } S, \text{ relative alg of } H^* \text{ on } \text{tot}(K) \)
e tot \( K = \text{Spec} \, \mathcal{O}_K^v \) over \( S \).

\( \Rightarrow \) data \( (E, \phi) \iff \text{module over } \mathcal{O}_K \iff \)

\( \text{quasi coherent sheaf on } \text{tot}(K) \)

\( \xrightarrow{\text{etale}} \quad S \)

(\( \mathcal{O}_K \)-coh sheaves on \( X \)) \( \iff \)

(\( \mathcal{O}_K \)-coh Higgs sheaves on \( S \))

(coherent \( \mathcal{O}_K \)-sheaves on \( S \))

(\( \phi \)-coherent \( \mathcal{O}_K \)-sheaves on \( S \))

ie coherent support \( S \)

\( \iff \) support finite over \( S \).

\( \exists \xrightarrow{\text{etale}} \quad \text{take } \quad E = \mathcal{O}_X \cdot \mathcal{E} \),

\( \mathcal{E} = \mathcal{O}_X (\lambda \cdot -) \quad \lambda : \Pi(X, \mathcal{O}_K^v) \) tautological section of \( \mathcal{O}_X \mathcal{O}_K^v \)

\( \mathcal{E} \xrightarrow{\mathcal{V}} \mathcal{E} \otimes \mathcal{O}_X \mathcal{O}_K^v \)

\( \mathcal{E} \xrightarrow{\mathcal{V}} \mathcal{E} \otimes \mathcal{O}_X \mathcal{O}_K^v \)

Conversely if \( (E, \phi) \) Higgs sheaf on \( S \)

- take \( \mathcal{E} = \ker \left( \mathcal{O}_X \mathcal{O}_K^v \xrightarrow{\phi - \lambda \cdot \text{Id}} \mathcal{O}_X \mathcal{O}_K^v \right) \)

"det \( \phi - \lambda \cdot \text{Id} \) is characteristic polynomial ... \( \ker \left( \phi - \lambda \cdot \text{Id} \right) \)

is eigenline ... spectral data of \( (E, \phi) \).

Supp \( (\mathcal{E}) = S \) spectral variety.

3. D-branes & Higgs bundles

(1) \( (S, \Sigma) \) compact Kähler manifold & \( \Sigma \) real analytic Kähler metric

\( K = \Sigma \quad X = \text{tot } K = \text{tot } (\Sigma^v) \xrightarrow{\mathcal{V}} S \)

holomorphic bundle \( B \quad X \) is algebraic symplectic manifold

\( S \subset X \) zero section : Lagrangian subvariety

1999: \( B. \) Feix & D. Kaledin — a formal neighborhood of \( S \subset X \) has a unique hyperkahler metric

which restricts to \( S \) ... integrates to a tubular neighborhood, for \( \Sigma \) real analytic.

Cebi: found his HK metric for \( \mathbb{CP}^n \), then it integrates to whole \( \mathbb{T} \) not complete, deduced ... can do for other Fano's ... (excluded by homogeneity)
(1) \( E \subset X \rightarrow (E, \varphi : E \rightarrow E \otimes \Omega^2) \)

(Sigma--Calabi-Yau) \( \rightarrow \) flat complex connections on \( S \) \( \rightarrow \) Rep \( \mathcal{N} \)

Explicit bundle on \( B \)-branes on \( X \).

(2) \( Z \) three-dimensional CY, compact,
\( C \subset Z \) smooth curve. In M-theory: look at \( M \)-theory of BPS \( M \)-branes (M5-branes) \( \text{BPS}(Z, C, r) \).
BPS branes on \( Z \) with support of homology class \( r \cdot [C] \).
\( \text{Want to} \) count \( \text{intersect} \) virtual Euler class of BPS \( (Z, C, r) \).
\( \text{Not well-defined} \) rigorously.
3 ways disjoint invariants of such curves, non-compact curves of arbitrary genus ...
\( \text{So what are we counting?} \)

Proposition: replace \( Z \rightarrow C \) by its linearization.
\( X \) to \( f(N_{C/Z}) \rightarrow C \) non-nil. Non-nil... NC CY coming from \( Z \) to \( X \).
\( C \xrightarrow{\iota} X \) new con project back on \( C \):
\( \text{BPS}(C, X, r) = \text{moduli of} \) \( \text{coh sheaves} \)
on \( X \), fibres of degree \( r \) over \( C \).
Ideas: not all branes will survive in this deformation to normal cone - most concompact disappear, blend together in linearization. Still a set of compact, but in fact \( \) not \( \) only way by

.. forget map of curve \( Z \rightarrow X \), but only \( \text{Nekrasov-Okounkov} \) brane sheaf.
\( \text{BPS}(C, X, r) = \text{moduli of} \) \( \text{Nekrasov} \) \( \text{branes} \)
on \( X \) of rank \( r \) on \( C \) ... has \( C^r \)-adic
\( 
\) \( \text{con couple Euler characteristic} \) by localization.

(3) \( A \)-branes on CY: non-holomorphic Higgs bundles used as \( A \)-branes.
A-branes \( X \) smooth CYs, \( M \times X \) compact SLAC

\( n \)-types of A-branes wrapping \( M \) are classically described by a rank \( n \) complex flat connection on \( M \) [Glownika]

Idea: linearize \( X \) near \( M \) \( X \times M \rightarrow T^*M \)

noncompact CY, YM field on \( T^*M \) satisfying SUSY equations

- do a dimensional reduction (Kahler-K"ahler) down to \( M \)
- for this need transverse Section or project to \( M \),
  which is why we linearize ...

Get \( \varphi, \Lambda \) \( := \frac{1}{2} \Lambda_r x \frac{1}{2} \varphi \) some \( M \)

\( \varphi^+ = - \varphi \) wrt hermitian metric provided by \( \alpha \)

\( \nabla \varphi = 0 \) \( \Leftrightarrow \) F-flatness \( DA \times \varphi = 0 \) \( \Rightarrow \) D-flatness

Note: take \( A = a + \mathcal{F} \varphi \) \( \Rightarrow \) A flat \( \Leftrightarrow \) F-flatness

Rank 1: interpreted \( \varphi \) as deformation of the SLAC,
\( \Rightarrow \) get complex moduli; \( F \)-flatness \( \Rightarrow \) combining \( \alpha \) \( (U(n)) \)-connection

\( \alpha \) deformation of SLAC

Higher rank: \( \Rightarrow \) get complex moduli; must take \( \varphi \) into account, no longer just deformations of SLACs.

- that's why we need all complex structures on \( M \), not strict ...

Conversely: \( A \) complex flat connec.

\( \nabla - \text{Hermitian metric on } V \Rightarrow \) can break \( A = \alpha + \mathcal{F} \varphi \)

\( \alpha \) = piece of \( A \) projectivity metric \( h \).

\( D \)-flatness for \( \varphi \) is now an equation for the metric \( h \)

\( \Rightarrow \) Theorem (Collette): \( M \) compact Riemannian manifold,
\( V \) complex vector bundle, \( A \) flat connection on \( V \)

\( \Rightarrow \) \( \exists h \) on \( V \) s.t. \( DA \times \varphi = 0 \) \[ A = \alpha + \mathcal{F} \varphi \] with ...
2. If $(VA)$ simple $\Rightarrow h$ is unique, $& a$ is flat.

Deform flat unitary to pairs $(g, \phi)$ satisfying F.D. relations $\Rightarrow$ just odd Higgs fields w. rep $h$, harmonic near unitary pts model; looks like harmonic Higgs fields for metrics on $\text{Rep}(\mathbb{C}^\infty_G, \text{Gauge})$.

$\text{K"ahler:}$ Higgs fields (a not real, flat) $\leftrightarrow$

all complex flat connections (not real, simple).

Questa: is there a quantization of the harmonic map equation on $h$ (universal form $\rightarrow$ symmetric space) taking $h$ account the quantum corrections to A-branes?

Interesting examples of moduli of harmonic maps on hyperbolic 3-manifolds $\dddot{0}$. $\dddot{0}$

**Example** $T = \mathbb{R}^3 / \mathbb{Z}^3$ with flat metric $g$

$x_1, x_2, x_3$ real on $\mathbb{R}^3$. $D = \mu_2 \times \mu_3$. Define set $\mathcal{P}$ of sets $\mathcal{P} = \{ (x, y) : x^2 = y^2 = 1, x_2, x_3 \in \mathbb{Z}^3 \}$

$0 \rightarrow \mathbb{Z}^3 \rightarrow D \rightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \rightarrow 0$ $\text{Ker} \phi$

covers $\mathcal{P}$. $\phi$ acts on $D$, $\phi(x) = (-x_1, -x_2, -x_3)$

$\phi(x) = (x_1, \frac{x_2}{2}, -x_3 + \frac{x_3}{2})$

$\phi$ acts on $T$, $\phi(x) = (x_1, \frac{x_2}{2}, -x_3 + \frac{x_3}{2})$

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Also cos to reach moduli of complex Cal. G₂ structures on M-theory dual. 

Moduli of flat connections on M:

\[ \text{Loc}_n(M) = \text{Hom}(\pi_1(M), G_2) \cong GL_n(C) / GL_n(C) \]

Center \( \langle \beta^2 \rangle \) acts as translation by a 2-torsion point on \( T : M = (T / \mathbb{Z}) / \text{Gal} \)

\[ 0 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(N) \to 0 \]

Write by genus & mutations \( \Rightarrow \) ample \( \text{Loc}_n(M) \)

eg \( n = 1 \):

\[ \text{Loc}_1(M) = 16 \text{ isor. 45 points} \]

\( n = 2 \):

\[ \text{Loc}_2(M) \] 41 connected components, 3 pts;

one trivial \( Y \) (complex) ; 3 copies of \( \mathbb{C} \) at each origin.

Other side of mirror symmetry: will be a gerbe over a CY, won't have many points!

Main problem: understand \( Y \) using T-duality ...

\[ X = \text{tot}(T^*M) = \text{tot}(T^*T / \mathbb{Z}) / \text{Gal} \]

Very natural SLAG form fibration:

\[ T^*T \to T^*T_0 \]

Project trivial bundle on fibers to

\[ T^*M \to T^*T / \mathbb{Z} = T^*T / \text{Gal} = \mathbb{C}^* \] acting trivially on translation invariant forms

acting on \( T^*T \) is product of affine action on \( T \)

\( \tilde{T} \) is linearized on the fibers ... \( X \to B \) is SLAG form fibration with \( \tilde{T} \mapsto 2 \tilde{T} / \mathbb{Z} \)

... acts a field of cycle shifts of \( \phi \) (horizontal)

\( \phi = \text{dimension of } X \)
Theorem: use covering $T^*T$ where there are no singular fibers, then double the result by $D$ acting dually.

$T$-dual of $T^*T \cong T \times T^* T$ is

$T \times T^* T \quad T$ dual tori $= \text{univ!}$ of flat $U(1)$ connections on $T$

$T$-dual of $X := \frac{(T \times T^* T)}{D}$ as a complex manifold $\mathbb{C}^3$ acts by $\alpha(x, y, z) = (x^{-1}, y^{-1}, z^{-1})$... in this case acts faithfully $\cong \mathbb{Z}/2\mathbb{Z}$

$Y := \frac{[\mathbb{C}^3]}{D} = \mathbb{Z}/2\mathbb{Z}$—Serre's orbifold

Now compute $\mathbb{H}$ of points on $X$. Since $X$ is a stack, $\mathbb{H} = \frac{\mathbb{Z}}{2\mathbb{Z}}$—Serre's orbifold

Try to replace stack $\frac{[\mathbb{C}^3]}{\mathbb{Z}/2\mathbb{Z}}$ or singular space $\mathbb{C}^3/\mathbb{Z}/2\mathbb{Z}$ by crepant resolution.
category but to get moduli spaces need to specify
stability conditions & these do depend on choice of
comport resolution !

- try to work $Y$ as $\mathbb{Z}/2\mathbb{Z}$-geometric

On start also need to choose which
are physical bases, and is stability condition -
are on comport resolution, need correct $\mathbb{Z}$-stability --
could be relevant previous e.g.

Natural guess: Bridgeled - King - Reid resolution ,
$K = \text{Hilb}(\mathbb{C}^3)$ $K$-compactly with regular
map of $K$ as fiber

Problem: Loc. $(\mathfrak{n}) \neq \text{Hilb}, (\mathbb{Z}/2\mathbb{Z}$-geometric on $\mathbb{C}^3$)

Outside singular cases we know what's going on:
if be $B -$ determined, $X_b = \Pi$,
and the corresponding point $\Rightarrow Y = \text{a flat}
\text{over } \mathbf{U}(\mathfrak{n}) - \text{connected on } \Pi$.

$\mathfrak{m} = X \to B$ the maps 1:1 to $M$:
- specified cover for $M$

$V = \mathbf{P}^1 \mathfrak{L}$ is a flat $\mathfrak{L}$ local system
on $M$, in flat $SO(4)$-sym.

Re-write $\mathbf{P}^1 \mathfrak{L}$ (via Cartier) as pair $(a, \phi)$
$a = \text{flat } SO(4,\mathbb{R})$-connection & $\phi$ flat Higgs field

Fix $\mathfrak{L}$ be for.

$\phi \in \Gamma(M, \text{ad } V \otimes A^m) \Rightarrow \text{flat}$ $\text{Higgs}$

$\mathfrak{m} \times \mathfrak{m} \otimes A^m \Leftrightarrow \text{flat}$
Proposed: $Y = \text{model of } (SU(4) \times SU(4))$ Higgs bundles on $M$ (more precisely a compact) ---

birotational to what we had before. $(\mathbb{C}^*)^5/\mathbb{C}$

Here: this is smooth.

H/16: $\mathfrak{H}/16_2$ are indeed the corners: 16 pts

$[\text{on } \mathbb{Z}/16 \text{ get } a Y, \text{ unire}]$

Note: $Y \rightarrow B$ is the Hitchin map

on Higgs fields $q \rightarrow$ invariant roots of $g$ as special case of $q$.

$B \subset \mathbb{R}^4$ as you see from
Quadratic Duality for Algebras

Prototype of spectral correspondence...

A = \mathbb{R}^n; graded algebra, assume A \text{ commutative}; A_0 = \mathbb{C}\; \exists \lambda = \mathbb{Z}

unit, & A \text{ locally finite dimensional}

Augmentation ideal \ A_+ = \{ \lambda \in \mathbb{C}; \lambda \mathbb{C} \}

Def. A is called quadratic if generated in deg 1 & relates in deg 2

\begin{align*}
& \text{tensor algebra } \ T^*(A_1) \xrightarrow{\text{can}} A \text{ remains map subject,} \\
& R = \ker \text{can} \cap T^2(A_1) \\
& \text{these two pair generates ideal } \ker \text{can} \Delta T^*(A_1) \\
\end{align*}

- i.e. A is really described by a pair of vector spaces:

\[ V = A_1, \; \quad R = A_0 \otimes A_1, \quad \text{write } \ A = \{ V, R \} \]

- really equivalent of categories

Def. The quadratic dual \ A^! \ of a quadratic algebra \ A = \{ V, R \}

is the graded algebra \ A^! = \{ V^*, R^! \}\;\quad R^! = V^* \text{ on } \quad \text{this is where we need } A_1 \text{ finite dimensional, to say } \ A^! = A \otimes A^* \text{ instead of } \text{graded completion} \\

\quad \text{is an autoequivalence (contravariant) of } (\text{graded}) \mathcal{C} \text{, whose square is the identity.}

Examples:

\begin{align*}
& V \text{ finite vector space, } A = \{ V, 0 \} \Rightarrow \\
& A^! = C \oplus V^* \text{ superalgebra} \\
& \text{V vector space } \quad A = S^*V = \{ V, V^2 \} \\
& \Rightarrow \quad A^! = V \otimes V^* = \{ V^*, S^2V^* \} \\
\end{align*}

If \ M \text{ is a module over an algebra } \ A = \{ V, R \}

say \ M \text{ is quadratic if}

\begin{itemize}
  \item M is \ G\mathbb{Z} M_i \text{ graded module}
  \item \exists L \subset A, 6M_0 \text{ s.t. } \ M = (A \otimes M_0) / A \cdot L
\end{itemize}
$\mathcal{M}$ quadr. $\Rightarrow \mathcal{M} = \mathcal{E} \mathcal{M}_0, L^0$ linear algebra data

Quadratic dual module $M^! A$ is

$\quad M^! = (\mathcal{A}^! \otimes L^0) / (\mathcal{A}^! \cdot L^1) \quad : \quad M^! = \{ \mathcal{M}_0, L^0 \}$

$\mathcal{A}^! : (A \text{-quad mod.}) \rightarrow (A^! \text{-quad mod.}) \quad \text{op}$

Example

- $\mathcal{C}^{\mathcal{A}^!} = A^!$
- $(A^!)^\mathcal{A} = \mathcal{C}$

Koszul algebras

Def A quadratic algebra $A$ is Koszul if we have

an isomorphism of graded algebras $A^i = \text{Ext}^i_A (C, C)$

Yoneda Ext algebra of trivial module in algebra of graded algebras

[Problem: graded module category doesn't have enough
injectives or projectives -- need to "resolve" $A$ by
a huge algebra...]

$\text{Ext}^i_A$ is bigraded initially but here

the condition is that all non-diagonal Ext vanish:

In fact: (under assumption) $A$ Koszul if $\text{Ext}^i_A (C, C) = 0 \text{ for } i \neq 0$

Def A quadratic module $M/A$ is Koszul if

$M^! = \text{Ext}^0_A (M, C)$

Note $T^! V$ is Koszul, as are $5^! V, \wedge^! V$ [Steenrod algebra Koszul]

Want like to compare all objects over quadratic $A, A^!$

not just modules over algebras!

React: If $A, A^!$ are quadratic dual algebras

functor $D^b (A \text{-mod } F^0) \rightarrow D^b (A^! \text{-mod } F^0)$ greatest module

$M \rightarrow \text{cobar} (A, M)$

$\text{Cobar} (A) \text{ is a c.d.g.a. associated to } A:$

- $\text{Cobar} (A) = \pi (A^!, [-])$ as a graded algebra

(cobar bar said

algebraic codgds

so dualize)
Differential characterized by $A^+ \rightarrow A^+ \otimes A^+$ dual to $A^+ \otimes A^+ \rightarrow A^+$ multiplication... extended uniquely (essentially) to derivation $\text{Der}(A) = \text{Hochschild} \text{ cochains valued in } \text{trivial module}$.

(Scalar $(A, M)$ = scalar $(A) \otimes M$ with different excinding vector of $A$ on $M$)

Claim (Brenner-Gelfand-Gelfand) If $A$ is finite dimensional & Koszul $A$ is finite then $(\text{scalar } (A, M) \subset M^2$)

So derived category of quadratic algebras to all modules...

Filtered quadratic algebraically

L. A. Popov (ski), Funk (funct. anal.)

(Polishchuk - Popovski: Quadratic algebras - Polishchuk website)

A filtered algebra with $F^n A \subset F^m A \subset \cdots$ is finite dimensional (i.e. $F^n A$ is)

Spec $A$

A is filtered quadratic if $A$ is generated in degree 1 with relations in degree two.

Start with $1 \in F^1 A$. Look at $n \rightarrow \text{Tor}_1 (1 \in F^1 A) = \text{Tor}_1 (1 \in F^1 A)$.

To $(1 \in F^1) \subset \text{Tor}_1 (1 \in F^1) \subset \text{Tor}_2 (1 \in F^1) \subset \cdots$

[images of graded pieces in $F^1 (A)$]

$\rightarrow A$ is filtered quadratic if $\text{Tor}_1 (1 \in F^1 A) \rightarrow A$

Surjective, with kernel generated by its subspaces $J_a = \ker (\text{Tor}_2 (1 \in F^1 A))$.

So $A$ is filtered quadratic if span by $\ast W$, $\ast$ fixed vector $\ast J \subset \text{Tor}_2 (1 \in W)$

$A = \{ \ast W, \ast J \}$

--- originally due to Priddy
Remark: A filtered quadratic algebra \( A^{(0)} = \{ W/C_e, J \mod V, (e_0 W) \} \) 

Quadratic part of \( A \) = quadratic part of \( \text{gr}_F A \)

Def: \( A \) is called Koszul if \( A^{(0)} \) Koszul.

In this case: \( \text{gr}_F (A) = A^{(0)} \oplus \text{soc} A \)

Q: Is there a duality of filtered quadratic algebras which lifts the duality of graded quadratic algebras?

A: \( \text{No} \): data of extensions in \( A \) is translated into something else, namely a differential: get curved algebra as quadratic algebra - curvature measuring lack of augmentation. [Augmented case: Frohlich, John (1973)]

Def: An \( \text{cdga} \) \( B \) is a triple \( B = (B, d_B, h_B) \)

\( B = \text{graded algebra} \) \( B = \bigotimes \bigotimes \)

\( d_B : \bigotimes \bigotimes \rightarrow \bigotimes \) derivative of degree one

\( h_B \in B_2 \) \& \( d_B h_B = 0 \).

A morphism of cdgas \( B \rightarrow C \) is a map \( f : (f, \alpha) \)

\( f : B \rightarrow C \), \( f \in C \),

\( f(\alpha B) = f(\alpha) = \alpha B \)

Typical example: \((\text{End } E \otimes \Omega^*, \nabla E, \text{Curvature})\)

\( E \) vector bundle with connection.

Def: A cdg module over a cdga \( B \) is a pair \( N = (N, d_N) \)

\( N \) is a graded \( B \)-module, \( d_N \) is an odd derivation \( d_N : N \rightarrow N \)

\( d_N^2 = 0 \) \& \( \forall x \in N \).

Example: \( E \) vector bundle \( N = (E \otimes \Omega^*, \nabla) \) is cdg module on \((\text{End } E \otimes \Omega^*, d^{\nabla}, \text{F})\).
If \( A \) is filtered quadratic & \( V = W \) complexify for \( e \) \( \\{ e \in W, J \} \)

\[ W = V \otimes C e \]

\[ \Rightarrow A^{(0)} = \{ V, R \} \quad R = J / T(\{ e \in W \}) < V \otimes V \]

\[ J \in T_2(\{ e \in W \}) = G \otimes V(\{ e \} \otimes V) \]

is a graph of a linear map \( R \rightarrow C \otimes V \quad \gamma = (3, \gamma) \)

which satisfies \((*) \quad (4^{12} - 4^{23}) (\{ e \} \otimes V, R \otimes R \otimes V) \leq \gamma \quad \gamma \)

Now if \( B = A^{(0)} \) \( B_2 = R^{P(2)} \)

\( (e, b) \) \( \rightarrow \) to \( g = g e \) \( \cdot B \rightarrow B_2 \)

\( b = h \) \( \cdot C \rightarrow B_2 \)

\( (*) \quad \Rightarrow \quad (B, d, b, h) \quad cdg a \)

**Def.** A **filtered quadratic** \( A = \{ e \in W, J \} \)

will be called **almost split** if it is equivalent with a splitting \( W \rightarrow W / e \)

**Theorem (Positivity)** The **filtered quadratic duality**

\[ ! : \text{filtered almost split} \rightarrow \text{quadratic} \quad [\text{red in even}] \]

gives an equivalence between almost-split Koszul filtered quadratic algebras & Koszul cdgas

Augmented filtered algebras \( \rightarrow \) cdgas (Positivity)

... see statement for modules
Spectral Construction Revisited

Let $S$ be a variety over $K$, $K ightarrow S$ a (fixed) algebraic map, and $X = \mathrm{tot} K \rightarrow S$.

Physical applications: want $\Lambda^{\mathrm{tor}} K = K_S$ so total space of $K$ has trivial canonical class --- info will assume will be a CY!

Spectral correspondence:

\begin{align*}
\text{(coherent sheaves on } X) & \leftrightarrow \text{(coherent } K\text{-valued)} \\
\text{(finite over } S) & \leftrightarrow \text{(Higgs sheaves on } S) \\
\end{align*}

Idea:

- Coherent sheaves on $X$, finite over $S$.

As Koszul duality:

\begin{align*}
\text{LHS} &= \text{fin gen. mod-} X \text{ algebras over } S^\ast K = \text{fil}^\wedge X \\
\text{RHS} &= \text{Higgs sheaves over } \Lambda^\ast K \text{ as } \Omega^\ast \text{ graded object} \\
E \xrightarrow{\partial} E \otimes K \xrightarrow{\partial} E \otimes K \otimes \Lambda^2 K & \rightarrow \cdots \\
\end{align*}

Integrability of Higgs sheaves $\Rightarrow$ this is a vector.

Differential is actually $O$-linear: $\partial$.

$\Rightarrow$ dg algebra over $\Omega^\ast K$.

$S^\ast K$ as filtered NC algebra:

\begin{align*}
\Lambda^\ast K \text{ as graded NC algebra} & \xrightarrow{\text{Koszul}} S^\ast K \text{ as graded} \cr
\text{filtered Koszul duality} & \Rightarrow S^\ast K \text{ as graded} \\
\end{align*}

Higgs fields: free modules over $\text{RHS}$.

Now deform both sides: $S^\ast K \rightarrow \text{filtered NC algebra}$.

Examples:

1) Deform $X \rightarrow S^\ast K$ as an algebraic variety.

- Three types: deform base $S$, deform vector bundle $K$, or deform vector bundle to an affine bundle $\pi$.

The latter destroys one subvarieties, e.g. $O$ section...
These affine bundles are parameterized by $H^*(S, K)$. Given $w \in H^*(S, K)$ get affine bundle $X_w \to P_w \to S$ 

$U_{X_w} = S \cdot w \cdot K$ filtration commutative algebras

$\text{gr}(S \cdot w \cdot K) = S \cdot K \cdot$.  [Xw still CY if Xw os]

B-branes on Xw = coherent sheaves (with compact support)

$cdg$ module over a cdg def of $(\Lambda^0 K, 0, 0)$ $S \cdot w \cdot K$

Explicitly:

$w \in H^*(S, K) \Rightarrow$ exists $0 \to K \to F_w \to Q \to 0$

$X_w =$ fiber of $F_w$ over $1 \in G$

$\therefore X_w \to 1$

Geometrically:

$P(F_w) \to S$

$S \cdot K \cdot = P_{X_w} \cdot C_{X_w}$ are zero functors on

$X_w = P(F_w) \cdot P(K)$

$P(F_w)$ with $\mu_{P,K} \circ \log P(f)$,

$S \cdot K \cdot = \Pi_{X_w} \cdot \log P(F_w)(\infty \cdot P(f))$

$
\Gamma$ (filtered by order of pole along $P(K)$.

Algebraically:

$S \cdot K \cdot = S \cdot F_w / \langle 1 \cdot S \cdot F_w \rangle$

$F^{-1}(S \cdot K \cdot) = S \cdot F_w : U_s \subset F_w \subset S \cdot F_w \subset S \cdot F_w \subset ...$

$K \cdot \quad S \cdot K \cdot \quad S \cdot K \cdot$

Koszul dual: $(\Lambda^0 K, 0, 0)$ $w$

well we needed a splitting of the first step of the Poincaré

$0 \to U_s \to F_w \to K \to 0 \ldots$ but its nonsplit!

So no hope for global sheaf of $cdg a \ldots$

BUT issue of add-ons include not just isomorphism

but gauge transformations: use these to shift

$cdg a$ on patches

like in derived algebraic geometry! can't believe

global sheaf of $cdg a$, or define sheaves

locally defined kahls glued by gives: isomorphing

here we only allow homotopies of $cdg a$,

since Koszul duality really behoal under grm.
Choose a Čech covering of $S$ & recycle $w_j \in \hat{H}(U, \mathcal{E})$ representing $w_i$.

Each $U_i$ has gives a section $X w_i \rightarrow U_i$.

Koszul dual is $(\Lambda^0 K, 0, 0)_U$:

$(\Lambda^0 K, 0, 0)_U = \frac{1}{2} (\Lambda^1 K, 0, 0)_U / \sim$

where

$(\Lambda^1 K, 0, 0)_U$; glue $\sim (\Lambda^1 K, 0, 0)_U$ by gauge transformation in $(\text{id}, c_{ij})$.

Recall $B \in \text{c} \in \mathcal{B} \xrightarrow{\epsilon} \mathcal{C} \xrightarrow{f} \mathcal{C}$ $\xrightarrow{(\epsilon, x)}$

$\epsilon: B \rightarrow \mathcal{C}$ map of $\mathcal{C}$-algebras

$x \in \mathcal{C},$

$\epsilon(\mathcal{B} x) = \mathcal{C}(f x) - \mathcal{L}(f x)$

$\epsilon(\mathcal{B} x) = \mathcal{C} x + x^2\mathcal{C}$

Here $\mathcal{C} \in \mathcal{B} = \mathcal{C}$, $x \in \mathcal{C}_{ij}$, $x^2 = 0$

So as $K$ acts on $(\Lambda^0 K, 0, 0)$ as gauge transformation $(\text{id}, x)$, we just take $K$-twist of trivial cdga $(\Lambda^0 K, 0, 0)$ with $K$-torus $X w$.

(coherent sheaves on $X w$ with compact support) $\leftrightarrow$ $(\omega$-twisted $\mathcal{B}$-algebra $\mathcal{E}$, $\mathcal{A})$

$\phi_i: \mathcal{E}_U \rightarrow \mathcal{E}_U \otimes K w_i$

$\phi_i \circ \phi_j = 0$ & $\phi_j \circ \phi_i = c_{ij} \circ \text{id}$

Note if ram $E$ finite (compact support on $U$'s)

$(\Rightarrow \text{tr } \phi_i - \text{tr } \phi_j = \text{tr } c_{ij} \circ \text{id})$

so recycle is a coboundary!

So if $E$ torsion-free, $\Rightarrow$ recycle is a coboundary: such won't exist globally...

Only get sheaves supported on subvarieties where $[\omega] = 0$, i.e. where class of locally affine bundle is zero.
Example: \( S = \text{elliptic curve}, \quad X = T^*S, \quad \mathcal{X}_u = T^*_u S \)
\( \omega \in H^1(S, \mathcal{O}_S) - \text{a holomorphic class} \)
- twistor family of \( X \)

Twisted Higgs bundles: Serreym interpretation

Higgs bundles \( E \rightarrow E \otimes \mathcal{O}_X \) are sheaves on stack \( K^\nu \)
\( K^\nu \rightarrow S \) bundle of stack \( K^\nu \)

\( \mathcal{B}K^\nu \rightarrow S \). Sheaves on \( \mathcal{B}K^\nu = [S/K^\nu] \) are sheaves \( E \rightarrow S \) with \( K^\nu \)-action \( E \otimes K^\nu \rightarrow E \)

i.e., \( \mathcal{O}_X \) module \( \mathcal{O}_X \)
- set all nilpotent Higgs sheaves
- to get all
go to formal stacks \( \mathcal{B}K^\nu \)

Twisted Higgs sheaves \( \mathcal{E} \rightarrow S \) in \( 0 \)-gerbe

on \( \mathcal{B}K^\nu \):
\[ 0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}_0 \rightarrow K^\nu \rightarrow 0 \]
defines gerbe \( \mathcal{O} \rightarrow \mathcal{B}G \rightarrow \mathcal{B}K^\nu \rightarrow 0 \)

\( 0 \)-gerbe: looks like rational curve of \( K^\nu \)

Example ii) \( h \in H^0(S, \Lambda^2 K) \)
deform \((\Lambda^2 K, 0, 0) \) by \( h \): commutes with \( \lambda \)
\( \Rightarrow (\Lambda^2 K, 0, h) \) s.t. \( \Lambda^2 h = h \cdot \text{id} \)
Higgs fields with central value

Kopust\'niř ladov: Clifford algebra \( \text{Cliff} V \)
Koszul d-\( d \) to \((S^*V, 0, h) \)

\( h \in \text{Sym}^2 V \)

\( h \) gives a Heisenberg Lie algebra
\[ 0 \rightarrow \mathcal{O}_h \rightarrow K^\nu \rightarrow 0 \]
vector space direct sum
with Higgs relations, \( \{a, h\} = \langle a, a \wedge S \rangle \)
\( \mathcal{O}_h \) - Her bracket!

\( \langle \Lambda^2 K, 0, h \rangle \) = \( U_h = U(\mathcal{L}_h) / U(\mathcal{L}_h - 1) \)

filtered NC \( O_S \)-algebra
\[ \rightarrow NC \text{ branes, } \text{on } "X_h" \Rightarrow S \quad U = U_h G \]
Also can look at \((\Lambda^n E, d, h)\) in \(K\nu\) no longer commutative Lie algebra.

\[ cv : \Lambda^n K^\nu \to K^\nu \] Lie bracket.

Get exact sequence \(0 \to \Omega_\nu \to \Lambda^n E \to K^\nu \to 0\).

Koszul dual \((\Lambda^n E, d, c)^! = \Omega^n \nu \to 1-1\)

without \(\nu\) \(\Lambda^n E\) is just Cosp(H) in \(\mathbb{C}\)-valued cases.

Special case \(K = \Omega^n \nu\), \((\Lambda^n \Omega^n \nu, d, 0)^! = \Omega^n \nu\)

\(\Omega^n \nu\)-modules \(\to\) de Rham co-planes

\[(\Lambda^n \Omega^n \nu, d, h)^! = (\Omega^n \nu)_h, w\]

\(X = \text{tot } K\nu \subset \text{cone} \subset \mathcal{P}(K\nu \oplus \Omega^n \nu)\) Poisson surface.

Poisson structure vanishes twice at \(0\).

\(\Rightarrow\) NC deformation. Look at sheaves on \(X\) controlled at \(0\).