Perverse sheaves on cell-stratified space
(after R. MacPherson)
Notes for Math 278, Fall 1996

1. Extensions between simple objects

Let $X$ be a regular CW-complex. This means that $X$ is a CW-complex such that
the closure of each cell $C$ is homeomorphic to the closed ball (in a way compatible
with the structural homeomorphism of $C$ with the open ball). Let $p : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$
be a perversity function, such that $p(0) = 0$, $0 \leq p(m) - p(n) \leq n - m$ for $m \leq n$. We
consider the category $\mathcal{C}$ of perverse sheaves on $X$ with respect to $p$, smooth along
the stratification by cells. Let $C$ be a cell in $X$, $i_C : C \to X$ be the inclusion.
Let $i_C^* : \mathcal{C} \to X$ be the inclusion of the closure of $C$. The simple objects of $\mathcal{C}$ are
Goresky-MacPherson extensions $P_C := i_C^* \mathbb{Q}[-p(C)]$, where $p(C) = p(\dim C)$. Let
us call an integer $i > 0$ of type $\ast$ if $p(i) = p(i - 1) - 1$, and of type ! otherwise (i.e. if
$p(i) = p(i - 1)$). It is convenient to assume that $0$ is of both types. We say that a
cell $C$ is of type $\ast$ (resp. of type !) if $\dim C$ is.

Lemma 1.1. For any cell $C$ we have $P_C \cong i_C^* \mathbb{Q}[-p(C)]$ if $C$ is of type $\ast$, and
$P_C \cong i_C^* \mathbb{Q}[-p(C)]$ if $C$ is of type !. In the latter case $P_C \cong i_C^* \mathbb{Q}[\ast p(-p(C))]$.

Proof. This follows easily from the fact that if $j : B \to \overline{B}$ is an inclusion of the open
ball into its closure, then $j_* \mathbb{Q}_B \cong \mathbb{Q}_{\overline{B}}$. □

In this situation R. MacPherson defines a perverse dimension of a cell $C$ as $\delta(C) =
\delta(dim C)$ where $|\delta(d)|$ is the number of $i$'s, $0 < i < d$, which are of the same type as
d (in particular, $\delta(0) = 0$). the sign of $\delta(d)$ for $d > 0$ is defined by the following rule:
$\delta(d) > 0$ if $d$ is of type $\ast$, and $\delta(d) < 0$ otherwise. Notice that $\delta$ maps every interval
$[0, d]$ ($d \geq 0$) to an interval of integers containing zero, and the above definition establishes
a bijective correspondence between perversities and maps $\delta : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$
with this property.

From now on we assume that the cell decomposition of $X$ satisfies the following
condition:
for every pair of cells $C$ and $C'$, the intersection $\overline{C} \cap \overline{C'}$ is either empty
or homeomorphic to a closed ball.

Below we compute Ext-groups in the derived category of sheaves on $X$ between
simple objects of $\mathcal{C}$.

Proposition 1.2. Let $C$ and $C'$ be cells. Then $\text{Ext}^i(P_C, P_{C'}) = 0$ unless $i \neq \delta(C) -
\delta(C')$. Furthermore, $\text{Ext}^i(P_C, P_{C'})$ is non-zero precisely in one of the following cases:
1) both $C$ and $C'$ are of $\ast$-type (resp. $\!$-type) and $C' \subset \overline{C}$ (resp. $C \subset \overline{C}$), 2) $C$
is of $\ast$-type, $C'$ is of $\!$-type and the intersection $\overline{C} \cap \overline{C'}$ is non-empty. In these cases
$\text{Ext}^i(P_C, P_{C'})$ is one-dimensional.
\[ DF \text{ - filtered derived category} \quad \text{(graded components have construcible homology)}. \]

\[ \text{structure on } D \quad \text{via} \quad \text{structure on } DF \]

\[ DF \overset{\text{st}}{=} \oplus_{i} \Gamma_{i} \quad \text{etc.} \quad \overset{\text{HF}}{\rightarrow} \]

Claim: \[ CF = DF \overset{\text{st}}{=} DF_{2} \overset{\text{st}}{=} \mathcal{C}_{0}(C) \quad \text{complexes in } \mathbb{K}[K^{\bullet}] \]

\[ K < CF \cong \Gamma_{i} \quad K < \mathcal{C}_{[i-1]} \]

\[ K_{i-1} \overset{K_{i+1}}{\rightarrow} \quad \text{we have truncation-def-filtation} \]

\[ \sigma_{\mathbb{Z}_{n}}, \sigma_{\mathbb{Z}_{k}} \quad \text{etc., as adjoint of inclusions} \]

\[ \Rightarrow \quad \sigma_{a,b}^{i} = \sigma_{a} \sigma_{b} \quad \sigma_{k} \Rightarrow \sigma_{a} \]

\[ \Rightarrow \quad \text{we have exact triangle} \quad \Gamma_{i-1} \overset{i}{\rightarrow} \sigma_{[i-1]} K \overset{e_{i}}{\rightarrow} \]

\[ \Rightarrow \quad H_{i}(\Gamma_{i} K) \rightarrow H_{i}(\Gamma_{i} K) \quad \text{boundary map} \]

\[ \Rightarrow \quad \text{give our differential}. \]

\[ H_{p} \Rightarrow \text{full and faithful} \quad \text{by spectral sequence from last line}: \]

\[ E_{1}^{p} \quad \text{is zero only in } p, q > 0. \]

\[ \begin{array}{c}
\text{Map is surjective on objects} \\
\text{get composition} \\
\text{Why is } H_{p} \text{ essentially surjective?} \\
0 \leq p < b \\
\end{array} \]

Write as \[ 0 \rightarrow K_{a} \rightarrow \ldots \rightarrow K_{p} \]

\[ \begin{array}{c}
\text{degree} \\
\text{degree plus 1} \\
\end{array} \]

\[ \rightarrow K_{b} \rightarrow 0 \quad \text{real} \]

\[ \begin{array}{c}
\text{degree} \\
\text{degree plus 1} \\
\end{array} \]

\[ \rightarrow K_{b} \rightarrow 0 \]
By induction on \( b-a \), \( f \) comes from map of \( \mathbb{F}^1 \)-graded complexes:

\[
\begin{align*}
\mathbb{F} &= \#_{\mathbb{F}} \left( f^* : C^* \to C^* \right) \\
(A, F^0, 1) &\to (A, F) \to (B, F) \\
\Rightarrow &\#_{\mathbb{F}} (\text{cone } g) = k^*
\end{align*}
\]

Remark: we had \( A = \otimes \text{Ext}^k (P, P) \)

\( S \) = ordered set of cells, Claim: \( A = A_S \)

before we only claimed \( A, A_S \), now \( A, A_S \)

\( C \) = type \( P_c = [C_1 \to C_2 \to \cdots] \)

\( P_c = [C_1 \to C_2 \to \cdots] \)

clue see this isomorphism, but

can write in ! case as \( P \therefore C \cdots \therefore \) self-obs.

\( 11/22 \)

Semi-small Stratified map:

\[
T < f^{-1}(S) 
\Rightarrow \text{loc trivial } f : X \to Y
\]

f is semi-small \( \forall S \subseteq f(T) \), \( x \in S \) \( \dim (f(T) \cap T) < \frac{1}{2} (\dim (f(T)) \cdot \dim S) \)

Claim: \( f \) semi-small \( \Rightarrow \) Perversity on \( X \) \( \& \) \( P \) perverse.

\( P \) suffices to consider \( & \) show \( f^* \text{D} \subseteq \text{D} \)

\( P \) the \( \Rightarrow \) \( f^* = f^! \), commutes with duality.

Condition \( P \in \text{D} \) is pointwise condition: sufficient to consider restriction to any stratum.

\( \Rightarrow \) \( P \subseteq \text{D} \) \( \leq -\dim T \)

Thus we have to show \( \Rightarrow \) \( f^* \text{D} \subseteq \text{D} \leq -\dim S \)

I be sys on \( T \) \( \Rightarrow \) \( \text{D} \leq 0 \) \( \Rightarrow \) \( \dim (f(T)) \cdot \dim S \)
\[ f(\mathcal{A}) \leq \dim T + \text{dim}(\mathcal{F}) - \dim s < D' \leq \dim s \]

\(X\) - triangulated, or more generally cell stratified:

- Every stratum is open 5-ball, \(5 = \text{order ball}\)
- Category of perverse sheaves on \(X\) w.r.t. stratification, some \(B\text{-model}\), \(B\) quadratic algebra.
- Ext occurs only when difference in perverse dimension is 1.
- Get filtration, morphism compatible with filtrations.
- \(B = \text{Aut}(\text{fiber functor } \mathcal{O}_G)\) (filtrations)

This functor has a geometric construction as in loop Grassmannian case.

**Perverse cells**

**Note**: in cell stratified case can construct barycentric subdivision by induction on dimension of strata. From center of ball to subdivision of boundary get simplicial triangulation of \(X\). \(X\) is geometric realization of poset of cells in subdivision.]

\[
Perv_{\text{filtration}}: \quad F_0 X = U \circlearrowleft \{ c_0 \cdots c_{k_i} \} \quad \text{for } k \text{ indexing }
\]

union of mapping in barycentric subdivision.

Perverse cells are connected components of \(F_1 X \setminus F_0 X\).

Example:

- \(S: 0 \to 0\), \(1 \to 1\), \(2 \to 1\)

- \(F_0 X = \text{centers of cells of } \dim 1\)
- \(F_0\): base from cells of \(\dim 0\)
- \(0\) - union of all stars of vertices

- \(F_2\) - union of core cells

Perverse cells are in bijective correspondence with usual cells.
\[ \delta \mathcal{C} \text{ need be d! cond. } C \neq \mathcal{C} \]
\[ \Rightarrow \mathcal{C} \cap \mathcal{C}' = \{ z : \mathcal{C} \cap z < \mathcal{C}', \exists \mathcal{C}(z) \subseteq \mathcal{C}(z) \} \]

which has a maximal element \( C \Rightarrow \) contractible.

\[ \delta \mathcal{C} \cap \mathcal{C}' \]

Sure set with cond. \( \mathcal{C} \neq \mathcal{C}' \), but \( C \) still gives maximal element \( \Rightarrow \) contractible.

b. \( \mathcal{C} \neq \mathcal{C}' \) (co-type !). Then \( \delta \mathcal{C} \cap \mathcal{C}' = \emptyset \)

c. \( \mathcal{C} = \mathcal{C}' \) \( \Rightarrow \mathcal{C} \cap \mathcal{C}' = \Phi, \mathcal{C} \cap \mathcal{C}' = \emptyset \)

\[ \text{Perversity & Purity} \]

\[ X_0 / \bar{\mathcal{F}}_2 \text{ scheme of finite type } \Rightarrow X / \bar{\mathcal{F}}_2 \]

\[ F_{0}/F_{0} = \bar{\mathcal{F}}_2 \text{ sheaf } \Rightarrow F / \bar{\mathcal{F}}_2 \text{ sheaf } \]

\[ F_{0}, F_{0} \text{ elements of } \bar{D}_c \left( X, \bar{\mathcal{F}}_2 \right) \]

\[ \text{RHom} (K_0, L_0) = R \prod R^{\bar{\mathcal{F}}}_a \text{ RHom } (K_0, L_0) \]

\[ \mathcal{C} : X_0 \Rightarrow \text{Spec } \bar{\mathcal{F}}_2 \]

\[ E_2^{k,0} = H^0 (Gal (\bar{\mathcal{F}}_2 / \mathcal{F}_2), H^2 M) \]

(Objects without \( C_0 \) are extended by scalars to \( \bar{\mathcal{F}}_2 \) schemes)

\[ \Rightarrow H^{2n-2} R \prod \mathcal{M}_0 \]

This is continuous cohomology - comes from inverse limit of such objects for \( \mathcal{F}_2 \) sheaves.
cell into perverse cell \( \overline{\mathcal{C}} = U \leq \mathcal{C} \leq \ldots \leq \mathcal{C}_k \) \\
\( s(\mathcal{C}) \leq s(\mathcal{C}) \) \\
\( \text{one of } \mathcal{C}_i \text{ is } \mathcal{C} \) \\
\( \Rightarrow \) get another stratification. 

**Theorem** 

\( H^*_C (X, \mathbb{C}) = \bigoplus \{ \mathbb{C} \text{ of degree } -\dim C, \mathcal{C} \subset C \} \)

(If we replaced \( s \leq s(C) \) by \( \geq \) we'd get \( H \), with compact support ...)

Thus \( F \mapsto \bigoplus H^*_C (X, F) \) will be our fibre functor – this is decomposition into isotropic components, graded piece of filtration corresponding to such object \( P_C \) ...

**Proof**

1. \( \mathcal{C}' \) of type \(*\) : \( \text{then } C \cap \mathcal{C}' = \emptyset \) unless \( \mathcal{C}' = \mathcal{C} \)

\( C = C' \Rightarrow C \cap \overline{\mathcal{C}} \), so taking \( H^*_C \Rightarrow P_C = \bigcap_{C \subset \mathcal{C}} \mathcal{C} \quad \mathcal{C} \subset \overline{\mathcal{C}} \) ...

2. \( \mathcal{C}' \) of type \(!\) : \( P_C \) is extension by zero – so by Verdier duality we need to compute \( H^*_C (\overline{\mathcal{C}}, i_\mathcal{C}'') \) \( \otimes \mathbb{C} \)

set constant strat on closure...

\( \text{case a: } C \subset \overline{\mathcal{C}} \). if \( C \ni \mathcal{C}' \) then \( \mathcal{C} \otimes \mathcal{C}' \) contractible

But \( H^*_C \) of difference of compact contractible spaces vanishes (by long exact sequence of \( H^*_C \)).

Why are these contractible? by long exact sequence of \( H^*_C \).

They are actually geometric realization of sets of cells:

\( \overline{\mathcal{C}} = U \leq \mathcal{C} \leq \ldots \leq \mathcal{C}_k \) \( \Rightarrow \{ \mathcal{C} : \mathcal{C} \in \overline{\mathcal{C}} \} \)

\( \overline{\mathcal{C}'} = U \leq \mathcal{C} \leq \ldots \leq \mathcal{C}_k \) \( \Rightarrow \{ \mathcal{C} : \mathcal{C} \leq \overline{\mathcal{C}}, \mathcal{C} \ni \mathcal{C}' \} \)

\( C \ni \mathcal{C}' \) : one is closure of the other
$\{0 \to (\text{Hom}^{-1}(K_0, L))_F \to \text{Hom}^{-1}(K_0, L) \to \text{Hom}(K_0, L)^F \to 0\}$

F is geometric. $\text{Frob} \in \text{Gal}(\mathbb{Q}_p/\mathbb{Q}_l)$, inverse to arithmetic $\text{Frob}^{-1}$.

**Definition**: $F/\lambda$ is called pure of weight $w \in \mathbb{Z}$ if for $X \in \mathbb{Z}$, $L \to \text{Hom}(\mathbb{Q}_p, \mathbb{Q}_l)$ is an isomorphism.

$F^\sigma : F \to \mathbb{C}$ has all eigenvalues algebraic numbers, all eigenconjugates have absolute value $\frac{2^w}{\lambda}$.

Complex absolute value, via some isomorphism $\mathbb{Q}_p \to \mathbb{C}$.

**Definition**: $F/\lambda$ is mixed if it is a filtration such that all graded factors are pure.

Conjecturally all $F^\sigma$ are mixed:

- Define $\text{D}^b_{\lambda}(X, \mathbb{Q}) = \{ K \in \text{K}: H^? K \text{ mixed} \} \subset \text{D}^b(X, \mathbb{Q})$.
- Stable under $f_+, f^\sigma, f^\vee, f^\vee, f^\vee, f^\vee, f^\vee$, $\text{RHom}$, Verdier duality, $\text{R}^? f_+ f_+$, $\text{R}^? f^\vee f^\vee$.
- perverse $f^\vee$-structure can talk about mixed perverse sheaves.
- Any subobject of a mixed perverse sheaf is mixed.
- $\text{D}^b_{\lambda}$ is triangulated & has mixed perverse sheaves.

1) $K_0$ has weights $\leq w$ if $\text{H}^i K_0$ has weight $\leq w + i$.

2) $K_0 \in \text{D}^b_{\lambda}$ if $\text{D}K_0 \in \text{D}^b_{\lambda}$.

**Proof**: $12/2$
Theorem (Bel'm). I) $f_1, f^*$ respect $D_{x,w}$

II) $f_1, f^*$ respect $D_{x,w}$

II) $RHom(D_{x,w}, D_{x,w}) < D_{x,w}$

III) $D : D_{x,w} \rightarrow D_{x,w}$

Hard part (Weil II): Show $f_1$ statement...

Proof. Denote $\alpha : x_0 \rightarrow Spec \hat{F}_2$.

Recall $\mathbb{R}Hom(K, L)_0 < D_{x,0}$ by Theorem.

We get weights $x_i$, $i > 0$ are non-negative.

Recall $x : 0 \rightarrow (Hom^{x_i}(K, L)) \rightarrow Hom^x(K, L)$ is 0.

In particular, $Hom^x(K, L)$ is zero.

Theorem (Bel'm). For mixed perverse sheaf $\mathcal{X}$, then $F \in D_{x,w}$

$\Rightarrow \forall V \rightarrow x_0$ affine étale, $H^0(V, F)$ has weight $w$ Caer alg closure.

Note: $\Rightarrow$ easy from previous thing: for étale map $f_1 = f^*$.

(\leq) hard: induction on dimension, resolving cycles...

Theorem. For perverse $F \in D_{x,w}$ same true for any subquotient.

Proof. Quotient case: $0 \rightarrow G : \forall W \rightarrow x_0$ affine étale.

$H^0(W, F) \rightarrow H^0(W, G)$ (corollaries from perverse

shoot on affine étale vanish alg closed $>$ 0)

So $F \in D_{x,w} \Rightarrow G \in D_{x,w}.$

Sub case $G_0 \rightarrow F_0$.
$H^1(U_0, F_0/G_0) \to H^0(U_0, G_0) \to H^0(U, F_0)$

can assume $u=0$ by shifting by constant systems on $U$, nonconstant on $U_0$ with given Frob action.

By coexact case $H^{-1}(U_0, F_0/G_0)$ has weight $\geq -1$. 

**Trick:** now take $G_0 \otimes F_0 \to F_0 \otimes F_0$

$\Rightarrow G_0 \otimes F_0$ has weight $\geq -1 \Rightarrow G_0$ has weight $\geq -\frac{1}{2}$

$\Rightarrow > 0$!

**Cor:** $j: U_0 \to X_0$ affine, $F_0$ perverse on $U_0$, $u \in [0, \frac{1}{2})$.

$\Rightarrow \dim F_0$ has same $\leq u (3w)$.

**Proof:** $j^* F_0 \to j^* F_0 \to j^* F_0$

preserves $sw$, $j^*$ preserves $\geq u$.

**Cor:** simple mixed perverse sheaves are pure.

**Proof:** $F_0 = \bigoplus \tilde{F}$, $\tilde{F}$/affine subspace of subobjects.

$\Rightarrow$ irreducible, mixed $\Rightarrow$ $\tilde{F}$ pure.

**Theorem:** Every mixed perverse sheaf $\tilde{F}$ has a unique increasing filtration $W$ s.t.

$\tilde{F} / W_0$ is pure of weight $\geq 0$.

$\Rightarrow$ every morphism is strictly compatible with $W$’s filtration.

**Proof:** use Fact. In abelian category $A^+$, every object has

finite length, $\mathcal{F}$ we have a partition of simple objects $\mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^-$ s.t.

$\text{Ext}^1(\mathcal{F}^+, \mathcal{F}^+) = 0$.

$\Rightarrow$ subcategory $\mathcal{F}^- \subset \mathcal{F}^+, \mathcal{F} \subset \mathcal{F}^+$.

$\Rightarrow$ filtration on every $\mathcal{F}^-$.

$\Rightarrow$ every morphism is strictly compatible.

So use $\mathcal{F} = \text{simple objects}$ weight $\leq 0$.

$\Rightarrow$ etc.

**Theorem:** $F_0 / U_0$ pure perverse $\Rightarrow \mathcal{F} = \oplus j^* F_0 [d]$, $u \in U_0$.
\textbf{Proof.} Take \((\mathcal{E}\) all rel. simple subobjects: \(\mathcal{F}')) \Rightarrow \mathcal{F}' \subseteq \mathcal{F}
\mathcal{F}'\text{ defined over }\mathcal{F}_0\text{ - closed under }\text{Frob}.\n0 \to \mathcal{F}_0 \to \mathcal{F}_0/\mathcal{F}' \to 0\text{ Both part of same weight - this extension thus splits over algebraic closure.}\n\text{since } \text{Hom}'(\mathcal{F}_0/\mathcal{F}', \mathcal{F}_0') \Rightarrow \text{Hom}'(\mathcal{F}/\mathcal{F}', \mathcal{F}_0')\n\Rightarrow \mathcal{F} = \mathcal{F}'\oplus\mathcal{E}\text{, we must have a simple subobject }\Rightarrow\text{ contradiction.}\n
\textbf{Theorem.} \text{K} < \mathcal{D}_n \iff \text{Hi}_k \text{ has weight } w_k \text{ (same for }\geq 0).\n
\textbf{Theorem.} \text{K} \text{ is pure of weight } \geq 0 \iff \text{Hi}_k \text{ pure of weight } \geq w_k;\text{ iff } \text{Hi}_k \text{ pure of weight } \geq w_k.\n
\textbf{Theorem.} \text{K}_0 \text{ pure } \Rightarrow \text{K} = \bigoplus \text{Hi}_k [-i] \text{ (Decomposition).}\n\text{Proof (using previous two): consider exact }\n\mathcal{P} < \text{K}_0 \to \mathcal{P} \subseteq \mathcal{F} \to \mathcal{P} \text{ Hi}_k [-i] \to \mathcal{P} \text{ K}_0[-i]\n\text{Assume } w_0. \text{ We wish to show middle term splits.}\n\text{All are pure of weight } \geq 0 \text{ by cohomology description.}\n\text{Hom}'(\mathcal{F} \otimes 0, \mathcal{F} \otimes 0) \text{ goes to } 0 \text{ over alg. closure}\n\text{So thick arrow goes to } 0 \text{ L sequence splits.}\n\text{Proof of } \ast : \ast \text{ is converse.}\n
\textbf{K. Rietsch} \text{ - Springer Correspondence 12/4}\n
\textbf{Setting:} \text{G reductive linear alg. grp } /k = \overline{k} \text{ (usually } \text{G}_0\text{)}\n\text{Originally (1979, Inventiones) Char } k > 0,\text{ unipotent } \Rightarrow\n\text{B}_n = \prod B^2 \text{ (if } n \text{ even) Springer Res}\n\text{A}_n (u) = \frac{Z_n (u)}{Z_n (u)}
$\mathcal{B} = \mathcal{C}/\mathcal{B}
$
Springer resolution: $\tilde{\mathcal{G}}_{\text{uni}} := \{ (u, B) \in \mathcal{G}_{\text{uni}} \times \mathcal{B} \mid u \in B \}$

Resolution: small map from smooth variety (birational map).

$\mathcal{G}_{\text{uni}} := T \ast \mathcal{B} = \mathcal{G}B \times \mathcal{B}^*.$

$H^1(\mathcal{B}_u, \mathcal{F}_L)$ gets $\mathcal{A}_2(n)$ action. Springer constructs rep of $\mathcal{A}_2(n) \times W$

Identity rep: $H^\text{top} (\mathcal{B}_u, \mathcal{F}_L) = \bigoplus_{\mathcal{W}} V_{u, \mathcal{W}} \otimes \rho$

$V_{u, \mathcal{W}}$ multiplicities of reps.

Theorem: The $V_{u, \mathcal{W}}$ are irreps (possibly 0) of $\mathcal{W}$ and $\{ (u, \mathcal{W}) \}/V_{u, \mathcal{W} \otimes 0}$ up to conjugacy $\sim \mathcal{W}$. We'll denote the set of such pairs and conjugacy $\sim$.

Lusztig: $\mathcal{G}_{\text{reg}} := \{ (g, B) \in \mathcal{G} \times \mathcal{B} \mid g \in B \}$

Greene: $\mathcal{G}_{\text{reg}} := \{ g \in \mathcal{G} \mid \text{reg s.s. elements} \}$

$\mathcal{G}_{\text{reg}} = \{ (g, xT) \in \mathcal{G} \times \mathcal{B} \mid x^{-1}gx \in \mathcal{G}_{\text{reg}} \}$

- determined for $s$ by pairs $(s \times B)$: stabilizing torus $W$ acts on $\mathcal{G}_{\text{reg}}$ by $(g, xT) \cdot W = (g, xwT)$

with orbits the fibers of the map.

$L := (\mathcal{G}_{\text{set}})$ pushforward of constant sheaf $\Rightarrow$

loc sys s.s. on $\mathcal{G}_{\text{reg}}$ with left $\mathcal{W}$ action

rank = 1 $\otimes$

$\Rightarrow$ take IC extension, $\mathcal{I}C (\mathcal{G}, L)$ $[\text{dim } s] = \text{jim } L$

is perverse sheaf on $\mathcal{G}$ with action

Prop (Lusztig) $\text{jim } L = \bigoplus \mathcal{F}_L$ in $J$

Proof: $\text{it is small.}$ (later..?)
\( \mathcal{V} \) \( \mapsto \mathcal{G} \) connected

\[ \Rightarrow \text{ get } W \text{ action on stacks } \mathcal{X} (\mathbb{P}_x, \Phi_x) = H^i (\mathcal{O}_x, \Phi_x) \]

\[ x_0 = \mathcal{X} (\mathcal{O}_x, \Phi_x) \]

**Special case** \( x = e \) (identity) \( \Rightarrow \mathcal{O}_x = \mathcal{O} \), set

\( W \) action on \( H^j (\mathcal{O}) \xrightarrow{\phi} H^j (\mathcal{O}_x) \) (left side: \( x \)-forms)

- classical \( W \) action. This is isomorphic to \( \Phi_e \) \([ W] \)
- as \( W \)-module get regular rep from \( H^* (\mathcal{C}(T)) \)
- and \( \Phi_e \) is equivalent under \( W \) action

**Berho-MacPherson's Theorem**

- Set \( V = \dim \mathcal{B} \)

1. The \( \Phi_e [2v] \) \( \mathcal{G} \) \( \cong \) \( \Phi(V) \) \( \cong \) \( \mathcal{X}(C, \mathcal{E}) \) \( [ \dim C ] \)

   where \( C \) is unipotent class, \( \mathcal{E} \) is local system on \( C \), \( G \)-equiv

   - \( \phi : \Phi(V) \) \text{ semi-simple}

2. The \( \mathcal{V}, \mathcal{E} \) are images of \( W (C \circ 0) \) and

   \( \{ \mathcal{O}_C, \mathcal{E} \} \mapsto \mathcal{W} \)

**Remark** \( (C, \mathcal{E}) \leftrightarrow (w, \mathcal{P}) \)

- Note \( \mathcal{T}_I : \mathcal{Z} \rightarrow G \) is

   - \( G \)-equivariant

   - \( G \)-equiv loc systems on a homogeneous \( G \)-space \( C \); \( \mathcal{G} \)

   - reps of \( A_0 (w) = Z^0 (w) / Z^0 (w) \):

   \begin{align*}
   & 6 \circ \mathcal{G} \rightarrow C \quad \mathcal{T}_I (C) \rightarrow \mathcal{T}_I (C) \rightarrow \mathcal{T}_I (G_w) \rightarrow \mathcal{T}_I (G) = 1
   
   & - \text{explains this over } C.
   
   & A_0 (w)
   
   \end{align*}

**Lemma:** \( \mathcal{T}_I : \mathcal{G}_{\mathcal{M}_{\mathcal{N}}} \rightarrow \mathcal{G}_{\mathcal{M}_{\mathcal{N}}} \) is semi-simple

Consider \( \mathcal{E} = \{(g, B, \mathcal{E}_2) \mid g \in \mathcal{G}_{\mathcal{M}_{\mathcal{N}}}, \mathcal{E}_2 \mathcal{P}_{\mathcal{N}_{\mathcal{P}}} \} \)

\[ \mathcal{O} \xrightarrow{q} \phi \]

\( 6 \)-orbits:

\[ \mathcal{G}_{\mathcal{W}} = B \times \mathcal{B} \quad \mathcal{G}_{\mathcal{W}}: \text{vector bundles over } \mathcal{G}_{\mathcal{W}} \]

\[ \mathcal{Z} = \mathcal{I}_I \mathcal{Z}_w, \quad \mathcal{Z}_w = \mathcal{P}^{-1}(\mathcal{O}_w) \text{ decomposes into irreps } \]

\[ \dim \mathcal{Z}_w = \dim \mathcal{G}_{\mathcal{W}} + \dim (U \times W \mathcal{V}_{\mathcal{W}^{-1}}) = 2v \quad (U = N) \]

\[ \dim 6 = \dim (B \times \mathcal{B}_{\mathcal{W}^{-1}}) \]

\[ 2v \]

\[ \dim \mathcal{Z}_w = \dim \mathcal{Z}_w + \dim (U \times W \mathcal{V}_{\mathcal{W}^{-1}}) = 2v \quad (U = N) \]

\[ \dim \mathcal{Z}_w = \dim \mathcal{Z}_w + \dim (U \times W \mathcal{V}_{\mathcal{W}^{-1}}) \]

\[ \Rightarrow \dim \mathcal{Z}_w = \dim \mathcal{Z}_w \quad \text{in fact they're equal} \]
- Uses classification of unipotent classes.

Take \( u \) regular unipotent, over here map is 1-1.

\( \dim \mathcal{G}_u = \dim \mathcal{Z} = 2V \).

Semi-simplicity: and \( \dim \mathcal{C}_u / \dim \mathcal{B}_u \geq \frac{1}{2} \), \( \mathcal{G}_u \). 

- Suppose otherwise.

\( \Rightarrow \dim \mathcal{g}^{-1}(V) > 2^{\frac{1}{2}} + (2V - 1) > 2V \) contradiction.

**Proof of Borel-Matthieu:** we know \( \mathcal{G}_u \) is

6-2- \( \mathcal{G} \) proper set \( = \mathcal{T}, \mathcal{G} \) is 1:

(Ad/House change). It is semi-simple since \( \mathcal{T} \) is prime

- decompose it as in 1).

- \( \mathcal{V} \) acts on \( \mathcal{K}_x \) by aut., must preserve simple parts \( \Rightarrow \)

acts on multiplicity spaces.

\( \mathcal{Q}[\mathcal{V}] \rightarrow \mathcal{End} \mathcal{K}_x = \oplus \mathcal{End} \mathcal{K}_x \).

Claim: \( \mathcal{V} \) is isomorphic.

\( \Rightarrow \) each occurs once and each is irreducible, since it's regular.

**Inclusion:** localise at identity \( e \) of \( \mathcal{G} \):

\( \mathcal{Q}[\mathcal{U}] \rightarrow \mathcal{End} \mathcal{K}_x \) \( \rightarrow \mathcal{End} \mathcal{H}^1(\mathcal{V}) \).

\( \mathbf{B} \) is injective.

from before \( \Rightarrow \mathcal{U} \) is injective.

[\( \mathcal{B} \) is regular \( m \) has no kernel.]

Surjectivity: show \( \dim \mathcal{U} \mathcal{K}_x \leq 1 \)

\( \dim \mathcal{H}^1 \mathcal{K}_x = \Sigma \dim (\mathcal{V}_x)^2 \). Write \( \mathcal{d}_x = \dim \mathcal{V}_x \).

Look at tor cohomology, localise derivs. \( \mathcal{V}_x \) in

\( \mathcal{H}^2 \mathcal{K}_x (\mathcal{V}_x) = \mathcal{P} \mathcal{V}_x, \mathcal{E}_x \mathcal{K}_x \rightarrow \mathcal{U} \).

Numerator here = 0

\( \mathcal{H}^2 \mathcal{K}_x (\mathcal{V}_x) \) has \( A_\mathcal{P}(x) = \mathcal{K}_x \).

\( \text{Multipliciy (P :} \mathcal{H}^2 \mathcal{K}_x (\mathcal{V}_x))) \text{ dim } \mathcal{V}_x \).

\( \forall x, \mathcal{P} \rightarrow \mathcal{E}_x \mathcal{K}_x \rightarrow \mathcal{C} (\mathcal{E}_x, \mathcal{E}_x) \) for \( x \in C \), which

is the representation \( \mathcal{P} \); it is the sheaf of \( \mathcal{E}_x \).
\[ \sum \chi_{(x)} \left| \frac{I(\chi_x) \chi(\chi_x)}{\lambda(x)} \right| \]  

\[ = \dim(\mathcal{E}(H^{2d}, \mathcal{O}_x, \overline{\phi})) f(x) \]  

\[ I(\chi_x) = \text{irred components} \]  

\[ |W| \to \text{follows from study of } Z' \text{ of geometric struc} \]

\[ Z' \]  

Preimages of \( C \), \( Z'(C) = Z(C) \). So \( B, B' \to C \in \mathcal{Z} \), made up of \( \text{irred components} \)  

\[ \Rightarrow 1\nu \leq \sum \frac{1}{|I(\chi^{-1}(C))|} \]  

but \( \chi^{-1}(C) = G \times \mathcal{B}_x \times \mathcal{B}_x \) \( \Rightarrow A(\mathcal{E}) \) orbits of components of \( I(\chi_x) \)  

\[ I(\chi^{-1}(C)) = \left( \frac{I(\chi_x) \times I(\chi_x)}{|A(\mathcal{E})|} \right) \]

**Theorem**: \( \exists \xi \in L^b(C_0, \mathbb{C}) \) has weights \( s_\nu \implies \frac{1}{v_i} \) every \( H_i \xi \) has weights \( s_{\nu_i} \) (since \( \nu_i > \nu \)).

**Lemma**: Assume \( \mathfrak{S} \) perverse \( \text{incl } \xi \)  

Then \( \xi \) has weights \( s_{\nu} \implies \forall \xi' \subset \xi \) \( \text{irred} \) subvariety of \( \text{dim } d \) \( \exists u_0 < \xi \) open dense (Zariski), s.t. weights of \( \mathcal{H}^{i-\nu} u_0 \) \( \leq w-\delta \).  

[Can check weights point-wise, or open & closed \( \xi \); 

This tells us we only have to check 'dimension' \( \subseteq \) 

\(-\text{note rel. perversity of } \xi \)]

**Proof**: \( (\Rightarrow) \) is immediate.

\( (\Leftarrow) \) Assume this is not so \( \Rightarrow \exists \text{ simple perverse quotient } \xi_0 \), fibre of weight \( W_1 > W \) (by canonical filtration with pure quotients.)

\[ \xi_0 = j_1 \ast L_0 [\mathbb{D}] \]  

\( \text{irred loc systs} \) \( \text{on } u_0 < \xi_0 \subset \xi \) open smooth affine \( \to \) as we wish. \( L_0 \) pure of weight \( W_1 - \delta \).
Assume we have a cone $\lambda_0 \leq \lambda \Rightarrow \lambda_f$

$\text{H}^{-d} \rightarrow \text{H}^{d+1} \rightarrow \text{H}^{d+1} \rightarrow \lambda_f$

latter vanishes over general point of $\lambda_f \leq \lambda_0$ or $\lambda_0 = \lambda_f$

$\Rightarrow \text{H}^{-d} \rightarrow \text{H}^{d+1} \rightarrow \lambda_f \rightarrow 0$

So $\text{H}^{d+1} \lambda_f$ has weight $\leq \nu$, but we know it's pure of higher wt $\Rightarrow$ contradiction.

Proof of Theorem

$\Rightarrow$ immediate: $A \rightarrow A \rightarrow B$ exact

together with $A, B$ of wts $\geq \nu \Rightarrow$ same for $X$.

We can construct $K_0$ from its cohomology in this fashion ...

$\Rightarrow )$ consider proof by descending induction in $i$

that $\Phi: K_0 \in \text{D}^{w, i}$

Assume true $i > 0$. Then $\Phi \circ K_0 \in \text{D}^{w, i}$

Using shifts assume $n = 0$.

$K_0, \Phi \circ K_0 \in \text{D}^{w, i}$

Use exact sequence (4).

$\Rightarrow \Phi \circ K_0 \rightarrow K_0 \rightarrow \Phi \circ K_0$

$\Rightarrow \Phi \circ K_0 \in \text{D}^{w, i}$

Consider $\Phi \circ K_0 \rightarrow \Phi \circ \Phi \circ K_0 \rightarrow \Phi H^0 K_0$

Restrict to $\lambda \leq \lambda_0$ irreducible.

$H^{-d} \Phi \circ \Phi \circ K_0 \rightarrow H^{-d} \Phi H^0 K_0 \rightarrow H^{-d+1} \Phi \circ \Phi \circ K_0$

$\nu \leq w + d$

$\Rightarrow H^{-d} \Phi H^0 K_0$ has wts $\leq w + d$

$\Rightarrow$ (lemma) $\Phi H^0 K_0 \in \text{D}^{w, i}$.

Passing from characteristic $p$ to $C$

Suppose $B = \lim B_i$, all noetherian, $X \in \text{of finite presentation}$

(secure or morphism of fin. type, or finite sheaf, or...)
This is obtained by base change from some $X_i$, $i \gg 0$. $X_i$ essentially unique - two such choices become isomorphic for $i$ large enough.

$X/\mathcal{E}$ scheme of finite type, $\mathcal{E} = \text{lim} A$ over $A$ of finite type, $\mathcal{E}$ flat $A$-scheme $\mathcal{E}$ smooth
cross to cofinal family: pass from $A$ to $A[1/\ell]$, shrinking

$X$ comes from $X_s$, $s = \text{spec} A$, $A$ large enough.

$X_s$ scheme of finite type. If $X$ was smooth, connected

$c$ can make $X_s$ be such

$f : X \to Y$ also comes from such base change for $b$ large enough.

$T$ stratification of $X$ also comes from $T_s$ of $X_s$.

can assume strata smooth geometrically connected.

$J = X - \mathcal{E}/\mathcal{E}$ scheme comes from $F_s / X_s$

(constructible sheaves.) NOT true rnr for $\mathbb{Z}$ sheaves.

\[ \text{Then (4.42): } f : X \to Y / \mathcal{E} , F / X \text{ constructible } \mathbb{Z}/\mathcal{E} \text{ sheaf} \]

\[ \Rightarrow \quad \mathcal{E} \quad \text{ scheme } , \quad \text{Then } \exists U \subset S \text{ s.t. } V \ni Z \to Z \]

\[ X_s \to Y_s \]

\[ X_s' \to X_s \quad \text{We have } g^* \mathcal{R}^e f_* F_s = R^e g^* f_* F_s \]

\[ g^* F_s = F_s / U \quad \text{and} \]

\[ R^e g^* F_s \quad \text{constructible} . \]

Moral: our functors $f_*, f^!, f^*, f^! ; \text{Tor}_r ; \text{Ext}^r$

for $\mathbb{Z}/\mathcal{E}$ constructible sheaves descend to $S$ for

$A$ large enough, commutes with base change (on $U$).

Recall that the derived category of constructible sheaves

was a limit of $D^b(X, \mathbb{Z}) < D^b(X, \mathcal{E})$

$T$ stratification of $X$ is a family of irreducible

slices on strata of $T$. Also replacing $\mathbb{Q}$ by $\mathbb{Z}/\mathcal{E}$ in above.
X, T, L / S \rightarrow X_s, T_s, L_s

Need \forall F, G of the form \text{Si}(L), \text{Ext}_2^1(F, G) to be compatible with all base changes S \rightarrow S'.

\text{Ext}_2^1(F, G) is compatible with all base changes S \rightarrow S'.

\text{Loc. const. and compat. with base change.} — true if we shrink S.

S: Spec A, choose dvr, strictly Henselian V st A \subset V ⊆ C.

Spec A \subset Spec V
X \cong X_s with residue field k(x) is k(x).

Henselian: category of étale sheaves on Spec V.

\text{Take sheaf } E \text{ on closed point. Strictness: } k(S) alg closure, k(S) \supseteq k(x).

Now pull back to V \rightarrow X_V, T_V, L_V. Take special fiber \ X_s, T_s, L_s (small s).
X \rightarrow X_V \leftarrow X_s

\text{Now consider perverse sheaves } X, T, \text{ and } L(Kt)
ji \rightarrow X, \text{ needed for def of perverse t-structure (true after refinement).}

\text{Can then achieve t-structure over } S \text{ after shrink, so that } R^\bullet j_* \text{ commutes with base change } \rightarrow \text{ so our equivalences of categories (Leray) are compatible with perverse t-structures.}

(Perv. truncation uses } R^\bullet j^* \text{ in et al.)}
**Definition:** If $\mathcal{F}$ is of geometric origin if $\mathcal{F}$ is smallest set containing $\mathcal{C}$ on pt $L$ which is stable under taking constants of $\mathcal{H}^1T$, $\mathcal{F} = R\mathcal{F}_0, R\mathcal{F}_1, R\mathcal{F}_2, RF$,

$(f: X \to Y \mathbb{A}^1)$, $\mathcal{H}^1T(A \circ B), \mathcal{H}^1T(A \circ B)$.

**Definition:** $K \subset \mathcal{D}_{b}(X(\mathbb{A}), \mathcal{O})$ is semi-simple of geometric origin if $1 \subset \mathcal{F}_{1}(\mathbb{A})$, $\mathcal{F}_{1}$ set of geometric origin.

**Theorem (Decomposition Theorem):** If $f: X \to Y$ is proper, $K$ semi-simple of geometric origin $\Rightarrow$ $Rf_*K$ is semi-simple.

**Lemma:** If $f$ is semi-pure, $f_*$ is semi-simple of geometric origin.

If $S = \text{Spec}(A \circ L \circ C)$ is $f_c$-proximate, then

$\mathcal{O}_X \otimes \mathcal{O}_Y \leftarrow \mathcal{O}_Z \leftarrow \mathcal{O}_{X_2} \mathcal{O}_{X_3}$.

**Proof:** True for constant sheaf, modified after applying various functors, take components of resulting mixed perverse sheaves $\Rightarrow$ pure sheaf again.

If $K = K_1 \otimes K_2$, $f_*$ is pure $\Rightarrow f_*|_{K_1}$ is pure.

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S. Kirillov - Kazhdan-Lusztig Theory

$k = \#_F$, $F$ Frobenius, $G$-reductive with alg group $/k$.

$B = Borel, T$-torus, $W = W_b$, $G_{ST}$ split dual $\rightarrow W$.

Maximal split situation: Facts from [12].

$B$ - variety of all par. $B \times B / = W$.

$(B_1, B_2) \rightarrow (B_0, B_0)$: write $B_0 \rightarrow B_2$. 

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\( B \times B = U \Phi \)

\[ B \xrightarrow{\mu} B_2 \xrightarrow{\nu} B_3 \Rightarrow B \xrightarrow{\kappa \mu} B_3 \]  

if \( \text{length}(\mu \kappa) = \text{length}(\mu) + \text{length}(\kappa) \)

Hecke algebra -

\[ \mathcal{P} = \{ \text{factors } f : B^F \rightarrow \Phi \} \]

(1) \[ B^F \text{ \textbf{B} closed } / \text{F} \]

\[ T_\mu : B \rightarrow B \]

\[ (T_\mu f)(B) = B \rightarrow B \]

Theorem \( \mathcal{H} = \Phi \otimes \langle T_\mu \rangle \) satisfying:

1. \( \mathcal{H} = \text{End}_{B^F} \Phi \)
2. \( s \)-simple reflections \( \Rightarrow \mathcal{H} \) generated by \( T_s \)

\[ T_\mu : T_s = T_\mu T_s \]

\[ T_s^2 = 2 T_s + (q-1) T_s \]

(2) \( \mathcal{H} = \text{G-invariant functions on } B^F \times B^F \)

\[ (T_\mu f)(B) = \sum_{B \in \Phi B} f(B_2^B f) \]

\[ \Phi = B^F \times B^F \]

\[ B^F \]

Tw = char. functions of \( \mathcal{H} \), satisfy same relation as \( T_s \).

\( \phi : \mathcal{H} \rightarrow \mathcal{H} \). Structure constants are polynomials in \( \mathcal{H} \), so we can consider \( \mathcal{H} \) as an algebra over \( \Phi \otimes \mathbb{Z}^2 \).

\( \phi : T_\mu \rightarrow T_\mu^{-1} \)

\( \Rightarrow T_\mu^{-1} = q^{-1} T_\mu + (q^{-1} - 1) \)

Setting \( \tilde{T}_\mu = q^{-\phi} T_\mu \Rightarrow \) \( \tilde{T}_\mu = \tilde{T}_\mu + \sum_{\phi} (\ldots) \tilde{T}_\mu \)

Define \( R_{x \cdot \cdot \cdot} \in \mathbb{Z}[Z] \) by \( \tilde{T}_\mu = \sum_{x \cdot \cdot \cdot} q^{(\ldots)} R_{x \cdot \cdot \cdot} \tilde{T}_\mu \)

Theorem (Kohler-Labesse) \exists \text{ a basis } C' \text{ in } \Phi \text{ such that}

1. \( C' = C' \)
2. \( C' = \tilde{T}_\mu + \sum_{x \cdot \cdot \cdot} (\ldots) \tilde{T}_\mu \)

with \( (\ldots) \) polynomials in \( q^{-\phi} \mathbb{Z}[Z] \).
We rewrite the first formula as:

\[ C_i = \sum_{y \in \mathbb{Z}} \frac{f(x)-f(y)}{x^2} P_{y, n} T_y \]

where \( P_{y, n} \in \mathbb{Z}[z] \), \( \deg P_{y, n} \leq \lfloor (n) - \frac{1}{2} \rfloor \)

(follows from estimates on powers of \( z \) in \( C_i \).)

Kazhdan-Leitze's polynomials:

\( P_{y, n} \) describe multiplicity \( [M_{\text{reg}} : L_{y, x}] \)

\( y \in \mathbb{Z} \) regular.

\( B = U B_{\infty} \). Denote \( (\tilde{P}_x) \) constant step on \( B_{\infty} \).

\( B_{\infty} \xrightarrow{\text{step}} B_{\infty} \). Set \( F_{\infty} = \lim\sup (\tilde{P}_x) \).

\( H^i_x(F_{\infty}) \) \( x \in B_{\infty} \). \( y \in \mathbb{Z} \). \( F_{\infty} \) is coh. constructible wrt \( B_{\infty} \).

**Theorem 1:**

\( \tilde{P}_x, F_{\infty} = 0 \); add.

2. Eigenvalues of \( F_{\infty} \) on \( H^2_x(F_{\infty}) \)

\( x \in B_{\infty} \).

(3) \( P_{y, n}(q) = \sum \dim H^2_x(F_{\infty}) \cdot q^i \)

\( = \text{Tr}(F_{\infty}, H^*(F_{\infty})) \)

\( \Rightarrow \) all coefficients of KL polynomials are non-negative integers.

**Proof Idea:** Find \( U \)-nbhd of \( B_{\infty} \) in \( B_{\infty} \) and calculate \( \text{Tr} (F_{\infty}, H^*(U, F_{\infty})) \).

First, calculate \( B \)-cell of \( U \):

\[ \alpha^V = \{ B \cap B \text{ is opposite to } B \} \]

\( \psi_B = k \).

\( \psi_B = \text{long word} \)

- big cell -

\( \exists U = B_{\infty} \cap \alpha^V \) - nbhd of \( B_{\infty} \).

- can choose so that \( \alpha^V \Rightarrow \psi_B \)

\( U = \bigcap_{x \in \mathbb{Z}} B_x \cap \alpha^V \) (?)

\( \Rightarrow \) \( \Lambda = \sum_{Y \in \mathbb{Z}} \text{Tr}(F^*_{\infty}, H^*(F_{\infty})) = \sum_{Y \in \mathbb{Z}} \sum_{x \in B_x \cap \alpha^V} \text{Tr}(F^*_{\infty}, H^*(F_{\infty})) \)

(Lebesgue principle)
Prove by induction on $y$; descending from $w$, so far assume proven for all $y < z \leq w$.

\[ q = \sum_{y \leq z \leq w} x_{y,z} \cdot \text{Tr} \left( F^*, H^* _{y,z} (F_w) \right) \]

\[ + \sum_{y < z < w} x_{y,z} \cdot 1 \text{B}_x = \alpha y F^* \cdot \dim H^* _{y,z} (F_w) \]

Recall $F_w = \text{fix} \left( \overline{\alpha} \right) w$ and $B = \text{V} \overline{\alpha} w$.

We're proving: for $y < w$, $x \in B_{w \leq y} \neq 0$.

We've fixed $w$ and do induction on $y$.

$U = B_w \cap \alpha y$ is a bd of $B_y$ in $B_w$.

$U$ is stable under the torus action.

The only fixed point in the origin in the affine space of $\text{aff} U$, under some identification:

\[ (x_1, \ldots, x_k) \rightarrow (a, \lambda x_1, \ldots) \quad \lambda > 0 \]

We're calculating $\text{Tr} \left( F^*, H^* _{y,z} (U \otimes F_w) \right)$

\[ = \text{Tr} \left( F^*, H^* _{y,z} (F_w) \right) + \left( \sum_{y < z < w} x_{y,z} \cdot 1 \text{B}_x = \alpha y F^* \cdot \dim H^* _{y,z} (F_w) \right) \]

More precisely first term is $\text{Tr} \left( F^*, H^* _{y,z} (F_w) \right)$.

Let's calculate recursively: $\text{Tr} \left( F^*, H^* _{y,z} (F_{w-1}) \right)$.

$\#$ of fixed points - can be calculated recursively to be some polynomial in $a_2$ - rest of term, $x$ a so polynomial in $a_2$.

Now use Verger duality:

$F^* \rightarrow F^* \quad H^* _{y,z} \rightarrow \text{aff} U$
\[ \text{Tr} (CF^*, H^*_{x}(U, F_w)) = (q^{1/n}) \text{Tr} (CF^*, H^*_{x}(U, F_w)) \]

But \( F_w \) is conic sheaf for torus action \( \Rightarrow \) relativize global homology by local homology at origin

\[ \Rightarrow \quad q^{1/n} \text{Tr} (F^*, L^*_{\mathbb{S}} (F_w)) \]

\( B^* = \) standard Borel twisted by \( \chi \) - fixed point of torus

**Lemma** Evals of Frobenius in \( H^i_{B^*} (F_w) \) have absolute value \( q^{i/n} \)

**Proof** Uses Deligne's result on purity and canonical structure.

This identity holds for all \( r \)

**Corollary** \( x = q^{1/n} \), \( x \neq \frac{1}{n} \in \mathbb{Z} \) (so no odd homologs \( x \))

**Theorem** \( \exists \lim H^i_x (F_w) q^i = P_{V, w} (q) \)

**Proof** let \( E \) be derived category of mixed sheaves on \( B \times B \), constructible w.r.t. orbit stratification. (Constant on strata)

\[ E \to \text{functors on } B \times B \text{, } C \text{-invariant}. \]

\[ F \mapsto f_{\pm} (x) = \text{Tr} (F^*, L^*_{\pm} (F)) \]

\[ (f_{\pm}(x)) = \text{Tr} \text{ char. function of an orbit} \]

\[ L - \text{Take sheaf: constant sheaf with } F^{1/2} = \mathbb{C} \]

**Theorem** Under \( q \), \( B \)-finite identity is identified with the complex involution \( \tau \) on \( E \)

**Proof** for \( sl_2 \) case \( D(\tau(t)) = \mathbb{C} \cdot \mathbb{Q} \) \( \tau \) takes to functions \( \tau(t+1) = \mathbb{C}^{-1} (t+1) \).

\[ \tau \text{ gives our involution} \]

In general prove for reflections, extend by multiplication. (Reflection case reduces to \( sl_2 \)).
Define convolution: $\tilde{\mathcal{C}} \times \mathcal{C} \times \mathcal{C}$.

If $F, G$ are sheaves in $\mathcal{C}$, $F \times G \colonequals \mathcal{P}_{13} \times \mathcal{P}_{23} \times \mathcal{C}$.

Since all is proper, the claim $\varphi(\mathcal{C} \times \mathcal{C}) = \varphi(F) \otimes \varphi(G) \otimes D(F \times G) \cdot D(C) \cdot D(G)$.

First is equivalent to Bertini's point formula, trace theorem.

This our sheaves $\mathcal{F}_u \to \mathcal{F}_w$ with:

1. $\mathcal{F}_w = \mathcal{F}_u$
2. $\mathcal{F}_w = \mathcal{F}_u + \mathcal{F}_w$
3. Estimates on power of $q$ in coefficients; come from perversity of sheaf; homologies vanish for some indices. Also use eigenvalues of Frobenius on perverse sheaves.

$\Rightarrow$ $\mathcal{F}_w$ is the K-L basis.

Coefficients of K-L polynomials are non-negative.

$\mathcal{F}_w$ give nice basis of regular basis of Frobenius algebra. For other reps, described by purely combinatorial data ("W-graphs") get similar combinatorial basis.

In case 1: There are $\text{Ind}_{\mathcal{F}_w} \mathcal{F}_w^{	ext{tr}}$

$\Rightarrow$ Grassmannian.

Then we get formulas for Hodge rings which depend well on $q$, which we can then specialize to seek interesting values of $q$.

Elementary construction of perverse sheaves 12/13

MacPherson-Vilonen, Inv. Math. 09/1986

$X =$ tori, stratified space, $\mathcal{P} =$ middle rows, $\mathcal{Y}$-stability condition.

$\mathcal{P}(X) =$ perverse sheaves, constructible.
Inductive construction of $P(X)$ from category of local systems. Sufficient to construct $P(X)$ from $P(X\times S)$, let $S$ be $X$ with $S$ a closed stratum (remove stratum to make it closed)

\[ A \rightarrow B \text{ categories, } A \xrightarrow{F} B \text{ functor, } F \xrightarrow{\phi} \]

\[ \text{Nat. transformation } \Rightarrow \]

\[ C(F, G, T) = \begin{cases} \text{object} : (A, B, m, n) & \rightarrow \text{B}\text{\&B} \\ \text{Morphisms} : (A, B) & \rightarrow (A', B') \\ \text{pair, } G, \phi \text{ with} \\ A \rightarrow A', B \rightarrow B' \\ FA \rightarrow GA \\ m \rightarrow B, n \rightarrow \end{cases} \]

1. If $A, B$ abelian categories, $F$ right exact, $G$ left exact \[ \Rightarrow C(F, G, T) \text{ is abelian, and forgetful functor } \]

\[ \text{is exact.} \]

Proof: $C$ is additive from additivity of $A, B$. Construct kernels: take $\ker F a \rightarrow \ker F a$.

- Not in our category: $\ker b$.
- $\ker F a$ needs to come from category $B$.
- Try $C(\ker a) \rightarrow C(\ker a)$ in our category.

and moves to above diagram.

$G$ left exact \[ \Rightarrow C(\ker a) = \ker G \text{ of a, } \]

Ex. Given two open sets, take our functors to be $i_! \& i^! \text{ restricted to other set } \Rightarrow \text{this construction gives sheaves on the union.}$

Assume $S$ closed, contractible. Choose $x$ = basepoint, dim $S = 2d$. 

2. $\mathcal{D}^b_c(S) \to \mathcal{D}^b(\text{Vec spc})$: fiber at $x$ is an equivalence (S contractible). Inverse: take constant extension. Just need to prove that all extensions in $\mathcal{D}^b_c(S)$ are trivial.

- Ext of complexes only in degree 0 are cohomology, but
  $S$ contractible $\implies$ extension trivial.

3. Identify $\mathcal{D}^b_c(S)$ with $\mathcal{D}^b(\text{Vec spc})$:

\[ 0 \rightarrow \mathcal{O} \rightarrow \mathcal{O} \rightarrow \mathcal{O} \rightarrow 0 \]

$\mathcal{O}$ still contractible.

\[ i^* i^* \mathcal{O} \rightarrow \mathcal{O} \rightarrow j^* j_! \mathcal{O} \rightarrow 0 \]

\[ \Rightarrow \mathcal{O}^* \text{ is determined (up to noncanonical iso) by} \]

an element in $\text{Ext}^1_{\mathcal{D}(C)}(\mathcal{O}, i^* i^* \mathcal{O})$.

\[ \text{Ext}^1_{\mathcal{D}(S)}(i^* j^* \mathcal{O}, i^* \mathcal{O}) = \bigoplus_{n \geq 0} \text{Hom}(H^n(i^* j^* \mathcal{O}), H^n(i^* \mathcal{O})) \]

By Poincaré duality, $H^{-d}(i^* \mathcal{O}) = 0$, $H^{-d}(i^* j^* \mathcal{O}) = 0$.

\[ i^* \mathcal{O} \rightarrow i^* \mathcal{O} \rightarrow i^* j^* \mathcal{O} \rightarrow i^* \mathcal{O} \rightarrow 0 \]

\[ \Rightarrow 0 \rightarrow H^{-d}(i^* \mathcal{O}) \rightarrow H^{-d}(i^* j^* \mathcal{O}) \rightarrow H^{-d}(i^* \mathcal{O}) \rightarrow 0 \]

\[ \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \]

\[ \Rightarrow \mathcal{O}^* \text{ is determined up to iso by } \mathcal{O}_{|S \times 0} \text{ and by above} \]

exact sequence $(*)$.

4. $S$ contractible, $T = \text{(small enough) closed tub. neighborhood}$.

It is $T \to S \times X$.

Define $D^0 = T_x^{-1}(0)$: stratified space, closed in $X$. -normal slice

$T^0 = T^0 \text{ link at } x$.

A paracompact link $K^0 = \text{any closed subset of } T^0 \text{ s.t.}$

\[ \forall \rho \in \mathcal{P}(X \times S), H^d_{T^0}(K^0, \rho) = 0 = H^d_{T^0}(T^0, \rho) \]

relative: restrict to complement of $K^0$, take compact support.
5. **Theorem**: Perverse lfaces always exist.

- Let \( X = \mathbb{C}^* \cup \{0\} \)
- \( K^0 \) any point on circle
- LHS is cohomology of local system in vector groups
- LHS for RHT is \( \mathcal{O}^* \)-compact cohomology

6. \( \mathcal{A} = \mathcal{P}(X - \{\cdot\}) \quad \mathcal{B} = \text{Vect} \quad \mathcal{F} : \mathcal{G} \to \mathcal{H} \quad \mathcal{F}(\mathcal{P}^*) = H^{-d-1}(K^0, p) \quad \mathcal{G}(\mathcal{P}^*) = H^{-d}(L^0, K^0, p) \)

- \( F \) is right exact, \( G \) left exact by 4) (def. of pres. lft)
- \( \mathcal{C}(\mathcal{F}, \mathcal{G}) \) is abelian

**Theorem (5 connected)** \( \mathcal{P}(X) \xrightarrow{\mathcal{F}} \mathcal{C}(\mathcal{F}, \mathcal{G}) \) is equivalent:

\[
\xymatrix{ Q \ar[r]^-{\mathcal{F}} & \mathcal{C}(\mathcal{F}, \mathcal{G}) }
\]

\( Q \xrightarrow{\mathcal{F}} \mathcal{C}(\mathcal{F}, \mathcal{G}) \)

**Proof**

1. **exactness**: \( \mathcal{F} \) \( Q \to Q_{\text{Kes}} \) exact. \( \mathcal{F} \) also
- \( H^d(0^*, K^0, Q^*) \) vanishes in all degrees but \( -d \)
- \( d \) exact

2. **equivalence**: on objects: \( Q \leftrightarrow Q_{\text{Kes}} \) and exact
- sequence \( (X) \) as before

\[
\begin{array}{c}
\mathcal{A} \xrightarrow{\mathcal{F}} \mathcal{B} \quad 0 \xrightarrow{\ker n} \ker m \xrightarrow{\ker n} \ker n \xrightarrow{\text{Ker} n} \text{Ker} m \\
\text{Exact}
\end{array}
\]

- this gives our 5-term exact sequence

\[
\begin{array}{c}
H^d(0, L) = H^d(C^* \mathcal{L})
\end{array}
\]

Given \( A, B \) vector spaces; \( n \) to give \( C \) is same
as to give such an exact sequence.
- \( \text{faithfulness of functor: exact } \iff \text{equiv on objects} \)
- \( \text{equiv on morphisms} \quad \text{border} \)
Example: $X = C^* \cup 30^9$

Perverse shears $Q^\circ$ on $X$

\[ \longrightarrow (Q^\circ C^* \cup H^0((b, G^\circ_1)), Q^\circ_1) \longrightarrow H^0_{\text{comp}}(C^* \cup 30^9, G^\circ_1) \]

\[ \text{Denote } Q^\circ_1 = \mathbb{L} \text{ and } a = \text{monodromy } L_b \rightarrow L_b \]

\[ \text{(relative)} \]

\[ \Rightarrow \text{ diagram } L_b \xrightarrow{T} H^1_{\text{comp}}(C^* \cup 30^9, \mathbb{L}) = L_b' \text{ (Poincaré)} \]

\[ \text{Cox. - Map} \rightarrow H^1_L(U, \mathbb{L}) \oplus H^1_L(U_b, \mathbb{L}) \xrightarrow{\text{Cox.}} L_b' \text{ monodromy} \]

\[ T = -id + a \]

\[ \Rightarrow \text{ diagram } \begin{array}{ccc} A & \xrightarrow{\iota} & A \\ \downarrow m & & \downarrow m \\ B & \xrightarrow{n} & B \end{array} \]

\[ \Rightarrow \text{ (A, B, m, n) s.t. } A \xrightarrow{m} B, \text{ and } l = \text{ a invertible} \]

Re-presentations of a quiver.

\[ 12/16 \]

\[ \text{ Fourier-Deligne Transform } \& \text{ Kazhdan-Langlands } \]

\[ K - L : \text{gluing of perverse sheaves, } \text{ discrete series rep.} \]

\[ \text{J. G. L. P. Vol 5. 11. } \text{ 1980} \]

\[ \text{ Deligne-Fourier transform: (restrict b symmetric vector space)} \]

\[ \text{char } k = p \quad \Psi: F_p \rightarrow \mathbb{Q}^x \]

\[ \Rightarrow \text{ Artin-Schreier theory for constant } \mathbb{Q} \text{- sheaf} \]

\[ \text{ from } x \rightarrow x^p - x, \]

\[ \text{TFp - covering of } F_p, \text{ push forward by } \Psi. \]
Theorem 1. $F^2 = \text{Id}$ (sign comes from being symplectic $G_m$)

2. $D F_U = F_U^{-1} D$  
3. $F$ is exact but reverse structure

($\rightarrow$ $F$ induces involution of $\text{Per}(V)$).

Main technical lemma: get some transform if we replace $P_U$ by $P_V$ $\Rightarrow$ $D$ statement.

- this is a type of properness statement for $G_S$ correspondance.
- similar to impaling coming from an automorphism of $V$.

Not hard to see $F$ is right $t$-exact - using estimates on $e^t$, amplitude of $1$, $*$ vector. Then exactness follows from 1, or from 2.

Suppose $U \subset V$ open $\Rightarrow F_U : D(U) \Rightarrow \text{right exact}$

$F_U = j^* F_V$  
Have natural transformation $j^* \Rightarrow F_U$.

Can try $F_U : \text{Per}(V)$ right exact.

Consider category of pairs $(A, B)$, $A, B \in \text{Per}(V)$, $\circ : F_V A \rightarrow B$,

$\exists \circ : F_V B \rightarrow A$ with

\[ p F^2 \rightarrow \text{Id} \]

\[ F U \rightarrow \text{Id} \]

Consider category by gluing (as last time) $\text{Kel}(U)$

Example $\dim(V-U) < \frac{\dim(V)}{2}$

$\Rightarrow$ glue $(V, F) \rightarrow \text{Per}(V)$
Natural functor \[ \text{Glue}(U;F) \to \text{Per}(U) \]
\[ (\ast A, j; \ast FA) \to \mathcal{U} \]
which is equivalent when \( A \) is big enough.
(complement shouldn't contain anything lower dimensional)
(can't have \( \mathcal{U} \) supported on \( U \) with \( FA \) of size \( \mathcal{U} \).
)

\( G \) connected, simply conn., semi-simple split \( A \). \( T \subset B \), \( U = [B, D] \) def. \( U \).
\( X = G/U \).
\( U = \prod_{x \in \mathbb{R}^+} X_x \) \( R^+ \) pos. \( \mathbb{x} \) w.m.

\[ R(w) = \{ x \in \mathbb{R}^+/w \leq -x \} / \mathbb{R}^+ \]
\[ U(w) = \prod_{x \in \mathbb{R}^+} X_x \] \( \mathcal{U} \).
\( U = U_{s} \cdot U_{s \bar{s}} \)
\[ M_{s} = < X_{s \bar{s}}, X_{s} > \] subgroup \( \cong S_{2} < G. \)

If \( U \) fix \( x \).
\[ X_{s \bar{s}} \cong \mathbb{R} \to \text{unique} \to M_{s} < S_{2}. \]

Consider projection \( G/U \to G/M_{s}U_{s \bar{s}} \)
(Not \( M_{s}U_{s \bar{s}} = U_{s}M_{s} \))

\[ \begin{align*}
G/U &\to G/M_{s}U_{s \bar{s}} = Y_{s} \\
G/U &\to G/M_{s}U_{s \bar{s}} = Y_{s} \\
\end{align*} \]

- rank 2 bundle over \( G/M_{s}U_{s \bar{s}} \).

Inclusion above comes from \( G \to G/M_{s}U_{s \bar{s}} \times k^2 \)
(under \( Y_{s} \cong (G, 1) \))

\[ \begin{align*}
\text{e.g.} & \ G = S_{2}, \ G/U = k^2 \times 0, \ G/M_{s}U_{s \bar{s}} = pt \ldots
\end{align*} \]

In some \( j \), is complement to zero section

\( Y_{s} \) has canonical \( G \)-invariant symplectic form
\[ \langle, \rangle : Y_{s} \times Y_{s} \to \mathbb{R} \]
(from \( S_{2} \)-invariant \( \mathcal{U} \).
)
\[ F_{s, t} : D(X) \rightarrow (X = G/G) \]

\[ s \times F_{s, t} \text{ is!} \]

**Theorem** \( W = s_1, s_2, \ldots, s_k \) shortest decomposition \( (l = \text{longest}) \)

\[ \Rightarrow F_{s, t} = F_{s_1} \circ \ldots \circ F_{s_k} \text{ independent of choice or decomposition} \rightarrow \text{ action of braid group on } D^2(X) \]

**Discussion:** group action on category \( \gamma : G \rightarrow \text{Functor} \)

\[ F_{s, t} : \mathcal{C} \rightarrow \mathcal{C} \quad \gamma, \gamma' : \gamma \rightarrow \gamma' \quad \text{satisfying cocycle condition} \]

Braid monoid \( B_W \subseteq W \)

To give action of \( B_W \) on \( \mathcal{C} \) \( \Leftrightarrow \) (Deligne) data:

For \( w, w' \in W \)

\[ (w, w') = (w_1, \ldots, w_k) \rightarrow F_{w_1} \circ F_{w_2} \circ \ldots \circ F_{w_k} + \text{associativity condition for rings with } 1(\text{identity}) = 1 \]

We want to have: \( w, w' \in W \)

\[ F_{w, w'} \circ F_{w, w'} = F_{w' w} \]

morphism + associativity

Need to check for \( s \cdot s' = s s' \)

\[ l(s s') = (l(s) + 1) \]

\[ s \cdot s' = s s' \text{ unproven} \]

\[ s \cdot s' = s s' \rightarrow w \]

Proof of Theorem - write geometrical kernel(s) \( K(C) \) on \( X^+ \)

for \( F_{s, t} \)

\[ A - ab. \text{ category, glued from } \mathcal{C} \text{ of } \mathcal{C}_V \]

\[ \text{of } \text{Perf}(A/V) \text{ of factors } \text{PF}_{w, w} : \text{Perf}(X) \rightarrow \text{Perf}(A/V) \text{ exact} \]

\[ \text{W, W'} = \text{morphism } \text{PF}_{w, w} \rightarrow \text{A \cdot W' + commutivity} \]

\( w \text{ like rows of quiver with } \mathcal{C} \text{ vertices} \)

\( W \text{ acts on } A \text{ commuting with } \mathcal{C} \text{ action, } e \text{ action} \)

behaves correctly under \( W \).
we \( W \Rightarrow \mathcal{A} \) as \( \mathcal{A} \) Ext with \( \ast \rho \) \( W \mathcal{F} \mathcal{G}^f A \Rightarrow A \)

"Bézivin-Usui" variety type construction.

(4) \( \mathcal{A} \) has finite cotangential dimension, \( \text{Ext}^i(\mathcal{A}, \mathcal{A'}) \) for \( i \leq 0 \).

\( \Rightarrow \) action on Ext grans.

\( \Rightarrow \mathcal{K}^0(\mathcal{A}w), \langle \rangle \) \( \rho_{\mathcal{F} \mathcal{G}^f} \), action of twisted Frobenius

\( A, A' \) \( \in \mathcal{A}w \Rightarrow \mathcal{W} \mathcal{F} \mathcal{G}^f A \Rightarrow A \) and \( A' \)

\( \Rightarrow \rho_i : \text{Ext}^i(\mathcal{A}, \mathcal{A'}) \leq 0 \),

\( \langle \rangle = \Sigma(-1)^i \text{Tr} \rho_i \),

Conjecture: \( \mathcal{K}^0(\mathcal{A}w) / \ker \langle \rangle \) is finite-dimensional vector space with \( \rho \times \text{Tr} \) action. Given characters of \( \text{Tr} \) \( \Rightarrow \) reps of \( \rho \) : conjecturally whole discrete series.