Characters in Categorical Representation Theory

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Overview

Describe ongoing joint work with David Nadler (Berkeley) on categorical harmonic analysis, centered around the notion of characters of categorical representations.

- (with J. Francis) Integral Transforms and Drinfeld Centers in Derived Algebraic Geometry. JAMS 2010. Appendix, July 2012.
- The Character Theory of a Complex Group. arXiv:0904.1247.
- Beilinson-Bernstein Localization in Families. Preprint. (Summer)
- Traces, Fixed Points and Characters in Derived Algebraic Geometry. Preprint. (Fall)
- Geometry of Harish Chandra Characters. In preparation. (Spring 2013?)
- Elliptic Character Sheaves. In progress. (2014?)

Context

We will work in the context of J. Lurie's Higher Algebra but suppress ∞-categorical technicalities throughout.

For example: category will stand for an enhanced derived category (pre-triangulated cocomplete dg category).

The collection of dg categories form a symmetric monoidal ∞-category.

In particular we may speak of monoidal dg categories, module categories, etc.

For a scheme or stack X, $QC(X)$ and $\mathcal{D}(X)$ denote the (∞-)categories of quasicoherent sheaves and $\mathcal{D}$-modules on X, respectively.

G-categories

Fix a reductive algebraic group G over $\mathbb{C}$, B, N, H, W as usual.

Two types of G-actions on categories:

Algebraic G-category:

• $g \in G$ act coherently on M by functors, varying algebraically
• Comodule category M for "quasicoherent group coalgebra" $QC(G)$ (under pullback for multiplication $\mu : G \times G \to G$)

Smooth G-category:

• Algebraic G-category, with trivialization of action of Lie algebra $\mathfrak{g}$ or formal group $\widehat{G}$ (i.e., algebraic G_{dR}-category)
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Examples of G-categories

• Fix a G-space (scheme or stack) $X \Longrightarrow$

$QC(X)$ is an algebraic G-category,

$\mathcal{D}(X)$ is a smooth G-category.

Main examples: $\mathcal{D}(G/B)$ and $\mathcal{D}_H(G/N)$, $\mathcal{D}$-modules on basic affine space locally constant on torus orbits – categorifications of principal series and universal principal series representations.

• Motivating example: The adjoint action of G on $\mathfrak{g}$ gives rise to a smooth G-action on $U\mathfrak{g}-$ mod

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Algebraic Hecke Categories

For algebraic G-categories, have complete control of elementary "categorical harmonic analysis" – intertwiners, centers, Morita equivalences, characters, spectral decomposition etc.

For example, results of [BZFN] together with
• Gaitsgory, Sheaves of categories on prestacks (2012)
give the following:

Theorem: For any affine $H \subset G$,
• There are equivalences of monoidal categories
$QC(H\backslash G/H) \simeq End_{QC(G)}(QC(G/H)) \simeq End((-)^H)$
• The above algebraic Hecke category is Morita equivalent to $QC(G)$

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Smooth Hecke Categories

Fewer general results hold for $\mathcal{D}$-modules, but still have (in equivariant or monodromic settings):

Theorem: $End_{\mathcal{D}(G)}(\mathcal{D}(G/B)) \simeq \mathcal{H} := \mathcal{D}(B\backslash G/B)$, the *finite Hecke category*, which is a *2-dualizable Calabi-Yau algebra* in categories.

$\mathcal{H}$ is a categorified form of $\mathbb{C}[W]$ — analog of finite-dimensional semisimple Frobenius algebra.

Corollary: $\mathcal{H}-$ mod are smooth G-categories "appearing in principal series" — subcategory generated by module $\mathcal{D}(G/B)$.

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Beilinson-Bernstein Localization

The two types of smooth G-categories are related by Beilinson-Bernstein localization.

Localization and global sections functors (Δ, Γ) give adjunction between $\mathfrak{g}$-modules and $\mathcal{D}$-modules on flag varieties, linear over $\mathcal{D}(G)$.

To obtain an equivalence of categories we

- fix an infinitesimal character $[\lambda] \in \mathfrak{h}^*/W$;
- choose lift to weight $\lambda \in \mathfrak{h}^*$;
- modify construction at singular λ.

Beilinson-Bernstein in Families

Consider derived version in families:

$$\Delta : U\mathfrak{g}-\text{mod} \longleftrightarrow \mathcal{D}_H(G/N) : \Gamma$$

two-sided adjoints, and Δ conservative $\rightsquigarrow$ can readily apply Barr-Beck-Lurie theorem:

Theorem: $\boxed{U\mathfrak{g}-\text{mod} \simeq \mathcal{D}_H(G/N)_{\mathcal{W}} \simeq \mathcal{D}_H(G/N)^{\mathcal{W}},}$

modules (or comodules) over the (Frobenius) algebra in the Hecke category

$$\mathcal{W} = \mathcal{D}_{N\backslash G/N} \in End_{\mathcal{D}(G)}(\mathcal{D}_H(G/N))$$

Weyl sheaf $\mathcal{W}$: analog of symmetrizing idempotent in Weyl group,

$$\boxed{U\mathfrak{g}-\text{mod} = \text{"Weyl (co)invariants" in } \mathcal{D}_H(G/N).}$$

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Dimensions

Context: $\mathcal{A}$ symmetric monoidal (higher) category, $A \in \mathcal{A}$ dualizable object

Dimension of A: $\dim(A) \in End(1_{\mathcal{A}})$ defined by

$$1_{\mathcal{A}} \xrightarrow{unit = Id_A} End(A) \xrightarrow{\simeq} A \otimes A^{\vee} \xrightarrow{counit = Tr_A} 1_{\mathcal{A}}$$

$$\dim(A) = Tr_A(Id_A) : 1_{\mathcal{A}} \to 1_{\mathcal{A}}$$

Examples:

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Functoriality of Dimension

$\mathcal{A}$ a 2-category $\rightsquigarrow$ notion of adjunctions between morphisms

$$A \underset{\pi_!}{\overset{\pi^!}{\rightleftarrows}} B$$

$\rightsquigarrow$ notion of continuous morphism (has right adjoint)

Construction: Dim is functorial for continuous morphisms of dualizable objects: $\dim(A) \xrightarrow{\dim(\pi_!)} \dim(B)$

Corollary: A dualizable object of A (continuous $1_{\mathcal{A}} \xrightarrow{V} A$) has a character $\dim(V) \in \dim(A)$, satisfying "abstract GRR"

$$\begin{array}{ccc} 1_{\mathcal{A}} & \xrightarrow{\dim(V)} & \dim(A) \\ & \searrow^{\dim(\pi_! V)} & \downarrow \dim(\pi_!) \\ & & \dim(B) \end{array}$$

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Algebraic Examples

• $\mathcal{A}$=Morita theory of algebras, $A = \mathbb{C}G$ finite group algebra.
$\dim(A) = \mathbb{C}[\frac{G}{G}]$ class functions.

$V : 1_{\mathcal{A}} \to A$: representation of $G \rightsquigarrow \dim(V) \in \mathbb{C}[\frac{G}{G}]$: character of representation.

• $\mathcal{A}$=Categories, $A = QC(X)$ or $\mathcal{D}(X)$ sheaves on a scheme (or stack).
$\dim(A)$=Dolbeault/de Rham cohomology of the scheme (functions/de Rham cohomology of loop stack)

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Geometric Examples

- $\mathcal{A}$=(Derived) Varieties/stacks with correspondences: $\dim(X) = \mathcal{L}X$ loop space $(\Delta_X \cap \Delta_X = X \times_{X\times X} X.)$

$$\pi : X \to Y \rightsquigarrow \dim(\pi) = \mathcal{L}\pi : \mathcal{L}X \to \mathcal{L}Y.$$

- X a G-space, $\pi : X/G \to BG$.

$$\begin{array}{ccc} \mathcal{L}(X/G) & \xrightarrow{\simeq} & \dfrac{\{g \in G, x \in X^g\}}{G} \\ \Big\downarrow {\scriptstyle \mathcal{L}\pi} & & \Big\downarrow {\scriptstyle p_G} \\ \mathcal{L}(BG) & \xrightarrow{\simeq} & \dfrac{G}{G} \end{array}$$

– dimension given by fixed point loci classified over adjoint quotient.

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Trace and Fixed-Point Formulas

Formal consequence: in any theory of sheaves (assignment stack $\mapsto$ category satisfying base change and proper adjunction) pushforward on Hochschild homology is given by integration on loop maps, i.e., integration on fixed points in equivariant setting

Corollary:

- Grothendieck-Riemann-Roch in Hochschild homology for proper maps of geometric stacks
- Lefschetz trace formula for $\mathcal{D}$-modules on proper Deligne-Mumford (derived) stacks
- Atiyah-Bott fixed point theorem for quasicoherent sheaves on proper Deligne-Mumford (derived) stacks (conjecture of Frenkel-Ngô)

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Character Formulas

Frobenius character formula: Character of permutation representation $\mathbb{C}[X]$ is $Tr(g) = |X^g|$: pushforward of constant function under map $\mathcal{L}(X/G) \to \frac{G}{G}$.

$X = G/K$: character of induced representation $\mathbb{C}[G/K]$ given by pushforward along

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Flag variety $X = G/B \rightsquigarrow$ Weyl character formula à la Atiyah-Bott:

$$\frac{B}{B} \simeq \frac{\widetilde{G}}{G} \xrightarrow{\mathcal{L}\pi = Grothendieck-Springer} \frac{G}{G}$$

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Characters of algebraic G-categories

Arguments apply unmodified to sheaves of categories and categorical representations:

• $A = QC(G)$ or $QC(H \backslash G / H)$ quasicoherent group or Hecke algebra (monoidal category)

Theorem: $\dim(A) = QC(\frac{G}{G})$

V algebraic G-category $\rightsquigarrow$ $\dim(V) \in QC(\frac{G}{G})$: algebraic character sheaf.

Character of $QC(X)$ for a G-space calculated by fixed points map $\mathcal{L}\pi : \mathcal{L}X/G \to \frac{G}{G}$.

Characters of smooth G-categories

- $A = \mathcal{D}(G)$: Natural map $\dim(A) \to \mathcal{D}(\frac{G}{G})$. Calculate characters by fixed points:

Corollary: The character of $\mathcal{D}(G/B)$ is $\mathcal{L}\pi_*\mathcal{O}_{\frac{\widetilde{G}}{G}} \in \mathcal{D}(\frac{G}{G})$, the Grothendieck-Springer — i.e., [Hotta-Kashiwara] the Harish Chandra system HC on G.

- Characters are microlocal: for a G-space can calculate $\dim \mathcal{D}(X) \in \mathcal{D}(\frac{G}{G})$ near 1_G from moment map $\mu : T^*X \to \mathfrak{g}^*$.

- Characters for Hecke category $\mathcal{H} = \mathcal{D}(B\backslash G/B)$:

Theorem: $\dim(\mathcal{H})$ is the category of Lusztig character sheaves in $\mathcal{D}(\frac{G}{G})$ — i.e., differential equations on $\frac{G}{G}$ with same singularities as HC.

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Harish Chandra Theory of Characters

Harish Chandra introduced a character for admissible representations M of reductive groups:

- The character Θ_M exists as a distribution on $\frac{G}{G}$
- For M with infinitesimal character $[\lambda]$, Θ_M solves the Harish Chandra system — i.e., defines a morphism of $\mathcal{D}$-modules on $\frac{G}{G}$,

$$HC_{[\lambda]} := \mathcal{D}/\langle z - \lambda(z)\rangle_{z \in Z(U\mathfrak{g})} \xrightarrow{\Theta_M} C^{-\infty}$$

- Properties of HC system imply Θ_M analytic on G^{rss}, extends to $L^1(G)$. Beautiful algebraic and geometric formulas for Θ_M suggest it is innately algebraic object.

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Representations as Intertwiners

Admissible representations of real group $G_{\mathbb{R}}$
$\rightsquigarrow$ Harish Chandra $(\mathfrak{g}, K)$ modules for $K \subset G$ symmetric subgroup
$\rightsquigarrow$ [Beilinson-Bernstein] $\mathcal{D}_K(G/B)$: K-equivariant (twisted) $\mathcal{D}$-modules on G/B, i.e., $\mathcal{D}$-modules on

$$K\backslash G/B \simeq G\backslash(G/B \times G/K)$$

Suggests an interpretation as intertwiners for smooth G-categories:

Proposition: $\mathcal{D}(K\backslash G/B) \simeq Hom_G(\mathcal{D}(G/B), \mathcal{D}(G/K))$

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Harish Chandra Characters via Functoriality

Now apply functoriality of dim:

M a $(\mathfrak{g}, K)$-module $\rightsquigarrow$ M defines a G-map

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Character is a solution of Harish Chandra system valued in "K-Springer sheaf"

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Properties of the Character

Construction applies to any subgroup K (not necessarily symmetric).

On the regular semisimple locus of K, Θ_M is a section of $\mathcal{O}_{K^{rss}}$ solving pullback of HC system. — e.g., for $K = G$ recover Weyl character.

Gives refined information and algebraic interpretation of character over entire group.

Recover the two geometric character formulas of [Schmid-Vilonen] from general formalism:

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Construction applies to any subgroup K (not necessarily symmetric).

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Goal: Develop theory of characters for smooth LG-categories, in particular for modules for spherical and affine Hecke categories

$$\mathcal{H}^{sph} = \mathcal{D}(G(\mathcal{O})\backslash LG/G(\mathcal{O})) \xrightarrow{Satake} QC(BG^L)$$

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Motivation: Geometric Arthur-Selberg trace formula

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Elliptic Character Sheaves

What are character sheaves on LG? Hard to define directly as $\mathcal{D}$-modules on $\frac{LG}{LG}$. Two proposals:

• Algebraic definition: Consider $\dim(\mathcal{H}^{aff})$, characters of affine Hecke modules.

• Topological Field Theory definition: q-deform $\frac{LG}{LG}$ to $Bun_G(E_q)$, G-bundles on (Tate) elliptic curve $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$

$\rightsquigarrow$ Consider $\mathcal{D}_{nil}(Bun_G(E_q))$, $\mathcal{D}$-modules with nilpotent singular support.

Locally constant in q, can describe monadically in limit $q \to 0$.

Claim (proof in progress): $\boxed{\dim(\mathcal{H}^{aff}) \simeq \mathcal{D}_{nil}(Bun_G(E_q))}$
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Corollary: topological geometric Langlands for elliptic curves.

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