PRACTICE FINAL EXAM

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1. Notes

Good luck.

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(1) (50 pts) Please provide short answers to the following questions:
  (a) (5 pts) Explain the connection between linear transformations and matrices.
  (b) (5 pts) Explain why Gaussian elimination does not change the row space of a matrix.
  (c) (5 pts) Are all linear maps either injective or surjective?
  (d) (5 pts) Let V be a vector space. Can every subspace be written as the span of a finite set of vectors?
  (e) (5 pts) Explain what all the possible subspaces of $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$ are.
  (f) (5 pts) Does every matrix have eigenvectors?
  (g) (5 pts) Is the row echelon form of a matrix unique?
  (h) (5 pts) Explain the power method for finding the largest eigenvalue.
  (i) (5 pts) Suppose that $Ax = b$ has no solutions. Let z be approximate solution provided by regression. What subspace does z live in? What about the error term $b - z$?
  (j) (5 pts) What is the adjacency matrix of a graph?

(continued)

(2) (25 pts)

(a) (20 pts) Prove that the kernel of a matrix A is the orthogonal complement of the row space. (Hint: show that a vector is in the kernel if and only if it is perpendicular to the rows of A.)

(b) (5 pts) What can you say about the dimension of the row space? (Hint: Think about the dimension theorem.)

(3) (15 pts) Let $\{v_1, \dots, v_n\}$ be a linearly independent set of vectors in $\mathbb{R}^m$. Let z be in $\text{Span}(v_1, \dots, v_n)$. Show that $\{v_1, \dots, v_n, z\}$ is not a linearly independent set.

(4) (10 pts) Suppose that A is a matrix such that Ax is never zero unless $x = 0$. What can you conclude about the columns of A?

(5) (25 pts) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^4$ be the function specified by $(x, y, z) \mapsto (x+y, 3y-4z, 0, 0)$.

(a) Show that f is a linear transformation.
(b) Write the matrix for f in the standard basis for both $\mathbb{R}^3$ and $\mathbb{R}^4$.
(c) Write the matrix for f in the basis $\{[1, 0, 0], [1, 1, 0], [1, 1, 1]\}$ for $\mathbb{R}^3$ and the standard basis for $\mathbb{R}^4$.
(d) Explain the relationship between the two matrices you obtained in the previous problem.
(e) Is f injective? Surjective?

(continued)

(6) (25 pts) Compute the orthogonal projection of the vector $[1,1,0]$ onto the solution space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{pmatrix}.$$

(7) (15 pts) Find the eigenvectors and eigenspaces for the matrix

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix},$$

given that the eigenvalues are 4 and -2. Diagonalize this matrix if possible.

(8) (15 pts) For what values of a, b, and c are the vectors [1000], [01ab], and [cd10] linearly independent?

(9) (20 pts)

(a) Let V and W be vector spaces of dimension m and n respectively. Prove that the set of linear transformations $f \colon V \to W$ is a subspace of the set of all functions.

(b) (5 pts) What is its dimension?

(10) (30 pts)
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(a) (10 pts) Prove that the subset of polynomials of degree n such that $p(x) = p(-x)$ is a subspace of $\mathcal{P}_n$.

(b) (10 pts) Explain why the set of vectors in $\mathbb{R}^n$ such that all the entries are even is not a subspace.

(c) (10 pts) Let $W_1, W_2 \subset V$ be subspaces. Prove that set $\{w_1 + w_2 \,|\, w_1 \in W_1, w_2 \in W_2\}$ is a subspace of V.