ON 3-BRAIDS AND L-SPACE KNOTS.

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An L-space knot generalizes the notion of knots which admit lens space surgeries. A rational homology 3-sphere Y is an L-space if $|H_1(Y;\mathbb{Z})| = \text{rank } \widehat{HF}(Y)$, where $\widehat{HF}$ denotes the 'hat' version of Heegaard Floer homology, and the name stems from the fact that lens spaces are L-spaces. Besides lens spaces, examples of L-spaces include all connected sums of manifolds with elliptic geometry [OS05].

A knot, $K \subset S^3$, is an *L-space knot* if K or its mirror image admits a positive L-space surgery. One of the most prominent problems in relating low-dimensional topology and Heegaard Floer homology is to give a topological characterization to L-spaces and L-space knots. In this direction, Ozsváth and Szabó's result states that L-spaces admit no co-orientable taut foliations [OS04, Theorem 1.4]. It is also known that an L-space knot $K \subset S^3$ must be prime [Krc13] and fibered [Ni07], and that K supports the tight contact structure on S^3 [Hed10, Proposition 2.1]. In addition, the Alexander polynomial $\triangle_K(t)$ of an L-space knot K satisfies the following:

- The absolute value of a nonzero coefficient of $\triangle_K(t)$ is 1. The set of nonzero coefficients alternates in sign [OS05, Corollary 1.3].
- If g is the maximum degree of $\triangle_K(t)$ in t, then the coefficients of the term t^{g-1} is nonzero and therefore ± 1 [HW14].

The purpose of this manuscript is to study which 3-braids, that close to form a knot, admit L-space surgeries. We prove that:

Theorem 1. *Twisted $(3,q)$ torus knots are the only knots with 3-braid representations that admit L-space surgeries.*

Our proof of Theorem 1 uses the constraints on the Alexander polynomial of an L-space knot that has previously been studied to give the classification of L-space knots among pretzel knots [LM]. We show that, except for the twisted $(3,q)$ torus knots, the Alexander polynomials of all of the knots with 3-braid representations violate the constraints mentioned for the Alexander polynomial of L-space knots.

We begin by computing certain coefficients of the Jones polynomials of closed 3-braids. The Alexander polynomial of a closed 3-braid may be written in terms of the Jones polynomial [Bir85]. This allows us to use our computation of the Jones polynomials to rule out closed 3-braids whose Alexander polynomials violate the aforementioned conditions.

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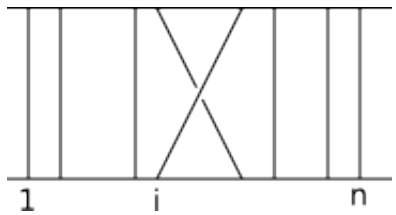

FIGURE 1. the generator σ_i. Note that this convention is the opposite of that of [Bir85].

The Jones polynomial, 3-braids, and the Alexander polynomial

We will first derive an expression of the Alexander polynomial of a closed 3-braid in terms of the Jones polynomial. Let B_n be the n-string braid group. The *Burau representation* of B_n is a map ψ from B_n to $n-1 \times n-1$ matrices with entries in $\mathbb{Z}[t, t^{-1}]$.

$$\psi : B_n \to GL(n-1, \mathbb{Z}[t, t^{-1}]).$$

For $n = 3$, ψ is defined explicitly on the generators σ_1, σ_2 (see Figure 1) as

$$\psi(\sigma_1^{-1}) = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix},$$

$$\psi(\sigma_2^{-1}) = \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix}.$$

Let a be an element of B_3, $\hat{a}$ be the closed braid, and e_a be the exponent sum of a. Note that when $\hat{a}$ is a knot, $2 \pm e_a$ is even and therefore e_a is even. The Jones polynomial $J_{\hat{a}}(t)$ of $\hat{a}$ can be written in terms of ψ [Jon85]:

$$J_{\hat{a}}(t) = (-\sqrt{t})^{-e_a}(t + t^{-1} + \text{trace } \psi(a)) \tag{1}$$

The sign change on e_a is due to the difference in convention as indicated in Figure 1. When $n = 3$, the Alexander polynomial of $\hat{a}$ may be written in terms of the trace of ψ. [Bir85, Eq. (7)]

$$(t^{-1} + 1 + t)\triangle_{\hat{a}}(t) = (-1)^{-e_a}(t^{-e_a/2} - t^{e_a/2}\text{trace } \psi(a) + t^{e_a/2}) \tag{2}$$

Rearranging equations (1) and (2) above, we have the symmetric Alexander polynomial of a closed 3-braid re-written in terms of the Jones polynomial.

$$(t^{-1}+1+t)\triangle_{\hat{a}}(t) = (-1)^{-e_a}(-(-1)^{e_a}t^{e_a}J_{\hat{a}}(t)+t^{e_a/2+1}+t^{e_a/2-1}+t^{-e_a/2}+t^{e_a/2}). \tag{3}$$

This expression allows us to compute certain coefficients of the Alexander polynomial from the Jones polynomial. By Birman and Menasco's solution [BM92] to the classification of 3-braids, there are finitely many conjugacy classes of B_3, and each 3-braid is isotopic to a representative of a conjugacy class. Schreier's work [Sch24] puts each representative of a conjugacy class in a normal form.

Theorem 2. *(Schreier) Let $b \in B_3$ be a braid on three strands, and C be the 3-braid $(\sigma_1\sigma_2)^3$. Then b is conjugate to a braid in exactly one of the following forms:*

(1) $C^k\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s}$, *where $k \in \mathbb{Z}$ and p_i, q_i and s are all positive integers,*

(2) $C^k\sigma_1^p$, *for* $k, p \in \mathbb{Z}$,
(3) $C^k\sigma_1\sigma_2$, *for* $k \in \mathbb{Z}$,
(4) $C^k\sigma_1\sigma_2\sigma_1$, *for* $k \in \mathbb{Z}$, *or*
(5) $C^k\sigma_1\sigma_2\sigma_1\sigma_2$, *for* $k \in \mathbb{Z}$.

It suffices to study the 3-braids among the conjugacy representatives above to determine which closed 3-braids is an L-space knot. It is straightforward to check that $C^k\sigma_1^p$ and $C^k\sigma_1\sigma_2\sigma_1$ represent links for any $k, p \in \mathbb{Z}$. Also, by noting that $C \sim (\sigma_1\sigma_2)^3$, we get that, for any $k \in \mathbb{Z}$, $C^k\sigma_1\sigma_2$ and $C^k\sigma_1\sigma_2\sigma_1\sigma_2$ represent the $(3, 3k+1)$ and $(3, 3k+2)$ torus knots, respectively. Thus we will only need to study class (1) of conjugacy representatives of Theorem 2.

Recall that if a knot K is a L-space knot, then the absolute value of a nonzero coefficient of the Alexander polynomial $\triangle_K(t)$ is 1, and the nonzero coefficients alternate in sign. Moreover, let g be the maximum degree of $\triangle_K(t)$ in t, then the coefficients of the term t^{g-1} is nonzero and therefore ± 1. Since the Alexander polynomial is symmetric, it has the two possible forms given below for an L-space knot.

$$t^g - t^{g-1} + \cdots + \text{terms in-between} - t^{-(g-1)} + t^{-g}$$

or

$$-t^g + t^{g-1} + \cdots + \text{terms in-between} + t^{-(g-1)} - t^{-g}.$$

Either way, when we take the product

$$\triangle_K(t) \cdot (t^{-1} + 1 + t).$$

The result is a symmetric polynomial with coefficients in $\{-1, 0, 1\}$, which do not necessarily alternate in sign, and the second coefficient and the second-to-last coefficient are zero.

The conjugacy representatives of class (1) in Theorem 2 are called *generic* 3-braids. That is, $b \in B_3$ is *generic* if it has the following form.

$$b = C^k\sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s},$$

where $p_i, q_i, k \in \mathbb{Z}$, with $p_i, q_i > 0$, and $C = (\sigma_1\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2)^3$ by the braid relations $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. The braid $a = \sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s}$ is called an *alternating 3-braid*. The first three coefficients and the last three coefficients of the Jones polynomial for this class of 3-braids, as well as the degree, are explicitly calculated in [FKP10]. We assemble below the results we will need.

Definition 1. *For an alternating braid* $a = \sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s}$, *let*

$$\mathbf{p} := \sum_{i=1}^{s} p_i, \text{ and } \mathbf{q} := \sum_{i=1}^{s} q_i,$$

so the exponent sum $e_a = \mathbf{p} - \mathbf{q}$.

Lemma 1. [FKP10, Lemma 6.2] *Suppose that a link $\hat{a}$ is the closure of an alternating 3-braid a:*

$$a = \sigma_1^{p_1}\sigma_2^{-q_1}\cdots\sigma_1^{p_s}\sigma_2^{-q_s},$$

with $p_i, q_i > 0$ and $\mathbf{p} > 1$ and $\mathbf{q} > 1$, then the following holds

(a) *The highest and lowest powers, $M(\hat{a})$ and $m(\hat{a})$ of $J_{\hat{a}}(t)$ in t are*

$$M(\hat{a}) = \frac{3\mathbf{q}-\mathbf{p}}{2} \text{ and } m(\hat{a}) = \frac{\mathbf{q}-3\mathbf{p}}{2}.$$

(b) *The first two coefficients α, β from $M(\hat{a})$, and the last two coefficients β', α' in $J_{\hat{a}}(t)$ from $m(\hat{a})$ are*

$$\alpha = (-1)^{\mathbf{p}}, \beta = (-1)^{\mathbf{p}+1}(s-\epsilon_{\mathbf{q}}), \beta' = (-1)^{\mathbf{q}+1}(s-\epsilon_{\mathbf{p}}), \alpha' = (-1)^{\mathbf{q}}.$$

where $\epsilon_{\mathbf{p}} = 1$ if $\mathbf{p} = 2$ and 0 if $\mathbf{p} > 2$, and similarly for $\epsilon_{\mathbf{q}}$.

(c) [FKP10, in the proof of Lemma 6.2] *Let γ, γ' denote the third and the third-to-last coefficient of $J_{\hat{a}}(t)$, respectively. We have*

$$(-1)^{\mathbf{p}}\gamma = \frac{s^2+3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{\mathbf{q}=3},$$

and

$$(-1)^{\mathbf{q}}\gamma' = \frac{s^2+3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{\mathbf{p}=3},$$

where $\delta_{\mathbf{q}=3}$ is zero if $\mathbf{q} \neq 3$ and 1 otherwise, and $\delta_{\mathbf{p}=3}$ is similarly defined.

The next result writes the Jones polynomial of a generic 3-braid in terms of the Jones polynomial of an alternating braid.

Lemma 2. [FKP10] *If b is a generic braid of the form*

$$b = C^k a,$$

where a is an alternating 3-braid, and let $J_{\hat{b}}(t)$ denote the Jones polynomial of $\hat{b}$, then

$$J_{\hat{b}}(t) = t^{-6k}J_{\hat{a}}(t) + (-\sqrt{t})^{-e_a}(t+t^{-1})(t^{-3k}-t^{-6k}).$$

If K is a knot which is the closure of a generic 3-braid $b = C^k a$, then by equation (3), we have

$$\begin{aligned}&\triangle_K(t)\cdot(t^{-1}+1+t)\\&= (-1)^{-e_b}(-(-1)^{e_b}t^{e_b}J_{\hat{b}}(t) + t^{e_b/2+1} + t^{e_b/2-1} + t^{-e_b/2} + t^{e_b/2}).\end{aligned}$$

Putting this together with Lemma 2 and noting that e_b must be even in order for $\hat{b}$ to be a knot, we get that the right side is equal to

$$\begin{aligned}&= -t^{e_b}(t^{-6k}J_{\hat{a}}(t) + (-\sqrt{t})^{-e_a}(t+t^{-1})(t^{-3k}-t^{-6k}))\\&+ t^{e_b/2+1} + t^{e_b/2-1} + t^{-e_b/2} + t^{e_b/2}.\end{aligned}$$

Since $b = C^k a$ and $e_b = 6k + e_a$, we have

$$\begin{aligned}
&\triangle_K(t)(t^{-1} + 1 + t) \\
&= -t^{(6k+e_a)}(t^{-6k} J_{\hat{a}}(t) + (-\sqrt{t})^{-e_a}(t + t^{-1})(t^{-3k} - t^{-6k})) \\
&+ t^{(6k+e_a)/2+1} + t^{(6k+e_a)/2-1} + t^{-(6k+e_a)/2} + t^{(6k+e_a)/2} \\
&= -t^{e_a} J_{\hat{a}}(t) + t^{e_a/2-1} + t^{e_a/2+1} + t^{-3k-e_a/2} + t^{3k+e_a/2}. \qquad (4)
\end{aligned}$$

We are now ready to determine which closed 3-braids are L-space knots.

Proof. For $K = \hat{b}$ to be an L-space knot, the right side of equation (4) has to be have coefficients in ± 1. This immediately restricts s to be less than 4, since $s \geq 4$ implies that there will be two coefficients β, β', the second and the penultimate, whose absolute values are greater than or equal to 4 in $J_{\hat{a}}(t)$ by (b) of Lemma 1. Based on (4), This will result in at least one coefficient whose absolute value is greater than or equal to 2 in $\triangle_K(t)(t^{-1} + 1 + t)$, even after possible cancellation from the terms $t^{e_a/2-1}, t^{e_a/2+1}, t^{-3k-e_a/2}$, and $t^{3k+e_a/2}$. Similarly, if $s = 3$, then $\mathbf{p}, \mathbf{q} \geq 3$, and $|\beta|$ and $|\beta'|$ are both greater than or equal to 3, each of which would need to be cancelled out by at least two terms of $t^{e_a/2-1}, t^{e_a/2+1}, t^{-3k-e_a/2}$, and $t^{3k+e_a/2}$. In addition, part (c) of Lemma 1 gives that the absolute values of the third and third-to-last coefficients γ, γ' are greater than or equal to 2, which would also need to be cancelled out in the sum of the right side of (4). This is impossible, so $s \neq 3$.

Now assume that $s = 2$. If $\mathbf{p} > 2$, then the penultimate coefficient $|\beta'| = 2$ and the third coefficient $|\gamma| \geq 2$. Since $\mathbf{p}, \mathbf{q}$ are either both even or both odd, β' and γ have opposite signs. One of them is positive, which would not cancel out with any of the terms $t^{e_a/2-1}, t^{e_a/2+1}, t^{-3k-e_a/2}$, and $t^{3k+e_a/2}$. The case is similar for $\mathbf{q} > 2$, so we must have that $\mathbf{p} = 2$ and $\mathbf{q} = 2$. This means that $a = \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$. The Jones polynomial of this alternating closed braid is

$$J_{\hat{a}(t)} = t^{-2} - t^{-1} + 1 - t + t^2,$$

for a braid $b = C^k a$,

$$\begin{aligned}
(t^{-1} + 1 + t)\triangle_{\hat{b}}(t) &= -J_{\hat{a}}(t) + \frac{1}{t} + t + t^{-3k} + t^{3k} \\
&= -(t^{-2} - t^{-1} + 1 - t + t^2) + \frac{1}{t} + t + t^{-3k} + t^{3k} \\
&= -t^{-2} + 2t^{-1} - 1 + 2t - t^2 + t^{-3k} + t^{3k}
\end{aligned}$$

For all $k \neq 0$, this shows that the product on the left side of this equation has nonzero coefficients that are not ± 1. This rules out the possibility that a closed 3-braid of this form can be an L-space knot.

Therefore, we need only to consider the case when $s = 1$. Assuming that both $\mathbf{p}, \mathbf{q}$ are greater than 1, the absolute values of the third coefficient γ and the third-to-last coefficient γ' of $-t^{e_\alpha} J_{\hat{\alpha}(t)}$ have the form

$$\underbrace{\left|\left(\frac{s^2+3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{\mathbf{q}=3}\right)\right|}_{\gamma} t^{(\mathbf{q}+\mathbf{p})/2-2}$$

and

$$\underbrace{\left|\left(\frac{s^2+3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{\mathbf{p}=3}\right)\right|}_{\gamma'} t^{-(\mathbf{q}+\mathbf{p})/2+2},$$

respectively. If $|\gamma|$ or $|\gamma'|$ is 2, they need to be canceled out by at least one term out of

$$t^{e_a/2-1},\ t^{e_a/2+1},\ t^{-3k-e_a/2},\ t^{3k+e_a/2}.$$

If $\mathbf{q} > 3$ and $\mathbf{p} > 1$, then $\gamma = 2$, and $(\mathbf{q}+\mathbf{p})/2-2$ needs to be equal to $(\mathbf{p}-\mathbf{q})/2-1$, $(\mathbf{p}-\mathbf{q})/2+1$, $-3k-(\mathbf{p}-\mathbf{q})/2$, or $3k+(\mathbf{p}-\mathbf{q})/2$. Similarly, we have the constraints on $-(\mathbf{q}+\mathbf{p})/2+2$. We examine the resulting equations and rule out values of $\mathbf{p}$ and $\mathbf{q}$ which lead to contradictions. Setting $(\mathbf{q}+\mathbf{p})/2-2$ equal to $(\mathbf{p}-\mathbf{q})/2-1$ or $(\mathbf{p}-\mathbf{q})/2+1$ gives $\mathbf{q} = 1$ or $\mathbf{q} = 3$. Setting $(\mathbf{q}+\mathbf{p})/2-2$ equal to $-3k-(\mathbf{p}-\mathbf{q})/2$ or $3k+(\mathbf{p}-\mathbf{q})/2$ gives $\mathbf{p} = -3k+2$ or $\mathbf{q} = 3k+2$. Similarly, if $\mathbf{p} > 3$ and $\mathbf{q} > 1$, then we must have $\mathbf{p} = 3$, $\mathbf{p} = 1$, $\mathbf{q} = 3k+2$, or $\mathbf{p} = -3k+2$. We dismiss the cases where $\mathbf{p}$ or $\mathbf{q} \leq 3$ for now, and suppose that $k \neq 0$. We cannot have that $\mathbf{p} = -3k+2$ *and* $\mathbf{q} = 3k+2$ since they are both supposed to be positive. Therefore we suppose that $\mathbf{p} = -3k+2$ or $\mathbf{q} = 3k+2$. In the first case, k is negative. In the second case, k is positive. Either way, we end up having, for $k < 0$,

$$\begin{aligned}
&\triangle_K(t) \cdot (t^{-1}+1+t) \\
&= (\pm t^{-\frac{-3k+2+\mathbf{q}}{2}} \mp t^{-\frac{-3k+2+\mathbf{q}}{2}+1} \pm 0 \mp \cdots \pm 0 \mp t^{\frac{-3k+2+\mathbf{q}}{2}-1} \pm t^{\frac{-3k+2+\mathbf{q}}{2}}) \\
&+ t^{\frac{-3k+2-\mathbf{q}}{2}-1} + t^{\frac{-3k+2-\mathbf{q}}{2}+1},
\end{aligned}$$

or, for $k > 0$,

$$\begin{aligned}
&\triangle_K(t) \cdot (t^{-1}+1+t) \\
&= (\pm t^{-\frac{3k+2+\mathbf{p}}{2}} \mp t^{-\frac{3k+2+\mathbf{p}}{2}+1} \pm 0 \mp \cdots \pm 0 \mp t^{\frac{3k+2+\mathbf{p}}{2}-1} \pm t^{\frac{3k+2+\mathbf{p}}{2}}) \\
&+ t^{\frac{-(3k+2)+\mathbf{p}}{2}-1} + t^{\frac{-(3k+2)+\mathbf{p}}{2}+1}
\end{aligned}$$

One of the conditions on the product $\triangle_K(t) \cdot (t^{-1}+1+t)$ is that the second and the penultimate coefficients are equal to zero. When $k < 0$, the terms $t^{-\frac{-3k+2+\mathbf{q}}{2}+1}, t^{\frac{-3k+2+\mathbf{q}}{2}-1}$ are the second and the penultimate coefficient which is not zero since we assume $\mathbf{p}, \mathbf{q} > 3$, this is impossible so this case cannot happen. The same argument applies to rule out the second case when $k > 0$.

When both $p, q = 3$, we have that the alternating 3-braid a takes the form $\sigma_1^3\sigma_2^{-3}$. The Alexander polynomial of this alternating 3-braid is

$$\triangle_a(t) = 3 + \frac{1}{t^2} - \frac{2}{t} - 2t + t^2,$$

obtained by multiplying the Alexander polynomial of the trefoil by itself, since this 3-braid is a connected sum of two (right-hand and left-hand) trefoils. It is clear from the Alexander polynomial that this knot cannot be an L-space knot due to the fact that several of its nonzero coefficients are not ± 1. Now we consider a generic 3-braid $b = C^k a$ with $a = \sigma_1^3\sigma_2^{-3}$. Since $e_a = 0$, the highest degree and the lowest degree of the Jones polynomial of $\hat{a}$ are 3 and -3. By equation (4),

$$(t^{-1} + 1 + t)\triangle_{\hat{b}}(t) = -J_{\hat{a}}(t) + \frac{1}{t} + t + t^{-3k} + t^{3k},$$

where

$$J_{\hat{a}}(t) = 3 - \frac{1}{t^3} + \frac{1}{t^2} - \frac{1}{t} - t + t^2 - t^3.$$

When $k \neq 0$, it is clear that the constant term 3 of $J_{\hat{a}}(t)$ will not be canceled out by the terms $\frac{1}{t}, t, t^{-3k}$, or t^{-3k}. Thus none of the closure of braids of the form $C^k\sigma_1^3\sigma_2^{-3}$ will be an L-space knot. We may also rule out the case $\mathbf{p}$ or $\mathbf{q} = 2$ since this would give a link rather than a knot. Thus, the only generic 3-braids whose closure can be an L-space knot are given below.

$C^k\sigma_1^1\sigma_2^{-q}$	for q odd.
$C^k\sigma_1^p\sigma_2^{-1}$	for p odd.

We now claim that $C^k\sigma_1^p\sigma_2^{-1}$, for p odd and $k > 0$, represents an L-space knot. Note that:

$$\begin{aligned}(\sigma_1\sigma_2\sigma_1)^{2k}\sigma_1^p\sigma_2^{-1} &\sim (\sigma_2\sigma_1)^{3k}\sigma_1^p\sigma_2^{-1}\\ &\sim (\sigma_2\sigma_1)^{3k-1}\sigma_1^{p+1}.\end{aligned}$$

The latter braid is the twisted torus knot, $K(3, 3k-1; 2, 1)$, which is known to be an L-space knot [Vaf14, Corollary 3.2]. Now if $k < 0$ then

$$\begin{aligned}(\sigma_1\sigma_2\sigma_1)^{2k}\sigma_1^p\sigma_2^{-1} &\sim (\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})^{-k}\sigma_1^p\sigma_2^{-1}\\ &\sim \sigma_2^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})^{-k}\sigma_1^p\\ &\sim \sigma_2^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1})^{-k}\sigma_1^p\\ &\sim \sigma_2^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1})^{-k}\sigma_1^{-1}\sigma_1\sigma_1^{p+1}\\ &\sim (\sigma_2^{-1}\sigma_1^{-1})^{-3k+1}\sigma_1^{p+1}\\ &\sim (\sigma_1\sigma_2)^{3k-1}\sigma_1^{p+1}.\end{aligned}$$

Using [Vaf14, Corollary 3.2], we get that the closure of the latter braid represents an L-space knot. We should point out that [Vaf14, Corollary 3.2], as stated, only holds for positive knots. However, it turns out that every twisted (p, q) torus knot,

where the twisting happens between $p-1$ strands, admits an L-space surgery from the proof of [Vaf14, Theorem 3.1]. A similar argument shows that the closure of $C^k\sigma_1^1\sigma_2^{-q}$, for q odd, also represents an L-space knot. Notice that in this case the knot is isotopic to the closure of $(\sigma_2\sigma_1)^{1-3k}\sigma_1^{-p-1}$. So its mirror image admits a positive L-space surgery.

□

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