ON 3-BRAIDS AND L-SPACE KNOTS.

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An L-space knot generalizes the notion of knots which admit lens space surgeries. A rational homology 3-sphere $Y$ is an L-space if $|H_1(Y;\mathbb{Z})| = \text{rank } \widehat{HF}(Y)$, where $\widehat{HF}$ denotes the ‘hat’ version of Heegaard Floer homology, and the name stems from the fact that lens spaces are L-spaces. Besides lens spaces, examples of L-spaces include all connected sums of manifolds with elliptic geometry [OS05].

A knot, $K \subset S^3$, is an L-space knot if $K$ or its mirror image admits a positive L-space surgery. One of the most prominent problems in relating low-dimensional topology and Heegaard Floer homology is to give a topological characterization to L-spaces and L-space knots. In this direction, Ozsváth and Szabó’s result states that L-spaces admit no co-orientable taut foliations [OS04, Theorem 1.4]. It is also known that an L-space knot $K \subset S^3$ must be prime [Krc13] and fibered [Ni07], and that $K$ supports the tight contact structure on $S^3$ [Hed10 Proposition 2.1]. In addition, the Alexander polynomial $\triangle_K(t)$ of an L-space knot $K$ satisfies the following:

- The absolute value of a nonzero coefficient of $\triangle_K(t)$ is 1. The set of nonzero coefficients alternates in sign [OS05, Corollary 1.3].
- If $g$ is the maximum degree of $\triangle_K(t)$ in $t$, then the coefficients of the term $t^{9-g-1}$ is nonzero and therefore $\pm 1$ [HW14].

The purpose of this manuscript is to study which 3-braids, that close to form a knot, admit L-space surgeries. We prove that:

**Theorem 1.** Twisted $(3,q)$ torus knots are the only knots with 3-braid representations that admit L-space surgeries.

Our proof of Theorem 1 uses the constraints on the Alexander polynomial of an L-space knot that has previously been studied to give the classification of L-space knots among pretzel knots [LM]. We show that, except for the twisted $(3,q)$ torus knots, the Alexander polynomials of all of the knots with 3-braid representations violate the constraints mentioned for the Alexander polynomial of L-space knots.

We begin by computing certain coefficients of the Jones polynomials of closed 3-braids. The Alexander polynomial of a closed 3-braid may be written in terms of the Jones polynomial [Bir85]. This allows us to use our computation of the Jones polynomials to rule out closed 3-braids whose Alexander polynomials violate the aforementioned conditions.

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THE JONES POLYNOMIAL, 3-BRAIDS, AND THE ALEXANDER POLYNOMIAL

We will first derive an expression of the Alexander polynomial of a closed 3-braid in terms of the Jones polynomial. Let $B_n$ be the $n$-string braid group. The Burau representation of $B_n$ is a map $\psi$ from $B_n$ to $n-1 \times n-1$ matrices with entries in $\mathbb{Z}[t, t^{-1}]$.

$$\psi : B_n \to GL(n-1, \mathbb{Z}[t, t^{-1}]).$$

For $n = 3$, $\psi$ is defined explicitly on the generators $\sigma_1, \sigma_2$ (see Figure 1) as

$$\psi(\sigma_1^{-1}) = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix},$$

$$\psi(\sigma_2^{-1}) = \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix}.$$

Let $a$ be an element of $B_3$, $\hat{a}$ be the closed braid, and $e_a$ be the exponent sum of $a$. Note that when $\hat{a}$ is a knot, $2 \pm e_a$ is even and therefore $e_a$ is even. The Jones polynomial $J_{\hat{a}}(t)$ of $\hat{a}$ can be written in terms of $\psi$ [Jon85]:

$$J_{\hat{a}}(t) = (-\sqrt{t})^{-e_a}(t + t^{-1} + \text{trace } \psi(a))$$  \hspace{1cm} (1)

The sign change on $e_a$ is due to the difference in convention as indicated in Figure 1.

When $n = 3$, the Alexander polynomial of $\hat{a}$ may be written in terms of the trace of $\psi$. [Bir85, Eq. (7)]

$$J_{\hat{a}}(t) = (-\sqrt{t})^{-e_a}(t + t^{-1} + \text{trace } \psi(a) + t^{e_a/2})$$  \hspace{1cm} (2)

Rearranging equations (1) and (2) above, we have the symmetric Alexander polynomial of a closed 3-braid re-written in terms of the Jones polynomial.

$$J_{\hat{a}}(t) = (-1)^{-e_a}(-1)^{t^{-e_a/2} - t^{e_a/2}\text{trace } \psi(a) + t^{e_a/2})$$  \hspace{1cm} (3)

This expression allows us to compute certain coefficients of the Alexander polynomial from the Jones polynomial. By Birman and Menasco’s solution [BM92] to the classification of 3-braids, there are finitely many conjugacy classes of $B_3$, and each 3-braid is isotopic to a representative of a conjugacy class. Schreier’s work [Sch24] puts each representative of a conjugacy class in a normal form.

**Theorem 2.** (Schreier) Let $b \in B_3$ be a braid on three strands, and $C$ be the 3-braid $(\sigma_1 \sigma_2)^3$. Then $b$ is conjugate to a braid in exactly one of the following forms:

(1) $C^k \sigma_1^{p_1} \sigma_2^{-q_1} \cdots \sigma_1^{p_s} \sigma_2^{-q_s}$, where $k \in \mathbb{Z}$ and $p_i, q_i$ and $s$ are all positive integers,
(2) $C^k \sigma_1^p$, for $k, p \in \mathbb{Z}$,
(3) $C^k \sigma_1 \sigma_2$, for $k \in \mathbb{Z}$,
(4) $C^k \sigma_1 \sigma_2 \sigma_1$, for $k \in \mathbb{Z}$, or
(5) $C^k \sigma_1 \sigma_2 \sigma_1 \sigma_2$, for $k \in \mathbb{Z}$.

It suffices to study the 3-braids among the conjugacy representatives above to determine which closed 3-braids is an L-space knot. It is straightforward to check that $C^k \sigma_1^p$ and $C^k \sigma_1 \sigma_2 \sigma_1$ represent links for any $k, p \in \mathbb{Z}$. Also, by noting that $C \sim (\sigma_1 \sigma_2)^3$, we get that, for any $k \in \mathbb{Z}$, $C^k \sigma_1 \sigma_2$ and $C^k \sigma_1 \sigma_2 \sigma_1 \sigma_2$ represent the $(3, 3k+1)$ and $(3, 3k+2)$ torus knots, respectively. Thus we will only need to study class [1] of conjugacy representatives of Theorem 2.

Recall that if a knot $K$ is a L-space knot, then the absolute value of a nonzero coefficient of the Alexander polynomial $\Delta_K(t)$ is 1, and the nonzero coefficients alternate in sign. Moreover, let $g$ be the maximum degree of $\Delta_K(t)$ in $t$, then the coefficients of the term $t^{g-1}$ is nonzero and therefore $\pm 1$. Since the Alexander polynomial is symmetric, it has the two possible forms given below for an L-space knot.

$$t^g - t^{g-1} + \cdots + \text{terms in-between} - t^{-(g-1)} + t^{-g}$$

or

$$-t^g + t^{g-1} + \cdots + \text{terms in-between} + t^{-(g-1)} - t^{-g}.$$

Either way, when we take the product

$$\Delta_K(t) \cdot (t^{-1} + 1 + t).$$

The result is a symmetric polynomial with coefficients in $\{-1, 0, 1\}$, which do not necessarily alternate in sign, and the second coefficient and the second-to-last coefficient are zero.

The conjugacy representatives of class (1) in Theorem 2 are called generic 3-braids. That is, $b \in B_3$ is generic if it has the following form.

$$b = C^k \sigma_1^{p_1} \sigma_2^{-q_1} \cdots \sigma_1^{p_s} \sigma_2^{-q_s},$$

where $p_i, q_i, k \in \mathbb{Z}$, with $p_i, q_i > 0$, and $C = (\sigma_1 \sigma_2 \sigma_1)^2 = (\sigma_1 \sigma_2)^3$ by the braid relations $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$. The braid $a = \sigma_1^{p_1} \sigma_2^{-q_1} \cdots \sigma_1^{p_s} \sigma_2^{-q_s}$ is called an alternating 3-braid. The first three coefficients and the last three coefficients of the Jones polynomial for this class of 3-braids, as well as the degree, are explicitly calculated in [FKP10]. We assemble below the results we will need.

**Definition 1.** For an alternating braid $a = \sigma_1^{p_1} \sigma_2^{-q_1} \cdots \sigma_1^{p_s} \sigma_2^{-q_s}$, let

$$p := \sum_{i=1}^s p_i, \text{ and } q := \sum_{i=1}^s q_i,$$

so the exponent sum $e_a = p - q$. 
Lemma 1. [FKP10] Lemma 6.2] Suppose that a link $\hat{a}$ is the closure of an alternating 3-braid $a$:

$$a = \sigma_1^{p_1} \sigma_2^{-q_1} \cdots \sigma_1^{p_9} \sigma_2^{-q_9},$$

with $p_i, q_i > 0$ and $p > 1$ and $q > 1$, then the following holds

(a) The highest and lowest powers, $M(\hat{a})$ and $m(\hat{a})$ of $J_{\hat{a}}(t)$ in $t$ are

$$M(\hat{a}) = \frac{3q - p}{2} \quad \text{and} \quad m(\hat{a}) = \frac{q - 3p}{2}.$$

(b) The first two coefficients $\alpha, \beta$ from $M(\hat{a})$, and the last two coefficients $\beta', \alpha'$ in $J_{\hat{a}}(t)$ from $m(\hat{a})$ are

$$\alpha = (-1)^p, \quad \beta = (-1)^{p+1}(s - \varepsilon_q), \quad \beta' = (-1)^{q+1}(s - \varepsilon_p), \quad \alpha' = (-1)^q.$$

where $\varepsilon_p = 1$ if $p = 2$ and 0 if $p > 2$, and similarly for $\varepsilon_q$.

(c) [FKP10] in the proof of Lemma 6.2] Let $\gamma, \gamma'$ denote the third and the third-to-last coefficient of $J_{\hat{a}}(t)$, respectively. We have

$$(-1)^p \gamma = \frac{s^2 + 3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{q=3},$$

and

$$(-1)^q \gamma' = \frac{s^2 + 3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{p=3},$$

where $\delta_{q=3}$ is zero if $q \neq 3$ and 1 otherwise, and $\delta_{p=3}$ is similarly defined.

The next result writes the Jones polynomial of a generic 3-braid in terms of the Jones polynomial of an alternating braid.

Lemma 2. [FKP10] If $b$ is a generic braid of the form

$$b = C^k a,$$

where $a$ is an alternating 3-braid, and let $J_b(t)$ denote the Jones polynomial of $b$, then

$$J_b(t) = t^{-6k} J_{\hat{a}}(t) + (-\sqrt{t})^{e_a} (t + t^{-1})(t^{-3k} - t^{-6k}).$$

If $K$ is a knot which is the closure of a generic 3-braid $b = C^k a$, then by equation (3), we have

$$\triangle_K(t) \cdot (t^{-1} + 1 + t)$$

$$= (-1)^{-e_b} (-1)^{e_b} t^{e_b} J_b(t) + t^{e_b/2+1} + t^{e_b/2-1} + t^{-e_b/2} + t^{e_b/2}.$$ 

Putting this together with Lemma 2 and noting that $e_b$ must be even in order for $b$ to be a knot, we get that the right side is equal to

$$= -t^{e_b} (t^{-6k} J_{\hat{a}}(t) + (-\sqrt{t})^{e_a} (t + t^{-1})(t^{-3k} - t^{-6k}))$$

$$+ t^{e_b/2+1} + t^{e_b/2-1} + t^{-e_b/2} + t^{e_b/2}.$$
Since $b = C^k a$ and $e_b = 6k + e_a$, we have

$$
\Delta_K(t)(t^{-1} + 1 + t) = -t^{6k+e_a}(t^{-6k}J_\hat{a}(t) + (\sqrt{t})^{-e_a}(t + t^{-1})(t^{-3k} - t^{-6k})) \\
+ t^{(6k+e_a)/2+1} + t^{(6k+e_a)/2-1} + t^{-(6k+e_a)/2} + t^{(6k+e_a)/2} \\
= -t^{e_a}J_\hat{a}(t) + t^{e_a/2-1} + t^{e_a/2+1} + t^{-3k-e_a/2} + t^{3k+e_a/2}.
$$

(4)

We are now ready to determine which closed 3-braids are L-space knots.

**Proof.** For $K = \hat{b}$ to be an L-space knot, the right side of equation (4) has to be have coefficients in $\pm 1$. This immediately restricts $s$ to be less than 4, since $s \geq 4$ implies that there will be two coefficients $\beta, \beta'$, the second and the penultimate, whose absolute values are greater than or equal to 4 in $J_\hat{a}(t)$ by (b) of Lemma [1]. Based on (4), this will result in at least one coefficient whose absolute value is greater than or equal to 2 in $\Delta_K(t)(t^{-1} + 1 + t)$, even after possible cancellation from the terms $t^{e_a/2-1}, t^{e_a/2+1}, t^{-3k-e_a/2},$ and $t^{3k+e_a/2}$. Similarly, if $s = 3$, then $p, q \geq 3$, and $|\beta|$ and $|\beta'|$ are both greater than or equal to 3, each of which would need to be cancelled out by at least two terms of $t^{e_a/2-1}, t^{e_a/2+1}, t^{-3k-e_a/2},$ and $t^{3k+e_a/2}$. In addition, part (c) of Lemma [1] gives that the absolute values of the third and third-to-last coefficients $\gamma, \gamma'$ are greater than or equal to 2, which would also need to be cancelled out in the sum of the right side of (4). This is impossible, so $s \neq 3$.

Now assume that $s = 2$. If $p > 2$, then the penultimate coefficient $|\beta'| = 2$ and the third coefficient $|\gamma| \geq 2$. Since $p, q$ are either both even or both odd, $\beta'$ and $\gamma$ have opposite signs. One of them is positive, which would not cancel out with any of the terms $t^{e_a/2-1}, t^{e_a/2+1}, t^{-3k-e_a/2},$ and $t^{3k+e_a/2}$. The case is similar for $q > 2$, so we must have that $p = 2$ and $q = 2$. This means that $a = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$. The Jones polynomial of this alternating closed braid is

$$J_\hat{a}(t) = t^{-2} - t^{-1} + 1 - t + t^2,$$

for a braid $b = C^k a$,

$$(t^{-1} + 1 + t)\Delta_\hat{b}(t) = -J_\hat{a}(t) + \frac{1}{t} + t + t^{-3k} + t^{3k} \\
= -(t^{-2} - t^{-1} + 1 - t + t^2) + \frac{1}{t} + t + t^{-3k} + t^{3k} \\
= -t^{-2} + 2t^{-1} - 1 + 2t - t^2 + t^{-3k} + t^{3k}$$

For all $k \neq 0$, this shows that the product on the left side of this equation has nonzero coefficients that are not $\pm 1$. This rules out the possibility that a closed 3-braid of this form can be an L-space knot.
Therefore, we need only to consider the case when \( s = 1 \). Assuming that both \( p, q \) are greater than 1, the absolute values of the third coefficient \( \gamma \) and the third-to-last coefficient \( \gamma' \) of \(-t^\alpha J_{\hat{\alpha}(t)}\) have the form

\[
\left(\frac{s^2 + 3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{q=3}\right) \gamma^t(q+p)/2 - 2
\]

and

\[
\left(\frac{s^2 + 3s}{2} - \#\{i : p_i = 1\} - \#\{i : q_i = 1\} - \delta_{p=3}\right) \gamma'^t(q+p)/2 + 2,
\]

respectively. If \(|\gamma|\) or \(|\gamma'|\) is 2, they need to be canceled out by at least one term out of

\[
t_e^a/2 - 1, \; t_e^a/2 + 1, \; t^{-3k - e_a/2}, \; t^{3k + e_a/2}.
\]

If \( q > 3 \) and \( p > 1 \), then \( \gamma = 2 \), and \((q+p)/2 - 2\) needs to be equal to \((p-q)/2 - 1\), \((p-q)/2 + 1\), \(-3k - (p-q)/2\), or \(3k + (p-q)/2\). Similarly, we have the constraints on \(-(q+p)/2 + 2\). We examine the resulting equations and rule out values of \( p \) and \( q \) which lead to contradictions. Setting \((q+i)/2 - 2\) equal to \((p-q)/2 - 1\) or \((p-q)/2 + 1\) gives \( q = 1 \) or \( q = 3 \). Setting \((q+i)/2 - 2\) equal to \(-3k - (p-q)/2\) or \(3k + (p-q)/2\) gives \( p = -3k + 2 \) or \( q = 3k + 2 \). Similarly, if \( k > 3 \) and \( q > 1 \), then we must have \( p = 3 \), \( q = 1 \), or \( p = 3k + 2 \). We dismiss the cases where \( p \) or \( q \leq 3 \) for now, and suppose that \( k \neq 0 \). We cannot have that \( p = -3k + 2 \) and \( q = 3k + 2 \) since they are both supposed to be positive. Therefore we suppose that \( p = -3k + 2 \) or \( q = 3k + 2 \). In the first case, \( k \) is negative. In the second case, \( k \) is positive. Either way, we end up having, for \( k < 0 \),

\[
\triangle_K(t) \cdot (t^{-1} + 1 + t) = \left(\pm t^{-\frac{3k + 2 + a}{2}} \mp t^{-\frac{3k + 2 + a}{2} + 1} \pm 0 \mp \cdots \pm 0 \mp t^{-\frac{3k + 2 + a}{2} - 1} \pm t^{-\frac{3k + 2 + a}{2}}\right)
\]

or, for \( k > 0 \),

\[
\triangle_K(t) \cdot (t^{-1} + 1 + t) = \left(\pm t^{\frac{3k + 2 + p}{2}} \mp t^{\frac{3k + 2 + p}{2} + 1} \pm 0 \mp \cdots \pm 0 \mp t^{\frac{3k + 2 + p}{2} - 1} \pm t^{\frac{3k + 2 + p}{2}}\right)
\]

One of the conditions on the product \( \triangle_K(t) \cdot (t^{-1} + 1 + t) \) is that the second and the penultimate coefficients are equal to zero. When \( k < 0 \), the terms \( t^{-\frac{3k + 2 + a}{2}}, t^{-\frac{3k + 2 + a}{2} - 1} \) are the second and the penultimate coefficient which is not zero since we assume \( p, q > 3 \), this is impossible so this case cannot happen. The same argument applies to rule out the second case when \( k > 0 \).
When both \( p, q = 3 \), we have that the alternating 3-braid \( a \) takes the form \( \sigma_1^3 \sigma_2^{-3} \). The Alexander polynomial of this alternating 3-braid is \[
\triangle_a(t) = 3 + \frac{1}{t^2} - \frac{2}{t} - 2t + t^2,
\]
obtained by multiplying the Alexander polynomial of the trefoil by itself, since this 3-braid is a connected sum of two (right-hand and left-hand) trefoils. It is clear from the Alexander polynomial that this knot cannot be an L-space knot due to the fact that several of its nonzero coefficients are not \( \pm 1 \). Now we consider a generic 3-braid \( b = C^k a \) with \( a = \sigma_1^3 \sigma_2^{-3} \). Since \( e_a = 0 \), the highest degree and the lowest degree of the Jones polynomial of \( \hat{a} \) are 3 and \( -3 \). By equation (4),
\[
(t^{-1} + 1 + t) \triangle_b(t) = -J_b(t) + \frac{1}{t} + t + t^{-3k} + t^{3k},
\]
where
\[
J_b(t) = 3 - \frac{1}{t^3} + \frac{1}{t^2} - \frac{1}{t} - t^2 - t^3.
\]
When \( k \neq 0 \), it is clear that the constant term 3 of \( J_b(t) \) will not be canceled out by the terms \( \frac{1}{t}, t, t^{-3k}, \) or \( t^{-3k} \). Thus none of the closure of braids of the form \( C^k \sigma_1^3 \sigma_2^{-3} \) will be an L-space knot. We may also rule out the case \( p \) or \( q = 2 \) since this would give a link rather than a knot. Thus, the only generic 3-braids whose closure can be an L-space knot are given below.

\[
\begin{array}{|c|c|}
\hline
C^k \sigma_1^q \sigma_2^{-q} & \text{for } q \text{ odd.} \\
C^k \sigma_1^p \sigma_2^{-1} & \text{for } p \text{ odd.} \\
\hline
\end{array}
\]

We now claim that \( C^k \sigma_1^p \sigma_2^{-1} \), for \( p \) odd and \( k > 0 \), represents an L-space knot. Note that:
\[
(\sigma_1 \sigma_2 \sigma_1)^{2k} \sigma_1^p \sigma_2^{-1} \sim (\sigma_2 \sigma_1)^{3k} \sigma_1^p \sigma_2^{-1} \\
\sim (\sigma_2 \sigma_1)^{3k-1} \sigma_1^{p+1}.
\]

The latter braid is the twisted torus knot, \( K(3,3k-1;2,1) \), which is known to be an L-space knot [Vaf14 Corollary 3.2]. Now if \( k < 0 \) then
\[
(\sigma_1 \sigma_2 \sigma_1)^{2k} \sigma_1^p \sigma_2^{-1} \sim (\sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1})^{-k} \sigma_1^p \sigma_2^{-1} \\
\sim \sigma_2^{-1} (\sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1})^{-k} \sigma_1^p \\
\sim \sigma_2^{-1} (\sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1})^{-k} \sigma_1^p \\
\sim \sigma_2^{-1} (\sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1})^{-k} \sigma_1^{-1} \sigma_1 \sigma_1^{p+1} \\
\sim (\sigma_1^{-1} \sigma_1^{-1})^{-3k+1} \sigma_1^{p+1} \\
\sim (\sigma_1 \sigma_2)^{3k-1} \sigma_1^{p+1}.
\]

Using [Vaf14 Corollary 3.2], we get that the closure of the latter braid represents an L-space knot. We should point out that [Vaf14 Corollary 3.2], as stated, only holds for positive knots. However, it turns out that every twisted \((p, q)\) torus knot,
where the twisting happens between $p - 1$ strands, admits an L-space surgery from the proof of [Vaf14 Theorem 3.1]. A similar argument shows that the closure of $C^k \sigma_1 \sigma_2^{-q}$, for $q$ odd, also represents an L-space knot. Notice that in this case the knot is isotopic to the closure of $(\sigma_2 \sigma_1)^{1-3k} \sigma_1^{-p}$. So its mirror image admits a positive L-space surgery.

\[ \square \]

References


