Furthermore, k maps each edge of T into itself. For if x belongs to the edge vw of T, any open set U_i containing x intersects this edge, so that v_i must equal either v or w. The definition of k then shows that $k(x)$ belongs to vw.

Let $h : \operatorname{Bd} T \to \operatorname{Bd} T$ be the restriction of k to $\operatorname{Bd} T$. Since h can be extended to the continuous map k, it is nulhomotopic. On the other hand, h is homotopic to the identity map of $\operatorname{Bd} T$ to itself; indeed, since h maps each edge of T into itself, the straight-line homotopy between h and the identity map of $\operatorname{Bd} T$ is such a homotopy. But the identity map i of $\operatorname{Bd} T$ is *not* nulhomotopic. ∎

Exercises

1. Show that if A is a retract of B^2, then every continuous map $f : A \to A$ has a fixed point.
2. Show that if $h : S^1 \to S^1$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.
3. Show that if A is a nonsingular 3 by 3 matrix having nonnegative entries, then A has a positive real eigenvalue.
4. Suppose that you are given the fact that for each n, there is no retraction $r : B^{n+1} \to S^n$. (This result can be proved using more advanced techniques of algebraic topology.) Prove the following:
   (a) The identity map $i : S^n \to S^n$ is not nulhomotopic.
   (b) The inclusion map $j : S^n \to \mathbb{R}^{n+1} - \mathbf{0}$ is not nulhomotopic.
   (c) Every nonvanishing vector field on B^{n+1} points directly outward at some point of S^n, and directly inward at some point of S^n.
   (d) Every continuous map $f : B^{n+1} \to B^{n+1}$ has a fixed point.
   (e) Every $n+1$ by $n+1$ matrix with positive real entries has a positive eigenvalue.
   (f) If $h : S^n \to S^n$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

*§56 The Fundamental Theorem of Algebra

It is a basic fact about the complex numbers that every polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree n with real or complex coefficients has n roots (if the roots are counted according to their multiplicities). You probably first were told this fact in high school algebra, although it is doubtful that it was proved for you at that time.

The proof is, in fact, rather hard; the most difficult part is to prove that every polynomial equation of positive degree has *at least one* root. There are various ways