Exercises

1. Check the distributive laws for $\cup$ and $\cap$ and DeMorgan's laws.

2. Determine which of the following statements are true for all sets A, B, C, and D. If a double implication fails, determine whether one or the other of the possible implications holds. If an equality fails, determine whether the statement becomes true if the "equals" symbol is replaced by one or the other of the inclusion symbols $\subset$ or $\supset$.
   (a) $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cup C)$.
   (b) $A \subset B$ or $A \subset C \Leftrightarrow A \subset (B \cup C)$.
   (c) $A \subset B$ and $A \subset C \Leftrightarrow A \subset (B \cap C)$.
   (d) $A \subset B$ or $A \subset C \Leftrightarrow A \subset (B \cap C)$.
   (e) $A - (A - B) = B$.
   (f) $A - (B - A) = A - B$.
   (g) $A \cap (B - C) = (A \cap B) - (A \cap C)$.
   (h) $A \cup (B - C) = (A \cup B) - (A \cup C)$.
   (i) $(A \cap B) \cup (A - B) = A$.
   (j) $A \subset C$ and $B \subset D \Rightarrow (A \times B) \subset (C \times D)$.
   (k) The converse of (j).
   (l) The converse of (j), assuming that A and B are nonempty.
   (m) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.
   (n) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
   (o) $A \times (B - C) = (A \times B) - (A \times C)$.
   (p) $(A - B) \times (C - D) = (A \times C - B \times C) - A \times D$.
   (q) $(A \times B) - (C \times D) = (A - C) \times (B - D)$.

3. (a) Write the contrapositive and converse of the following statement: "If $x < 0$, then $x^2 - x > 0$," and determine which (if any) of the three statements are true.
   (b) Do the same for the statement "If $x > 0$, then $x^2 - x > 0$."

4. Let A and B be sets of real numbers. Write the negation of each of the following statements:
   (a) For every $a \in A$, it is true that $a^2 \in B$.
   (b) For at least one $a \in A$, it is true that $a^2 \in B$.
   (c) For every $a \in A$, it is true that $a^2 \notin B$.
   (d) For at least one $a \notin A$, it is true that $a^2 \in B$.

5. Let $\mathcal{A}$ be a nonempty collection of sets. Determine the truth of each of the following statements and of their converses:
   (a) $x \in \bigcup_{A \in \mathcal{A}} A \Rightarrow x \in A$ for at least one $A \in \mathcal{A}$.
   (b) $x \in \bigcup_{A \in \mathcal{A}} A \Rightarrow x \in A$ for every $A \in \mathcal{A}$.
   (c) $x \in \bigcap_{A \in \mathcal{A}} A \Rightarrow x \in A$ for at least one $A \in \mathcal{A}$.
   (d) $x \in \bigcap_{A \in \mathcal{A}} A \Rightarrow x \in A$ for every $A \in \mathcal{A}$.

6. Write the contrapositive of each of the statements of Exercise 5.

the set A has the least upper bound property. For, given any su[illegible] an upper bound in A, it follows that its least upper bound (in the real numbers) must be in A. For example, the subset $\{-1/2n \mid n \in \mathbb{Z}_+\}$ of A, though it has no largest element, does have a least upper bound in A, the number 0.

On the other hand, the set $B = (-1, 0) \cup (0, 1)$ does not have the least upper bound property. The subset $\{-1/2n \mid n \in \mathbb{Z}_+\}$ of B is bounded above by any element of $(0, 1)$, but it has no least upper bound in B.

Exercises

Equivalence Relations

1. Define two points (x_0, y_0) and (x_1, y_1) of the plane to be equivalent if $y_0 - x_0^2 = y_1 - x_1^2$. Check that this is an equivalence relation and describe the equivalence classes.
2. Let C be a relation on a set A. If $A_0 \subset A$, define the ***restriction*** of C to A_0 to be the relation $C \cap (A_0 \times A_0)$. Show that the restriction of an equivalence relation is an equivalence relation.
3. Here is a "proof" that every relation C that is both symmetric and transitive is also reflexive: "Since C is symmetric, aCb implies bCa. Since C is transitive, aCb and bCa together imply aCa, as desired." Find the flaw in this argument.
4. Let $f : A \to B$ be a surjective function. Let us define a relation on A by setting $a_0 \sim a_1$ if

$$f(a_0) = f(a_1).$$

   (a) Show that this is an equivalence relation.
   (b) Let A^* be the set of equivalence classes. Show there is a bijective correspondence of A^* with B.
5. Let S and S' be the following subsets of the plane:

$$S = \{(x, y) \mid y = x + 1 \text{ and } 0 < x < 2\},$$
$$S' = \{(x, y) \mid y - x \text{ is an integer}\}.$$

   (a) Show that S' is an equivalence relation on the real line and $S' \supset S$. Describe the equivalence classes of S'.
   (b) Show that given any collection of equivalence relations on a set A, their intersection is an equivalence relation on A.
   (c) Describe the equivalence relation T on the real line that is the intersection of all equivalence relations on the real line that contain S. Describe the equivalence classes of T.

5. Prove the following properties of $\mathbb{Z}$ and $\mathbb{Z}_+$:

(a) $a, b \in \mathbb{Z}_+ \Rightarrow a + b \in \mathbb{Z}_+$. [*Hint:* Show that given $a \in \mathbb{Z}_+$, the set $X = \{x \mid x \in \mathbb{R} \text{ and } a + x \in \mathbb{Z}_+\}$ is inductive.]

(b) $a, b \in \mathbb{Z}_+ \Rightarrow a \cdot b \in \mathbb{Z}_+$.

(c) Show that $a \in \mathbb{Z}_+ \Rightarrow a - 1 \in \mathbb{Z}_+ \cup \{0\}$. [*Hint:* Let $X = \{x \mid x \in \mathbb{R}$ and $x - 1 \in \mathbb{Z}_+ \cup \{0\}\}$; show that X is inductive.]

(d) $c, d \in \mathbb{Z} \Rightarrow c + d \in \mathbb{Z}$ and $c - d \in \mathbb{Z}$. [*Hint:* Prove it first for $d = 1$.]

(e) $c, d \in \mathbb{Z} \Rightarrow c \cdot d \in \mathbb{Z}$.

6. Let $a \in \mathbb{R}$. Define inductively

$$a^1 = a,$$
$$a^{n+1} = a^n \cdot a$$

for $n \in \mathbb{Z}_+$. (See §7 for a discussion of the process of inductive definition.) Show that for $n, m \in \mathbb{Z}_+$ and $a, b \in \mathbb{R}$,

$$a^n a^m = a^{n+m},$$
$$(a^n)^m = a^{nm},$$
$$a^m b^m = (ab)^m.$$

These are called the ***laws of exponents***. [*Hint:* For fixed n, prove the formulas by induction on m.]

7. Let $a \in \mathbb{R}$ and $a \neq 0$. Define $a^0 = 1$, and for $n \in \mathbb{Z}_+$, $a^{-n} = 1/a^n$. Show that the laws of exponents hold for $a, b \neq 0$ and $n, m \in \mathbb{Z}$.

8. (a) Show that $\mathbb{R}$ has the greatest lower bound property.

(b) Show that $\inf\{1/n \mid n \in \mathbb{Z}_+\} = 0$.

(c) Show that given a with $0 < a < 1$, $\inf\{a^n \mid n \in \mathbb{Z}_+\} = 0$. [*Hint:* Let $h = (1 - a)/a$, and show that $(1 + h)^n \geq 1 + nh$.]

9. (a) Show that every nonempty subset of $\mathbb{Z}$ that is bounded above has a largest element.

(b) If $x \notin \mathbb{Z}$, show there is exactly one $n \in \mathbb{Z}$ such that $n < x < n + 1$.

(c) If $x - y > 1$, show there is at least one $n \in \mathbb{Z}$ such that $y < n < x$.

(d) If $y < x$, show there is a rational number z such that $y < z < x$.

10. Show that every positive number a has exactly one positive square root, as follows:

(a) Show that if $x > 0$ and $0 \leq h < 1$, then

$$(x + h)^2 \leq x^2 + h(2x + 1),$$
$$(x - h)^2 \geq x^2 - h(2x).$$

(b) Let $x > 0$. Show that if $x^2 < a$, then $(x + h)^2 < a$ for some $h > 0$; and if $x^2 > a$, then $(x - h)^2 > a$ for some $h > 0$.

Exercises

1. Show that $\mathbb{Q}$ is countably infinite.
2. Show that the maps f and g of Examples 1 and 2 are bijections.
3. Let X be the two-element set $\{0, 1\}$. Show there is a bijective correspondence between the set $\mathcal{P}(\mathbb{Z}_+)$ and the cartesian product X^ω.
4. (a) A real number x is said to be ***algebraic*** (over the rationals) if it satisfies some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with rational coefficients a_i. Assuming that each polynomial equation has only finitely many roots, show that the set of algebraic numbers is countable.

(b) A real number is said to be ***transcendental*** if it is not algebraic. Assuming the reals are uncountable, show that the transcendental numbers are uncountable. (It is a somewhat surprising fact that only two transcendental numbers are familiar to us: e and π. Even proving these two numbers transcendental is highly nontrivial.)

5. Determine, for each of the following sets, whether or not it is countable. Justify your answers.
   (a) The set A of all functions $f : \{0, 1\} \to \mathbb{Z}_+$.
   (b) The set B_n of all functions $f : \{1, \dots, n\} \to \mathbb{Z}_+$.
   (c) The set $C = \bigcup_{n \in \mathbb{Z}_+} B_n$.
   (d) The set D of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$.
   (e) The set E of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$.
   (f) The set F of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$ that are "eventually zero." [We say that f is ***eventually zero*** if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$.]
   (g) The set G of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually 1.
   (h) The set H of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually constant.
   (i) The set I of all two-element subsets of $\mathbb{Z}_+$.
   (j) The set J of all finite subsets of $\mathbb{Z}_+$.
6. We say that two sets A and B ***have the same cardinality*** if there is a bijection of A with B.

(a) Show that if $B \subset A$ and if there is an injection

$$f : A \longrightarrow B,$$

then A and B have the same cardinality. [*Hint:* Define $A_1 = A$, $B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$. Define a bijection $h : A \to B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n, \\ x & \text{otherwise.]} \end{cases}$$

Order Relations

6. Define a relation on the plane by setting

$$(x_0, y_0) < (x_1, y_1)$$

if either $y_0 - x_0^2 < y_1 - x_1^2$, or $y_0 - x_0^2 = y_1 - x_1^2$ and $x_0 < x_1$. Show that this is an order relation on the plane, and describe it geometrically.

7. Show that the restriction of an order relation is an order relation.

8. Check that the relation defined in Example 7 is an order relation.

9. Check that the dictionary order is an order relation.

10. (a) Show that the map $f : (-1, 1) \to \mathbb{R}$ of Example 9 is order preserving.
(b) Show that the equation $g(y) = 2y/[1 + (1 + 4y^2)^{1/2}]$ defines a function $g : \mathbb{R} \to (-1, 1)$ that is both a left and a right inverse for f.

11. Show that an element in an ordered set has at most one immediate successor and at most one immediate predecessor. Show that a subset of an ordered set has at most one smallest element and at most one largest element.

12. Let $\mathbb{Z}_+$ denote the set of positive integers. Consider the following order relations on $\mathbb{Z}_+ \times \mathbb{Z}_+$:
(i) The dictionary order.
(ii) $(x_0, y_0) < (x_1, y_1)$ if either $x_0 - y_0 < x_1 - y_1$, or $x_0 - y_0 = x_1 - y_1$ and $y_0 < y_1$.
(iii) $(x_0, y_0) < (x_1, y_1)$ if either $x_0 + y_0 < x_1 + y_1$, or $x_0 + y_0 = x_1 + y_1$ and $y_0 < y_1$.
In these order relations, which elements have immediate predecessors? Does the set have a smallest element? Show that all three order types are different.

13. Prove the following:
Theorem. If an ordered set A has the least upper bound property, then it has the greatest lower bound property.

14. If C is a relation on a set A, define a new relation D on A by letting $(b, a) \in D$ if $(a, b) \in C$.
(a) Show that C is symmetric if and only if $C = D$.
(b) Show that if C is an order relation, D is also an order relation.
(c) Prove the converse of the theorem in Exercise 13.

15. Assume that the real line has the least upper bound property.
(a) Show that the sets

$$[0, 1] = \{x \mid 0 \le x \le 1\},$$
$$[0, 1) = \{x \mid 0 \le x < 1\}$$

have the least upper bound property.
(b) Does $[0, 1] \times [0, 1]$ in the dictionary order have the least upper bound property? What about $[0, 1] \times [0, 1)$? What about $[0, 1) \times [0, 1]$?