Exercises

1. Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.
2. Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.
   (a) Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.
   (b) Show that if Y is path connected, the set $[I, Y]$ has a single element.
3. A space X is said to be ***contractible*** if the identity map $i_X : X \to X$ is nulhomotopic.
   (a) Show that I and $\mathbb{R}$ are contractible.
   (b) Show that a contractible space is path connected.
   (c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.
   (d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

§52 The Fundamental Group

The set of path-homotopy classes of paths in a space X does not form a group under the operation $*$ because the product of two path-homotopy classes is not always defined. But suppose we pick out a point x_0 of X to serve as a "base point" and restrict ourselves to those paths that begin and end at x_0. The set of these path-homotopy classes does form a group under $*$. It will be called the *fundamental group* of X.

In this section, we shall study the fundamental group and derive some of its properties. In particular, we shall show that the group is a topological invariant of the space X, the fact that is of crucial importance in using it to study homeomorphism problems.

Let us first review some terminology from group theory. Suppose G and G' are groups, written multiplicatively. A ***homomorphism*** $f : G \to G'$ is a map such that $f(x \cdot y) = f(x) \cdot f(y)$ for all x, y; it automatically satisfies the equations $f(e) = e'$ and $f(x^{-1}) = f(x)^{-1}$, where e and e' are the identities of G and G', respectively, and the exponent -1 denotes the inverse. The ***kernel*** of f is the set $f^{-1}(e')$; it is a subgroup of G. The image of f, similarly, is a subgroup of G'. The homomorphism f is called a ***monomorphism*** if it is injective (or equivalently, if the kernel of f consists of e alone). It is called an ***epimorphism*** if it is surjective; and it is called an ***isomorphism*** if it is bijective.

Suppose G is a group and H is a subgroup of G. Let xH denote the set of all products xh, for $h \in H$; it is called a ***left coset*** of H in G. The collection of all such cosets forms a partition of G. Similarly, the collection of all right cosets Hx of H in G forms a partition of G. We call H a ***normal subgroup*** of G if $x \cdot h \cdot x^{-1} \in H$ for each $x \in G$ and each $h \in H$. In this case, we have $xH = Hx$ for each x, so that our two