is also a finite intersection of elements of $\mathcal{S}$, so it belongs to $\mathcal{B}$.

Exercises

1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

2. Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

3. Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

4. (a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?
   (b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.
   (c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

7. Consider the following topologies on $\mathbb{R}$:

$\mathcal{T}_1 =$ the standard topology,
$\mathcal{T}_2 =$ the topology of $\mathbb{R}_K$,
$\mathcal{T}_3 =$ the finite complement topology,
$\mathcal{T}_4 =$ the upper limit topology, having all sets $(a, b]$ as basis,
$\mathcal{T}_5 =$ the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

Determine, for each of these topologies, which of the others it contains.

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit on $\mathbb{R}$.

§14 The Order Topology

If X is a simply ordered set, there is a standard topology for X, defined using relation. It is called the *order topology*; in this section, we consider it and st of its properties.

Suppose that X is a set having a simple order relation $<$. Given elements of X such that $a < b$, there are four subsets of X that are called the *interv* mined by a and b. They are the following :

$$(a, b) = \{x \mid a < x < b\},$$
$$(a, b] = \{x \mid a < x \leq b\},$$
$$[a, b) = \{x \mid a \leq x < b\},$$
$$[a, b] = \{x \mid a \leq x \leq b\}.$$

The notation used here is familiar to you already in the case where X is the but these are intervals in an arbitrary ordered set. A set of the first type is ca ***open interval*** in X, a set of the last type is called a ***closed interval*** in X, and se second and third types are called ***half-open intervals***. The use of the term " this connection suggests that open intervals in X should turn out to be open se we put a topology on X. And so they will.

Definition. Let X be a set with a simple order relation; assume X has more th element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any)

(3) All int