6. Let A, B, and A_α denote subsets of a space X. Prove the following:
   (a) If $A \subset B$, then $\bar{A} \subset \bar{B}$.
   (b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
   (c) $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails.

7. Criticize the following "proof" that $\overline{\bigcup A_\alpha} \subset \bigcup \bar{A}_\alpha$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \overline{\bigcup A_\alpha}$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \bigcup \bar{A}_\alpha$.

8. Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds.
   (a) $\overline{A \cap B} = \bar{A} \cap \bar{B}$.
   (b) $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$.
   (c) $\overline{A - B} = \bar{A} - \bar{B}$.

9. Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}.$$

10. Show that every order topology is Hausdorff.

11. Show that the product of two Hausdorff spaces is Hausdorff.

12. Show that a subspace of a Hausdorff space is Hausdorff.

13. Show that X is Hausdorff if and only if the ***diagonal*** $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

14. In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

15. Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

16. Consider the five topologies on $\mathbb{R}$ given in Exercise 7 of §13.
   (a) Determine the closure of the set $K = \{1/n \mid n \in \mathbb{Z}_+\}$ under each of these topologies.
   (b) Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

17. Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $\mathcal{C}$ of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

18. Determine the closures of the following subsets of the ordered square:

$$A = \{(1/n) \times 0 \mid n \in \mathbb{Z}_+\},$$
$$B = \{(1 - 1/n) \times \tfrac{1}{2} \mid n \in \mathbb{Z}_+\},$$
$$C = \{x \times 0 \mid 0 < x < 1\},$$
$$D = \{x \times \tfrac{1}{2} \mid 0 < x < 1\},$$
$$E = \{\tfrac{1}{2} \times y \mid 0 < y < 1\}.$$

19. If $A \subset X$, we define the ***boundary*** of A by the equation

$$\operatorname{Bd} A = \bar{A} \cap \overline{(X - A)}.$$

(a) Show that $\operatorname{Int} A$ and $\operatorname{Bd} A$ are disjoint, and $\bar{A} = \operatorname{Int} A \cup \operatorname{Bd} A$.
(b) Show that $\operatorname{Bd} A = \varnothing \Leftrightarrow A$ is both open and closed.
(c) Show that U is open $\Leftrightarrow \operatorname{Bd} U = \bar{U} - U$.
(d) If U is open, is it true that $U = \operatorname{Int}(\bar{U})$? Justify your answer.

20. Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:
(a) $A = \{x \times y \mid y = 0\}$
(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$
(c) $C = A \cup B$
(d) $D = \{x \times y \mid x \text{ is rational}\}$
(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$
(f) $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$

***21.** (Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \bar{A}$ and complementation $A \to X - A$ are functions from this collection to itself.
(a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.
(b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained.

§18 Continuous Functions

The concept of continuous function is basic to much of mathematics. Continuous functions on the real line appear in the first pages of any calculus book, and continuous functions in the plane and in space follow not far behind. More general kinds of continuous functions arise as one goes further in mathematics. In this section, we shall formulate a definition of continuity that will include all these as special cases, and we shall study various properties of continuous functions. Many of these properties are direct generalizations of things you learned about continuous functions in calculus and analysis.

Continuity of a Function

Yet another method for constructing continuous functions that is familiar from analysis is to take the limit of an infinite sequence of functions. There is a theorem to the effect that if a sequence of continuous real-valued functions of a real variable converges uniformly to a limit function, then the limit function is necessarily continuous. This theorem is called the *Uniform Limit Theorem*. It is used, for instance, to demonstrate the continuity of the trigonometric functions, when one defines these functions rigorously using the infinite series definitions of the sine and cosine. This theorem generalizes to a theorem about maps of an arbitrary topological space X into a metric space Y. We shall prove it in §21.

Exercises

1. Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the ϵ-δ definition of continuity implies the open set definition.

2. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

3. Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.
(a) Show that i is continuous $\Leftrightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$.
(b) Show that i is a homeomorphism $\Leftrightarrow$ $\mathcal{T}' = \mathcal{T}$.

4. Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

6. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

7. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.
(b) Can you conjecture what functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$? As maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$? We shall return to this question in Chapter 3.

8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.
(a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.