Yet another method for constructing continuous functions that is familiar from analysis is to take the limit of an infinite sequence of functions. There is a theorem to the effect that if a sequence of continuous real-valued functions of a real variable converges uniformly to a limit function, then the limit function is necessarily continuous. This theorem is called the *Uniform Limit Theorem.* It is used, for instance, to demonstrate the continuity of the trigonometric functions, when one defines these functions rigorously using the infinite series definitions of the sine and cosine. This theorem generalizes to a theorem about maps of an arbitrary topological space X into a metric space Y. We shall prove it in §21.

Exercises

1. Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the ϵ-δ definition of continuity implies the open set definition.

2. Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

3. Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.
(a) Show that i is continuous $\Leftrightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$.
(b) Show that i is a homeomorphism $\Leftrightarrow$ $\mathcal{T}' = \mathcal{T}$.

4. Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

6. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

7. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.
(b) Can you conjecture what functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$? As maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$? We shall return to this question in Chapter 3.

8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.
(a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.

(b) Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [*Hint:* Use the pasting lemma.]

9. Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$; suppose that $f|A_\alpha$, is continuous for each α.

(a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.

(b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.

(c) An indexed family of sets $\{A_\alpha\}$ is said to be ***locally finite*** if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

10. Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = f(a) \times g(c).$$

Show that $f \times g$ is continuous.

11. Let $F : X \times Y \to Z$. We say that F is ***continuous in each variable separately*** if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k : Y \to Z$ defined by $k(y) = F(x_0 \times y)$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.

12. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0. \\ 0 & \text{if } x \times y = 0 \times 0. \end{cases}$$

(a) Show that F is continuous in each variable separately.

(b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.

(c) Show that F is not continuous.

13. Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f.

19 The Product Topology

e now return, for the remainder of the chapter, to the considera
s for imposing topologies on sets.

Exercises

1. Prove Theorem 19.2.
2. Prove Theorem 19.3.
3. Prove Theorem 19.4.
4. Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic with $X_1 \times \cdots \times X_n$.
5. One of the implications stated in Theorem 19.6 holds for the box topology. Which one?
6. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?
7. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.
8. Given sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation
$$h((x_1, x_2, \ldots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots).$$
Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?
9. Show that the choice axiom is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets, with $J \neq 0$, the cartesian product
$$\prod_{\alpha \in J} A_\alpha$$
is not empty.
10. Let A be a set; let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.
    (a) Show there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.
    (b) Let
    $$\mathcal{S}_\beta = \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\},$$
    and let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Show that $\mathcal{S}$ is a subbasis for $\mathcal{T}$.
    (c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous.
    (d) Let $f : A \to \prod X_\alpha$ be defined by the equation
    $$f(a) = (f_\alpha(a))_{\alpha \in J};$$
    let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z.