Exercises

1. Prove Theorem 19.2.
2. Prove Theorem 19.3.
3. Prove Theorem 19.4.
4. Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic with $X_1 \times \cdots \times X_n$.
5. One of the implications stated in Theorem 19.6 holds for the box topology. Which one?
6. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?
7. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.
8. Given sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \ldots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots).$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?
9. Show that the choice axiom is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets, with $J \neq 0$, the cartesian product

$$\prod_{\alpha \in J} A_\alpha$$

is not empty.
10. Let A be a set; let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.
   (a) Show there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.
   (b) Let

$$\mathcal{S}_\beta = \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\},$$

   and let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Show that $\mathcal{S}$ is a subbasis for $\mathcal{T}$.
   (c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous.
   (d) Let $f : A \to \prod X_\alpha$ be defined by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J};$$

   let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z.

Conversely, consider a basis element

$$U = \prod_{i \in \mathbb{Z}_+} U_i$$

for the product topology, where U_i is open in $\mathbb{R}$ for $i = \alpha_1, \dots, \alpha_n$ and $U_i = \mathbb{R}$ for all other indices i. Given $\mathbf{x} \in U$, we find an open set V of the metric topology such that $\mathbf{x} \in V \subset U$. Choose an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ in $\mathbb{R}$ centered about x_i and lying in U_i for $i = \alpha_1, \dots, \alpha_n$; choose each $\epsilon_i \leq 1$. Then define

$$\epsilon = \min\{\epsilon_i / i \mid i = \alpha_1, \dots, \alpha_n\}.$$

We assert that

$$\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U.$$

Let $\mathbf{y}$ be a point of $B_D(\mathbf{x}, \epsilon)$. Then for all i,

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon.$$

Now if $i = \alpha_1, \dots, \alpha_n$, then $\epsilon \leq \epsilon_i / i$, so that $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$; it follows that $|x_i - y_i| < \epsilon_i$. Therefore, $\mathbf{y} \in \prod U_i$, as desired. ∎

Exercises

1. (a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

(b) More generally, given $p \geq 1$, define

$$d'(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p \right]^{1/p}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

2. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

3. Let X be a metric space with metric d.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

(b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.