Exercises

1. (a) A G_δ *set* in a space X is a set A that equals a countable intersection of open sets of X. Show that in a first-countable T_1 space, every one-point set is a G_δ set.

   (b) There is a familiar space in which every one-point set is a G_δ set, which nevertheless does not satisfy the first countability axiom. What is it?

   The terminology here comes from the German. The "G" stands for "Gebiet," which means "open set," and the "δ" for "Durchschnitt," which means "intersection."

2. Show that if X has a countable basis $\{B_n\}$, then every basis $\mathcal{C}$ for X contains a countable basis for X. [*Hint:* For every pair of indices n, m for which it is possible, choose $C_{n,m} \in \mathcal{C}$ such that $B_n \subset C_{n,m} \subset B_m$.]

3. Let X have a countable basis; let A be an uncountable subset of X. Show that uncountably many points of A are limit points of A.

4. Show that every compact metrizable space X has a countable basis. [*Hint:* Let $\mathcal{A}_n$ be a finite covering of X by $1/n$-balls.]

5. (a) Show that every metrizable space with a countable dense subset has a countable basis.

   (b) Show that every metrizable Lindelöf space has a countable basis.

6. Show that $\mathbb{R}_\ell$ and I_o^2 are not metrizable.

7. Which of our four countability axioms does S_Ω satisfy? What about $\bar{S}_\Omega$?

8. Which of our four countability axioms does $\mathbb{R}^\omega$ in the uniform topology satisfy?

9. Let A be a closed subspace of X. Show that if X is Lindelöf, then A is Lindelöf. Show by example that if X has a countable dense subset, A need not have a countable dense subset.

10. Show that if X is a countable product of spaces having countable dense subsets, then X has a countable dense subset.

11. Let $f : X \to Y$ be continuous. Show that if X is Lindelöf, or if X has a countable dense subset, then $f(X)$ satisfies the same condition.

12. Let $f : X \to Y$ be a continuous open map. Show that if X satisfies the first or the second countability axiom, then $f(X)$ satisfies the same axiom.

13. Show that if X has a countable dense subset, every collection of disjoint open sets in X is countable.

14. Show that if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

15. Give $\mathbb{R}^I$ the uniform metric, where $I = [0, 1]$. Let $\mathcal{C}(I, \mathbb{R})$ be the subspace consisting of continuous functions. Show that $\mathcal{C}(I, \mathbb{R})$ has a countable dense subset, and therefore a countable basis. [*Hint:* Consider those continuous functions whose graphs consist of finitely many line segments with rational end points.]

This proof that $\mathbb{R}_\ell^2$ is not normal is in some ways not very satisfying. We showed only that there must exist some proper nonempty subset A of L such that the sets A and $B = L - A$ are not contained in disjoint open sets of $\mathbb{R}_\ell^2$. But we did not actually find such a set A. In fact, the set A of points of L having rational coordinates is such a set, but the proof is not easy. It is left to the exercises.

Exercises

1. Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.

2. Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

3. Show that every order topology is regular.

4. Let X and X' denote a single set under two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; assume that $\mathcal{T}' \supset \mathcal{T}$. If one of the spaces is Hausdorff (or regular, or normal), what does that imply about the other?

5. Let $f, g : X \to Y$ be continuous; assume that Y is Hausdorff. Show that $\{x \mid f(x) = g(x)\}$ is closed in X.

6. Let $p : X \to Y$ be a closed continuous surjective map. Show that if X is normal, then so is Y. [*Hint:* If U is an open set containing $p^{-1}(\{y\})$, show there is a neighborhood W of y such that $p^{-1}(W) \subset U$.]

7. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. (Such a map is called a *perfect map*.)

(a) Show that if X is Hausdorff, then so is Y.

(b) Show that if X is regular, then so is Y.

(c) Show that if X is locally compact, then so is Y.

(d) Show that if X is second-countable, then so is Y. [*Hint:* Let $\mathcal{B}$ be a countable basis for X. For each finite subset J of $\mathcal{B}$, let U_J be the union of all sets of the form $p^{-1}(W)$, for W open in Y, that are contained in the union of the elements of J.]

8. Let X be a space; let G be a topological group. An ***action*** of G on X is a continuous map $\alpha : G \times X \to X$ such that, denoting $\alpha(g \times x)$ by $g \cdot x$, one has:

(i) $e \cdot x = x$ for all $x \in X$.

(ii) $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$ for all $x \in X$ and $g_1, g_2 \in G$.

Define $x \sim g \cdot x$ for all x and g; the resulting quotient space is denoted X/G and called the ***orbit space*** of the action α.

Theorem. Let G be a compact topological group; let X be a topological space; let α be an action of G on X. If X is Hausdorff, or regular, or normal, or locally compact, or second-countable, so is X/G.

[*Hint:* See Exercise 13 of §26.]