This proof that $\mathbb{R}_\ell^2$ is not normal is in some ways not very satisfying. We showed only that there must exist some proper nonempty subset A of L such that the sets A and $B = L - A$ are not contained in disjoint open sets of $\mathbb{R}_\ell^2$. But we did not actually find such a set A. In fact, the set A of points of L having rational coordinates is such a set, but the proof is not easy. It is left to the exercises.

Exercises

1. Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.

2. Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

3. Show that every order topology is regular.

4. Let X and X' denote a single set under two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; assume that $\mathcal{T}' \supset \mathcal{T}$. If one of the spaces is Hausdorff (or regular, or normal), what does that imply about the other?

5. Let $f, g : X \to Y$ be continuous; assume that Y is Hausdorff. Show that $\{x \mid f(x) = g(x)\}$ is closed in X.

6. Let $p : X \to Y$ be a closed continuous surjective map. Show that if X is normal, then so is Y. [*Hint:* If U is an open set containing $p^{-1}(\{y\})$, show there is a neighborhood W of y such that $p^{-1}(W) \subset U$.]

7. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. (Such a map is called a *perfect map*.)

(a) Show that if X is Hausdorff, then so is Y.

(b) Show that if X is regular, then so is Y.

(c) Show that if X is locally compact, then so is Y.

(d) Show that if X is second-countable, then so is Y. [*Hint:* Let $\mathcal{B}$ be a countable basis for X. For each finite subset J of $\mathcal{B}$, let U_J be the union of all sets of the form $p^{-1}(W)$, for W open in Y, that are contained in the union of the elements of J.]

8. Let X be a space; let G be a topological group. An ***action*** of G on X is a continuous map $\alpha : G \times X \to X$ such that, denoting $\alpha(g \times x)$ by $g \cdot x$, one has:

(i) $e \cdot x = x$ for all $x \in X$.

(ii) $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$ for all $x \in X$ and $g_1, g_2 \in G$.

Define $x \sim g \cdot x$ for all x and g; the resulting quotient space is denoted X/G and called the ***orbit space*** of the action α.

Theorem. *Let G be a compact topological group; let X be a topological space; let α be an action of G on X. If X is Hausdorff, or regular, or normal, or locally compact, or second-countable, so is X/G.*

[*Hint:* See Exercise 13 of §26.]

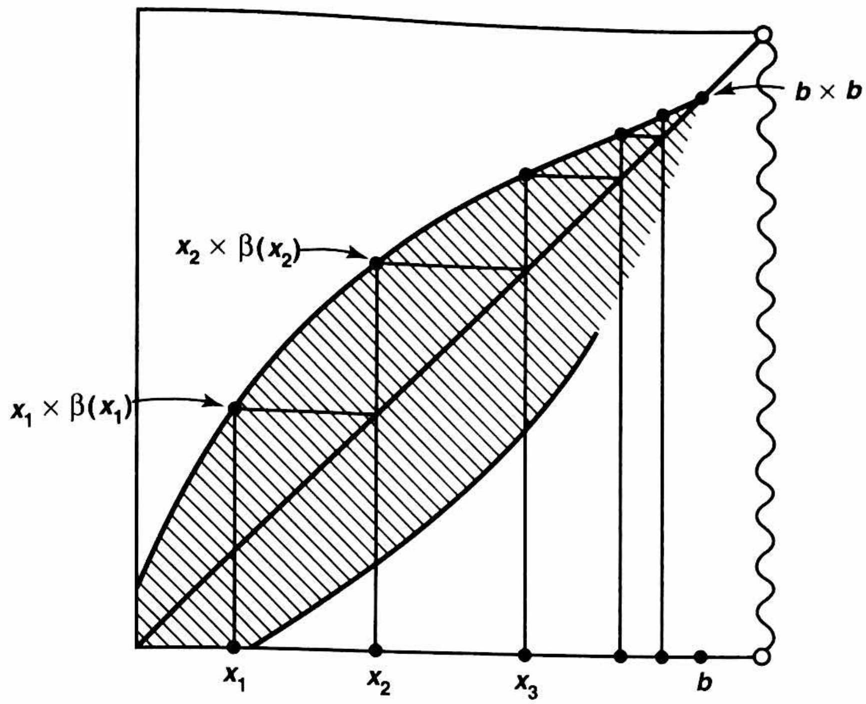

Figure 32.3

Exercises

1. Show that a closed subspace of a normal space is normal.
2. Show that if $\prod X_\alpha$ is Hausdorff, or regular, or normal, then so is X_α. (Assume that each X_α is nonempty.)
3. Show that every locally compact Hausdorff space is regular.
4. Show that every regular Lindelöf space is normal.
5. Is $\mathbb{R}^\omega$ normal in the product topology? In the uniform topology?
   It is not known whether $\mathbb{R}^\omega$ is normal in the box topology. Mary-Ellen Rudin has shown that the answer is affirmative if one assumes the continuum hypothesis [RM]. In fact, she shows it satisfies a stronger condition called *paracompactness*.
6. A space X is said to be ***completely normal*** if every subspace of X is normal. Show that X is completely normal if and only if for every pair A, B of separated sets in X (that is, sets such that $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$), there exist disjoint open sets containing them. [*Hint:* If X is completely normal, consider $X - (\bar{A} \cap \bar{B})$.]
7. Which of the following spaces are completely normal? Justify your answers.
   (a) A subspace of a completely normal space.
   (b) The product of two completely normal spaces.
   (c) A well-ordered set in the order topology.
   (d) A metrizable space.

Proof. Let X be completely regular; let Y be a subspace of X. Let x_0 be a point of Y, and let A be a closed set of Y disjoint from x_0. Now $A = \bar{A} \cap Y$, where $\bar{A}$ denotes the closure of A in X. Therefore, $x_0 \notin \bar{A}$. Since X is completely regular, we can choose a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(\bar{A}) = \{0\}$. The restriction of f to Y is the desired continuous function on Y.

Let $X = \prod X_\alpha$ be a product of completely regular spaces. Let $\mathbf{b} = (b_\alpha)$ be a point of X and let A be a closed set of X disjoint from $\mathbf{b}$. Choose a basis element $\prod U_\alpha$ containing $\mathbf{b}$ that does not intersect A; then $U_\alpha = X_\alpha$ except for finitely many α, say $\alpha = \alpha_1, \dots, \alpha_n$. Given $i = 1, \dots, n$, choose a continuous function

$$f_i : X_{\alpha_i} \to [0, 1]$$

such that $f_i(b_{\alpha_i}) = 1$ and $f_i(X - U_{\alpha_i}) = \{0\}$. Let $\phi_i(\mathbf{x}) = f_i(\pi_{\alpha_i}(\mathbf{x}))$; then ϕ_i maps X continuously into $\mathbb{R}$ and vanishes outside $\pi_{\alpha_i}^{-1}(U_{\alpha_i})$. The product

$$f(\mathbf{x}) = \phi_1(\mathbf{x}) \cdot \phi_2(\mathbf{x}) \cdot \dots \cdot \phi_n(\mathbf{x})$$

is the desired continuous function on X, for it equals 1 at $\mathbf{b}$ and vanishes outside $\prod U_\alpha$. ∎

EXAMPLE 1. The spaces $\mathbb{R}_\ell^2$ and $S_\Omega \times \bar{S}_\Omega$ are completely regular but not normal. For they are products of spaces that are completely regular (in fact, normal).

A space that is regular but not completely regular is much harder to find. Most of the examples that have been constructed for this purpose are difficult, and require considerable familiarity with cardinal numbers. Fairly recently, however, John Thomas [T] has constructed a much more elementary example, which we outline in Exercise 11.

Exercises

1. Examine the proof of the Urysohn lemma, and show that for given r,

$$f^{-1}(r) = \bigcap_{p>r} U_p - \bigcup_{q<r} U_q,$$

p, q rational.

2. (a) Show that a connected normal space having more than one point is uncountable.

(b) Show that a connected regular space having more than one point is uncountable.† [*Hint:* Any countable space is Lindelöf.]

3. Give a direct proof of the Urysohn lemma for a metric space (X, d) by setting

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

†Surprisingly enough, there does exist a connected *Hausdorff* space that is countably infinite. See Example 75 of [S-S].