The proof is almost a copy of Step 2 of the preceding proof; one merely replaces n by α, and $\mathbb{R}^\omega$ by $\mathbb{R}^J$, throughout. One needs one-point sets in X to be closed in order to be sure that, given $x \neq y$, there is an index α such that $f_\alpha(x) \neq f_\alpha(y)$.

A family of continuous functions that satisfies the hypotheses of this theorem is said to *separate points from closed sets* in X. The existence of such a family is readily seen to be equivalent, for a space X in which one-point sets are closed, to the requirement that X be completely regular. Therefore one has the following immediate corollary:

Theorem 34.3. *A space X is completely regular if and only if it is homeomorphic to a subspace of $[0,1]^J$ for some J.*

Exercises

1. Give an example showing that a Hausdorff space with a countable basis need not be metrizable.
2. Give an example showing that a space can be completely normal, and satisfy the first countability axiom, the Lindelöf condition, and have a countable dense subset, and still not be metrizable.
3. Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.
4. Let X be a locally compact Hausdorff space. Is it true that if X has a countable basis, then X is metrizable? Is it true that if X is metrizable, then X has a countable basis?
5. Let X be a locally compact Hausdorff space. Let Y be the one-point compactification of X. Is it true that if X has a countable basis, then Y is metrizable? Is it true that if Y is metrizable, then X has a countable basis?
6. Check the details of the proof of Theorem 34.2.
7. A space X is ***locally metrizable*** if each point x of X has a neighborhood that is metrizable in the subspace topology. Show that a compact Hausdorff space X is metrizable if it is locally metrizable. [*Hint:* Show that X is a finite union of open subspaces, each of which has a countable basis.]
8. Show that a regular Lindelöf space is metrizable if it is locally metrizable. [*Hint:* A closed subspace of a Lindelöf space is Lindelöf.] Regularity is essential; where do you use it in the proof?
9. Let X be a compact Hausdorff space that is the union of the closed subspaces X_1 and X_2. If X_1 and X_2 are metrizable, show that X is metrizable. [*Hint:* Construct a countable collection $\mathcal{A}$ of open sets of X whose intersections with X_i form a basis for X_i, for $i = 1, 2$. Assume $X_1 - X_2$ and $X_2 - X_1$ belong to $\mathcal{A}$. Let $\mathcal{B}$ consist of finite intersections of elements of $\mathcal{A}$.]

Clearly, F is continuous. To prove that F is an imbedding we need only to show that F is injective (because X is compact). Suppose that $F(x) = F(y)$. Then $\phi_i(x) = \phi_i(y)$ and $h_i(x) = h_i(y)$ for all i. Now $\phi_i(x) > 0$ for some i [since $\sum \phi_i(x) = 1$]. Therefore, $\phi_i(y) > 0$ also, so that $x, y \in U_i$. Then

$$\phi_i(x) \cdot g_i(x) = h_i(x) = h_i(y) = \phi_i(y) \cdot g_i(y).$$

Because $\phi_i(x) = \phi_i(y) > 0$, we conclude that $g_i(x) = g_i(y)$. But $g_i : U_i \to \mathbb{R}^m$ is injective, so that $x = y$, as desired. ∎

In many applications of partitions of unity, such as the one just given, all one needs to know is that the sum $\sum \phi_i(x)$ is positive for each x. In others, however, one needs the stronger condition that that $\sum \phi_i(x) = 1$. See §50.

Exercises

1. Prove that every manifold is regular and hence metrizable. Where do you use the Hausdorff condition?
2. Let X be a compact Hausdorff space. Suppose that for each $x \in X$, there is a neighborhood U of x and a positive integer k such that U can be imbedded in $\mathbb{R}^k$. Show that X can be imbedded in $\mathbb{R}^N$ for some positive integer N.
3. Let X be a Hausdorff space such that each point of X has a neighborhood that is homeomorphic with an open subset of $\mathbb{R}^m$. Show that if X is compact, then X is an m-manifold.
4. An indexed family $\{A_\alpha\}$ of subsets of X is said to be a ***point-finite indexed family*** if each $x \in X$ belongs to A_α for only finitely many values of α.
   Lemma (The shrinking lemma). Let X be a normal space; let $\{U_1, U_2, \dots\}$ be a point-finite indexed open covering of X. Then there exists an indexed open covering $\{V_1, V_2, \dots\}$ of X such that $\bar{V}_n \subset U_n$ for each n.
5. The Hausdorff condition is an essential part of the definition of a manifold; it is not implied by the other parts of the definition. Consider the following space: Let X be the union of the set $\mathbb{R} - \{0\}$ and the two-point set $\{p, q\}$. Topologize X by taking as basis the collection of all open intervals in $\mathbb{R}$ that do not contain 0, along with all sets of the form $(-a, 0) \cup \{p\} \cup (0, a)$ and all sets of the form $(-a, 0) \cup \{q\} \cup (0, a)$, for $a > 0$. The space X is called the ***line with two origins***.
   (a) Check that this is a basis for a topology.
   (b) Show that each of the spaces $X - \{p\}$ and $X - \{q\}$ is homeomorphic to $\mathbb{R}$.
   (c) Show that X satisfies the T_1 axiom, but is not Hausdorff.
   (d) Show that X satisfies all the conditions for a 1-manifold except for the Hausdorff condition.