GEOMETRIC ANALYSIS SUMMER COURSE

DANIEL WESER

These are lecture notes from a week-long summer course given at UT Austin in 2018. They’re currently missing the material from chapters 1, 2, and 4 of [Li12], and I may update them sometime later to include this content. So all of the information below is from [Lee97].

1 Background

Note: all manifolds are presumed to be smooth \((C^\infty)\), Hausdorff, and second countable.

**Definition 1.1.** Let \(V\) be a finite-dimensional vector space. \(V^*\) is called the space of **covectors** on \(V\) (a.k.a. the dual), and we denote the pairing \(V^* \times V \to \mathbb{R}\) by

\[
(\omega, X) \mapsto \langle \omega, X \rangle \quad \text{or} \quad (\omega, X) \mapsto \omega(X).
\]

A **covariant** \(k\)-**tensor** on \(V\) is a multilinear map

\[
F : V \times \cdots \times V \to \mathbb{R}.
\]

A **contravariant** \(l\)-**tensor** on \(V\) is a multilinear map

\[
F : V^* \times \cdots \times V^* \to \mathbb{R}.
\]

A **tensor of type** \(k,l\) is a multilinear map

\[
F : V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbb{R}.
\]

The **space of all covariant** \(k\)-**tensors** is denoted by \(T^k(V)\), and the **space of all contravariant** \(l\)-**tensors** is denoted by \(T_l(V)\). The **space of all mixed** \(k,l\)-**tensors** is denoted by \(T^{k,l}(V)\). The **rank** of a tensor is the number of arguments (vectors and/or covectors) it takes.

**Definition 1.2.** We define the **tensor product** as follows: if \(F \in T^k_l(V)\) and \(G \in T^p_q(V)\), then the tensor \(F \otimes G \in T^{k+p}_{l+q}(V)\) is defined by

\[
F \otimes G(\omega^1, \ldots, \omega^{k+q}, X_1, \ldots, X_{k+p})
= F(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k) G(\omega^{l+1}, \ldots, \omega^{l+q}, X_{k+1}, \ldots, X_{k+p}).
\]
Definition 1.3. If \((E_1, \ldots, E_n)\) is a basis for a vector space \(V\), then \((\varphi^1, \ldots, \varphi^n)\) denotes the dual basis for \(V^*\), defined by \(\varphi^i(E_j) = \delta^i_j\).

A basis for \(T^k_l(V)\) is given by the set of all tensors of the form
\[
E_{j_1} \otimes \ldots \otimes E_{j_k} \otimes \varphi^{i_1} \otimes \ldots \otimes \varphi^{i_l},
\]
where the indices \(i_p, j_q\) range from 1 to \(n\). These tensors act on basis elements by
\[
E_{j_1} \otimes \ldots \otimes E_{j_k} \otimes \varphi^{i_1} \otimes \ldots \otimes \varphi^{i_l} \varphi^s(E_{r_1}, \ldots, E_{r_l}) = \delta^s_j \delta^{i_1}_{r_1} \ldots \delta^{i_l}_{r_l}.
\]

Any tensor \(F \in T^k_l(V)\) can be written as
\[
F = F^{j_1 \ldots j_l}_{i_1 \ldots i_k} E_{j_1} \otimes \ldots \otimes E_{j_k} \otimes \varphi^{i_1} \otimes \ldots \otimes \varphi^{i_l},
\]
where
\[
F^{j_1 \ldots j_l}_{i_1 \ldots i_k} = F(\varphi^{j_1}, \ldots, \varphi^{j_l}, E_{i_1}, \ldots, E_{i_k}).
\]

Definition 1.4. A \(k\)-form is a covariant \(k\)-tensor on \(V\) that changes sign whenever two arguments are interchanged. The space of all such \(k\)-forms is denoted by \(\Lambda^k(V)\).

There is a natural bilinear, associative product on forms called the wedge product, defined on 1-forms \(\omega^1, \ldots, \omega^k\) by setting
\[
\omega^1 \wedge \ldots \wedge \omega^k(X_1, \ldots, X_k) = \det (\langle \omega^i, X_j \rangle)
\]
and extending by linearity.

Definition 1.5. Given a smooth manifold \(M\), local coordinates for a point \(p \in M\) are functions \(x^i : U \subset M \to \mathbb{R}\) for open \(U\) containing \(p\).

Definition 1.6. For any point \(p\) in a smooth manifold \(M\), we define the tangent space \(T_pM\) as follows: pick a coordinate chart \(\phi : U \to \mathbb{R}^n\), where \(U\) is an open subset of \(M\) containing \(p\). Suppose that two curves \(\gamma_1, \gamma_2 : (-1, 1) \to M\) are such that \(\gamma_1(0) = \gamma_2(0) = p\) and \(\phi \circ \gamma_1, \phi \circ \gamma_2 : (-1, 1) \to \mathbb{R}^n\) are differentiable. \(\gamma_1\) and \(\gamma_2\) are equivalent at 0 iff the derivatives of \(\phi \circ \gamma_1\) and \(\phi \circ \gamma_2\) at 0 agree. This defines an equivalence relation on the set of all differentiable curves initialized at 0, and the equivalence classes of all such curves are call the tangent vectors of \(M\) at \(p\). The tangent space of \(M\) at \(p\), denoted \(T_pM\), is defined as the set of all tangent vectors at \(p\).

Definition 1.7. We can find a basis of the tangent space at a point \(p\) as follows: given a chart \(\varphi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n\) with \(p \in U\), we can define an ordered basis
\[
\left( \left( \frac{\partial}{\partial x^i} \right)_p \right)_{i=1}^n
\]
of \(T_pM\) by
\[
\forall f \in C^\infty(M) : \left( \frac{\partial}{\partial x^i} \right)_p (f) := \left( \partial_i (f \circ \varphi^{-1}) \right)(\varphi(p)).
\]

Then, for every tangent vector \(v \in T_pM\), we have
\[
v = \sum_{i=1}^n v(x^i) \cdot \left( \frac{\partial}{\partial x^i} \right)_p.
\]
Definition 1.8. A vector bundle consists of:

1. a pair of smooth manifolds $E$ (the total space) and $M$ (the base)
2. a continuous surjective map $\pi : E \to M$ (the projection)
3. a vector space structure on each set $E_p := \pi^{-1}(p)$ (the fiber of $E$ over $p$)

such that for every $p \in M$, there exists an open neighborhood $U$ of $p$, an integer $k$, and a diffeomorphism $\varphi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ such that for all $q \in U$

- $(\pi \circ \varphi)(q, v) = q$ for all vectors $v \in \mathbb{R}^k$
- the map $v \mapsto \varphi(q, v)$ is a linear isomorphism between $\mathbb{R}^k$ and $\pi^{-1}(x)$.

Definition 1.9. The tangent bundle is given by $TM = \bigcup_{p \in M} T_p M$, and the cotangent bundle is given by $T^*M = \bigcup_{p \in M} (T_p M)^*$.

Definition 1.10. The bundle of $\binom{k}{l}$-tensors on $M$ is defined as

$$T^k_l M := \bigcup_{p \in M} T^k_l (T_p M),$$

and the bundle of $k$-forms is

$$\Lambda^k M := \bigcup_{p \in M} \Lambda^k (T_p M).$$

Definition 1.11. If $\pi : E \to M$ is a vector bundle over $M$, a section of $E$ is a map $F : M \to E$ such that $F(p) \in E_p = \pi^{-1}(p)$ for all $p \in M$. It is said to be a smooth section if it is smooth as a map between manifolds.

Definition 1.12. A tensor field on $M$ is a smooth section of some tensor bundle $T^k_l M$, and a differentiable $k$-form is a smooth section of some $\Lambda^k M$. The space of all $\binom{k}{l}$-tensor fields is denoted by $T^k_l (M)$, and the space of all covariant $k$-tensor fields (smooth sections of $T^k M$) is denoted by $T^k (M)$.

Definition 1.13. A vector field $X$ is a section of the tangent bundle, denoted $X \in \Gamma(TM)$. In coordinates,

$$X = X^i \partial_i, \quad X^i \in C^\infty(M),$$

where $\partial_i = \frac{\partial}{\partial x^i}$ is the coordinate basis as defined above.

Definition 1.14. A 1-form $\omega$ is a section of the cotangent bundle, denoted $\omega \in \Gamma(T^*M)$. In coordinates,

$$\omega = \omega_i dx^i, \quad w_i \in C^\infty(M).$$
2 Some Riemannian geometry

2.1 Riemannian metric

Definition 2.1. A Riemannian metric on a smooth manifold \( M \) is a family of positive definite inner products

\[
g_p : T_p M \times T_p M \to \mathbb{R}, \quad p \in M
\]
such that for all vector fields \( X, Y \) on \( M \)

\[
p \mapsto g_p(X(p), Y(p))
\]
defines a smooth function \( M \to \mathbb{R} \).

Equivalently, a Riemannian metric is a 2-tensor field \( g \in T^2(M) \) that is symmetric, e.g. \( g(X, Y) = g(Y, X) \), and positive definite, e.g. \( g(X, X) > 0 \) if \( X \neq 0 \).

In coordinates \((x^i)\), the components of \( g \) at a point \( p \) are given by

\[
g_{ij}(p) := g_p \left( \frac{\partial}{\partial x^i}_p, \frac{\partial}{\partial x^j}_p \right),
\]
so that

\[
g = g_{ij} \, dx^i \otimes dx^j,
\]
where \( \{dx^1, \ldots, dx^n\} \) is the dual basis of the tangent bundle.

Definition 2.2. A Riemannian metric thus determines an inner product on each tangent space \( T_p M \), which is typically written \( \langle X, Y \rangle := g(X, Y) \) for \( X, Y \in T_p M \).

We define the length of any tangent vector \( X \in T_p M \) to be \( |X| := \langle X, X \rangle^{1/2} \). We define the angle between any two nonzero vectors \( X, Y \in T_p M \) to be the unique \( \theta \in [0, \pi] \) such that \( \cos \theta = \langle X, Y \rangle / (|X||Y|) \). Finally, \( X \) and \( Y \) are orthogonal if \( \langle X, Y \rangle = 0 \). \( E_1, \ldots, E_n \) are orthonormal if \( \langle E_i, E_j \rangle = \delta_{ij} \).

Definition 2.3. Finally, a Riemannian manifold is a smooth manifold \( M \) together with a Riemannian metric \( g \), written \((M, g)\).

2.2 The musical isomorphisms

Definition 2.4. The metric \( g \) gives an isomorphism between \( TM \) and \( T^*M \) called the flat operator,

\[
\flat : TM \to T^*M
\]
defined by

\[
\flat(X)(Y) = g(X, Y).
\]

The inverse map is denoted by \( \sharp : T^*M \to TM \). The cotangent bundle is then endowed with the metric

\[
\langle \omega_1, \omega_2 \rangle = g(\sharp \omega_1, \sharp \omega_2).
\]
If $X \in \Gamma(TM)$, then
\[ b(X) = X_i dx^i, \]
where $X_i = g_{ij} X^j$. Hence the flat operator “lowers” an index, or equivalently converts a vector into a covector.

If $\omega \in \Gamma(T^*M)$, then
\[ \#(\omega) = \omega^i \partial_i, \]
where $\omega^i = g^{ij} \omega_j$ (and $g^{ij}$ are the elements of the inverse matrix $(g_{ij})^{-1}$). Hence the sharp operator “raises” an index, or equivalent converts a covector into a vector.

2.3 Connections on vector bundles

**Definition 2.5.** Let $\pi : E \to M$ be a vector bundle, and let $\Gamma(E)$ denote the space of all smooth sections of $E$. A **connection** is a map
\[ \nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (X,Y) \mapsto \nabla_X Y \]
with the properties

1. $\nabla_X Y$ is linear over $C^\infty(M)$ in $X$:
\[ \nabla_{fX}Y = f\nabla_X Y + g\nabla_X Y \quad \text{for } f, g \in C^\infty(M) \]

2. $\nabla_X Y$ is linear over $\mathbb{R}$ in $Y$:
\[ \nabla_X (aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2 \quad \text{for } a, b \in \mathbb{R} \]

3. $\nabla$ satisfies the following product rule:
\[ \nabla_X (fY) = f\nabla_X Y + (Xf)Y \quad \text{for } f \in C^\infty(M). \]

for all $f \in C^\infty(M)$, $X \in \Gamma(TM)$, and $Y \in \Gamma(E)$.

**Definition 2.6.** A connection on the tangent bundle is called a **linear connection**:
\[ \nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM). \]

We note that $\nabla_X Y$ is called the **covariant derivative of $Y$ in the direction of $X$**. In coordinates $(x^i)$, we can expand $\nabla_{\partial_i} \partial_j$ in terms of the coordinates:
\[ \nabla_{\partial_i} \partial_j = \Gamma^{k}_{ij} \partial_k \]

These functions $\Gamma^{k}_{ij}$ are called the **Christoffel symbols** of $\nabla$ with respect these coordinates.

**Proposition 2.7.** Let $\nabla$ be a linear connection, and let $X, Y \in \Gamma(TU)$ (a smooth section of the tangent space in an open set $U$). Write $X = X^i \partial_i$ and $Y = Y^j \partial_j$. Then,
\[ \nabla_X Y = (XY^k + X^i Y^j \Gamma^{k}_{ij}) \partial_k. \]

Hence, the action of a linear connection on $U$ is completely determined by its Christoffel symbols.
2.4 Riemannian connection

**Definition 2.8.** Let \( g \) be a Riemannian metric on a manifold \( M \). A linear connection \( \nabla \) is said to be **compatible with** \( g \) if it satisfies the following product rule for all vector fields \( X, Y, Z \):

\[
\nabla_X (Y, Z) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle.
\]

**Definition 2.9.** The **torsion tensor** of a linear connection is

\[
\tau : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)
\]

defined by

\[
\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],
\]

where \([X, Y]\) is the smooth vector field given by

\[
[X, Y](f) = X(Y(f)) - Y(X(f))
\]

for all \( f \in C^\infty(M) \).

**Definition 2.10.** A linear connection \( \nabla \) is said to be **torsion free** (or symmetric) if its torsion vanishes identically, e.g.

\[
\nabla_X Y - \nabla_Y X \equiv [X, Y].
\]

**Theorem 2.11.** [Fundamental theorem of Riemannian geometry] Let \((M, g)\) be a Riemannian manifold. There exists a unique linear connection \( \nabla \) on \( M \) that is compatible with \( g \) and torsion free.

**Proof.** We only do a sketch of the proof:

By a computation, the connection is determined by

\[
\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left( X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle \right).
\]

Using coordinates \((x^i)\) and letting \( X = \partial_i, Y = \partial_j, \) and \( Z = \partial_k, \) we can reduce the above to

\[
\langle \nabla \partial_i \partial_j, \partial_k \rangle = \frac{1}{2} \left( \partial_i \langle \partial_j, \partial_k \rangle + \partial_j \langle \partial_k, \partial_i \rangle - \partial_k \langle \partial_i, \partial_j \rangle \right),
\]

where we used the fact that the Lie brackets of coordinate vector fields are zero. Now, recalling the definitions of the metric coefficients and Christoffel symbols:

\[
g_{ij} = \langle \partial_i, \partial_j \rangle, \quad \nabla \partial_i \partial_j = \Gamma^m_{ij} \partial_m,
\]

we can insert these above to find

\[
\Gamma^k_{ik} = \frac{1}{2} g^{kl} \left( \partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij} \right).
\]

This is the formula for the Riemannian Christoffel symbols. \(\square\)
2.5 Curvature in the tangent bundle

**Definition 2.12.** The curvature endomorphism is a map

\[ R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \]

defined by

\[ R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z. \]

If \( X = \partial/\partial x_i \) and \( Y = \partial/\partial x_j \) are coordinate vector fields, then \( [X,Y] = 0 \), so

\[ R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z, \]

i.e. the curvature endomorphism measures the noncommutativity of the covariant derivative.

We can write the curvature endomorphism in coordinates \((x^i)\) by

\[ R = R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l, \]

where the coefficients \( R^l_{ijk} \) are determined by

\[ R(\partial_i,\partial_j)\partial_k = R^l_{ijk} \partial_l. \]

**Definition 2.13.** The curvature tensor is then given by

\[ Rm(X,Y,Z,W) := \langle R(X,Y)Z,W \rangle \]

for vector fields \( X,Y,Z,W \).

We can write it in local coordinates by

\[ Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l, \]

where \( R_{ijkl} = g_{lm} R^m_{ijkl} \).

**Proposition 2.14.** We have the following symmetries:

\[ R(X,Y)Z = -R(Y,X)Z \]

\[ 0 = R(X,Y)Z - +R(Y,Z)X - R(Z,X)Y \]

\[ Rm(X,Y,Z,W) = -Rm(X,Y,W,Z) \]

\[ Rm(X,Y,Z,W) = Rm(W,Z,X,Y). \]

**Definition 2.15.** The Ricci tensor as follows: given an orthonormal basis \( \partial_i \) for \( T_pM \) with respect to \( g(p) \), we define the Ricci curvature by

\[ Ric_p(X,Y) = \sum_i \langle R(\partial_i,X)Y,\partial_i \rangle. \]

The components of \( Ric \) are denoted by \( R_{ij} \), so that

\[ Ric = R_{ij} dx^i \otimes dx^j, \]

where \( R_{ij} := R^k_{kij} \) is from the above definition of the curvature endomorphism (we’re taking the trace of the curvature endomorphism on its first and last components – hence the \( k \) in both the sub- and super-scripts).
Definition 2.16. Finally, the **scalar curvature** is the function $S$ defined as the trace of the Ricci tensor:

$$S := g^{ij} R_{ij}.$$ 

2.6 Riemannian submanifolds and the second fundamental form

Definition 2.17. Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold of dimension $m$, let $M$ be a manifold of dimension $n$, and let $\iota : M \to \tilde{M}$ be an embedding. If $M$ is given the induced Riemannian metric $g := \iota^* \tilde{g}$, then $f$ is said to be an **isometric embedding**, and $M$ is said to be a **Riemannian submanifold** of $\tilde{M}$. $\tilde{M}$ is called the **ambient manifold**.

We henceforth assume that $M$ is an embedded Riemannian submanifold of $\tilde{M}$. We will use the standard symbols, e.g. $\nabla$, for operations in $M$, and we will use add tildes, e.g. $\tilde{\nabla}$, to write these operations in $\tilde{M}$. However, we can unambiguously use the inner-product notation $\langle X, Y \rangle$ to refer to either $g$ or $\tilde{g}$, since $g$ is just the restriction of $\tilde{g}$ to $TM$.

Definition 2.18. We can see that the set $T\tilde{M}|_M := \bigcup_{p \in M} T_p \tilde{M}$ is a smooth vector bundle over $M$. We call it the **ambient tangent bundle** over $M$.

At each $p \in M$, the ambient tangent space $T_p \tilde{M}$ splits as orthogonal direct sum $T_p \tilde{M} = T_p M \oplus N_p M$, where $N_p M := (T_p M)^\perp$ is the **normal space** at $p$ with respect to the inner product $\tilde{g}$ on $T_p M$.

The set $NM := \bigcup_{p \in M} N_p M$ is called the **normal bundle** of $M$. This is a smooth vector bundle over $M$. Given a point $p \in M$, there is an open set $\tilde{U}$ of $p$ in $\tilde{M}$ and a smooth orthonormal frame $(E_1, \ldots, E_m)$ on $\tilde{U}$ called an **adapted orthonormal frame** satisfying the following: the restrictions of $(E_1, \ldots, E_n)$ to $M$ form a local orthonormal frame for $TM$ and the last $m - n$ vectors $(E_{n+1}|_p, \ldots, E_m|_p)$ form a basis for $N_p M$ at each $p \in M$.

Definition 2.19. Projecting orthogonally at each point $p \in M$ onto the subspaces $T_p M$ and $N_p M$ gives maps called the **tangential** and **normal projections**

$$\pi^\top : T\tilde{M}|_M \to TM$$
$$\pi^\perp : T\tilde{M}|_M \to NM.$$ 

In terms of adapted orthonormals frames, these are the usual projections onto $\text{span}(E_1, \ldots, E_n)$ and $\text{span}(E_{n+1}, \ldots, E_m)$. If $X$ is a section of $T\tilde{M}|_M$, we use the shorthand notation $X^\top := \pi^\top X$ and $X^\perp := \pi^\perp X$ for its tangential and normal projections.
Definition 2.20. If $X, Y$ are vector fields in $\Gamma(TM)$, we can extend them to be vector fields on $\tilde{M}$, apply the ambient covariant derivative $\tilde{\nabla}$, and decompose them at points of $M$ to get

$$\tilde{\nabla}_X Y = (\tilde{\nabla}_X Y)^\top + (\tilde{\nabla}_X Y)^\perp.$$ 

We define the **second fundamental form** of $M$ to be the map $\Pi$ (read “two”)

$$\Pi : \Gamma(TM) \times \Gamma(TM) \to \Gamma(NM)$$

given by

$$\Pi(X, Y) := (\tilde{\nabla}_X Y)^\perp,$$

where $X$ and $Y$ have been extended arbitrarily to $\tilde{M}$. Since $\pi_\perp$ maps smooth sections to smooth sections, $\Pi(X, Y)$ is a smooth section of $NM$.

**Proposition 2.21.** The second fundamental form is

1. independent of the extensions $X$ and $Y$
2. bilinear over $C^\infty(M)$
3. symmetric in $X$ and $Y$.

**Theorem 2.22.** [Gauss formula] If $X, Y \in \Gamma(TM)$ are extended arbitrarily to vector fields on $\tilde{M}$, the following holds along $M$:

$$\tilde{\nabla}_X Y = \nabla_X Y + \Pi(X, Y).$$

Definition 2.23. Give a unit normal vector field $N$, we can define the **scalar second fundamental form** $h$ to be the symmetric 2-tensor on $M$ defined by

$$h(X, Y) := \langle \Pi(X, Y), N \rangle.$$

Definition 2.24. Raising one index of $h$, we get a tensor field $s \in T^1_1(M)$ called the **shape operator** of $M$. It is characterized by

$$\langle X, sY \rangle = h(X, Y) \quad \forall X, Y \in \Gamma(TM).$$

Because $h$ is symmetric, $s$ is self-adjoint:

$$\langle sX, Y \rangle = \langle X, sY \rangle.$$

At any point $p \in M$, the shape operator $s$ is a self-adjoint linear transformation on the tangent space $T_pM$. Hence, it has real eigenvalues $\kappa_1, \ldots, \kappa_n$ and an orthonormal basis $(E_1, \ldots, E_n)$ for $T_pM$ such that $sE_i = \kappa_i E_i$ (no summation). In this basis, both $h$ and $s$ are diagonal, and $h$ has the expression

$$h(X, Y) = \kappa_1 X^1 Y^1 + \ldots + \kappa_n X^n Y^n.$$

The eigenvalues of $s$ are called the **principal curvatures** of $M$ at $p$, and the corresponding eigenspaces are called the **principal directions**.
Definition 2.25. There are two combinations of the principal curvatures that are of particular importance: the Gaussian curvature is given by

\[ K = \kappa_1 \kappa_2 \cdots \kappa_n, \]

and the mean curvature is given by

\[ H = \frac{1}{n}(\kappa_1 + \kappa_2 + \ldots + \kappa_n). \]

3 Quick reference of dietary restrictions

<table>
<thead>
<tr>
<th>Object</th>
<th>Eats</th>
<th>Gives</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector fields</td>
<td>Functions</td>
<td>Functions</td>
</tr>
<tr>
<td>( \partial / \partial x^i )</td>
<td>Functions</td>
<td>Functions</td>
</tr>
<tr>
<td>k-forms</td>
<td>k-many vectors</td>
<td>Scalars</td>
</tr>
<tr>
<td>( dx^k )</td>
<td>Single vector fields</td>
<td>Functions</td>
</tr>
<tr>
<td>Riemannian metric</td>
<td>Pairs of vectors</td>
<td>Scalars</td>
</tr>
<tr>
<td>Covectors</td>
<td>Single vectors</td>
<td>Scalars</td>
</tr>
<tr>
<td>Covariant tensors</td>
<td>Many vectors</td>
<td>Scalars</td>
</tr>
<tr>
<td>Contravariant tensors</td>
<td>Many covectors</td>
<td>Scalars</td>
</tr>
<tr>
<td></td>
<td>Many covector fields</td>
<td>Functions</td>
</tr>
</tbody>
</table>

References
