AN INTRODUCTION TO VARIFOLDS

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1 Introduction

Why varifolds? A varifold is a measure-theoretic generalization of a differentiable submanifold of Euclidean space, where differentiability has been replaced by rectifiability. Varifolds can model almost any surface, including those with singularities, and they've been used in the following applications:

1. for Plateau's problem (finding area-minimizing soap bubble-like surfaces with boundary values given along a curve), varifolds can be used to show the existence of a minimal surface with the given boundary,

2. varifolds were used to show the existence of a generalized minimal surface in given compact smooth Riemannian manifolds,

3. varifolds were the starting point of mean curvature flow, after they were used to create a mathematical model of the motion of grain boundaries in annealing pure metals.

2 Preliminaries

We begin by covering the needed background from geometric measure theory.

Definition 2.1. [Mag12]
Given $k \in \mathbb{N}$, $\delta > 0$, and $E \subset \mathbb{R}^n$, the **k-dimensional Hausdorff measure of step δ** is

$$\mathcal{H}^k_\delta = \inf_{\mathcal{F}} \sum_{F \in \mathcal{F}} \omega_k \left(\frac{\operatorname{diam}(F)}{2} \right)^k,$$

where $\mathcal{F}$ is a covering of E by sets $F \subset \mathbb{R}^n$ such that $\operatorname{diam}(F) < \delta$.

Definition 2.2. [Mag12]
The **k-dimensional Hausdorff measure** of E is

$$\lim_{\delta \to 0^+} \mathcal{H}^k_\delta(E).$$

Definition 2.3. [Mag12]

Given a set $M \subset \mathbb{R}^n$, we say that M is k**-rectifiable** if (1) M is $\mathcal{H}^k$ measurable and $\mathcal{H}^k(M) < +\infty$, and (2) there exists countably many Lipschitz maps $f_j : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$\mathcal{H}^k\left(M \setminus \bigcup_{j\in\mathbb{N}} f_j(\mathbb{R}^k)\right) = 0.$$

Remark. We're saying that a set is k-rectifiable if piecewise it is a k-dimensional Lipschitz graph, except for sets of measure 0. Hence, recitifiable sets can have singularities. Also, recitifiable sets have tangent spaces a.e.

3 Introduction to varifolds

We begin with the definition of a varifold and a lengthy discussion of it.

Definition 3.1. [Lel12]

Let $U \subset \mathbb{R}^n$ be an open set. An **integral varifold** V of dimension k in U is a pair $V = (\Gamma, f)$, where (1) $\Gamma \subset U$ is a k-rectifiable set, and (2) $f : \Gamma \to \mathbb{N}\backslash\{0\}$ is a Borel map (called the **multiplicity function** of V).

We can naturally associate to V the following Radon measure:

$$\mu_V(A) \;=\; \int_{\Gamma\cap A} f \, d\mathcal{H}^k \qquad \text{for any Borel set } A.$$

We define the **area** of V to be $\mathcal{A}(V) := \mu_V(U)$.

Remark 1. Why the multiplicity function? Consider the sequence of spheres of radius one with center $1/n$, which we will label $B_{1/n} \subset \mathbb{R}^3$. Let B_0 be the ball of radius one centered at the origin. For each n, we have a 2-rectifiable set $B_0 \cup B_{1/n}$, whose surface area $\mathcal{H}^2(B_0 \cup B_{1/n}) = 8\pi$. Clearly, the sequence $B_0 \cup B_{1/n}$ converges to B_0. The sequence of surface areas $\mathcal{H}^2(B_0 \cup B_{1/n}) \equiv 8\pi$, yet the surface area of the limit of the sets is $\mathcal{H}^2(B_0) = 4\pi$. For the area functional to be continuous, we need to count B_0 as having multiplicity 2.

Another way of looking at the multiplicity function is by considering fibers. If you map one space to another ($\pi : X \to Y$), the fiber of a point $y \in Y$ is $\pi^{-1}(y)$. Since varifolds are measure theoretic, we really only care about the number of points in the fiber. So, if say $\pi : (B(0,1) \cup B(2,1)) \to B(0,1)$, then for $x \in B_0$ we could set $f(x) = |\pi^{-1}(x)|$ ($= 2$ if we use the simplest map). *Notation*: $B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}$.

Remark 2. So multiplicity allows an notion of sheets – why rectifiability? As noted above, rectifiability does not prevent singluarities. This is in contrast to the manifold setting, where singularities require surgery or some other handling. Hence, when you work with mean-curvature flow or Ricci flow, singularities developing on manifolds can break the flow since you leave the class of objects on which the flow was defined. However, with varifolds, singularities don't pose an issue – you don't leave the class of varifolds by a singularity forming. Is this desirable? It depends on the problem, of course. See the following figures for examples:

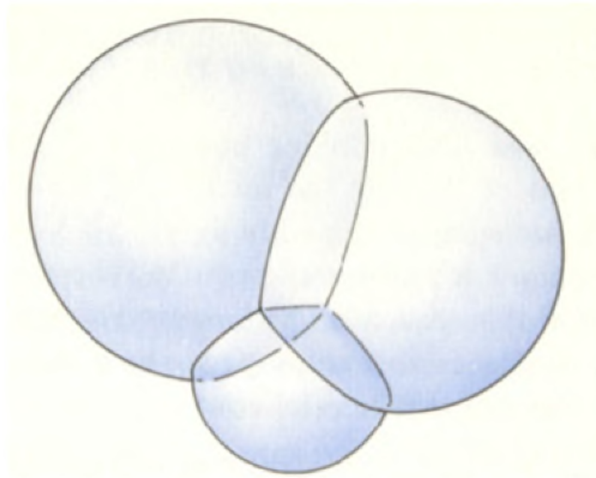

FIGURE 4-6. *A familiar soap bubble provides an example of a regular two-dimensional integral varifold in R^3.*

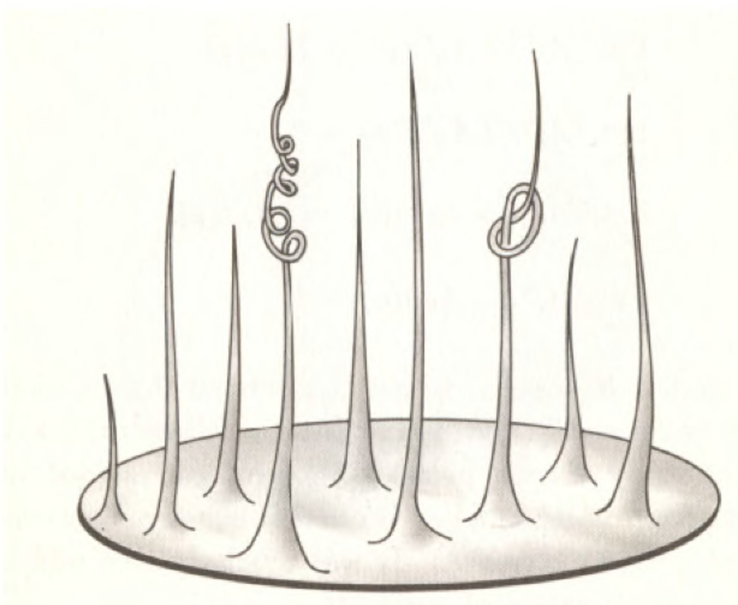

FIGURE 3-2. *A disk with spines.*

Images taken from [Alm66]

In the first surface, we see the triple intersection of bubbles producing singular sets, which is an example of physically relevant singularities. However, in the disk, we have a close approximation of what we know to be the minimal surface spanning a flat curve, except this approximation has singularities arising as spines. In both cases, manifolds would lose differentiability. However, by modeling as varifolds, we still have tangent spaces almost everywhere – plus there is ample theory for characterizing and handling these types of singularities – so we can work with these objects within the same general theory as, say, a smooth manifold considered as a varifold.

Remark 3. Finally, I want to mention that the definition given here will differ from some texts. In [Alm66], Almgren defines a varifold to be a positive real-valued function defined on the set of continuous differential k-forms satisfying some given axioms. In other contexts, authors will define a varifold as a measure over $\mathbb{R}^n \times G(k,n)$, where $G(k,n)$ is the Grassmannian manifold of all unoriented k-dimensional vector subspaces of $\mathbb{R}^n$. However, with certain assumptions, we can use differentiation theorems of measures to reduce to our case.

Now, we discuss some of the geometry of varifolds.

Definition 3.2. [Lel12]
If $\Phi : U \to W$ is a diffeomorphism and $V = (\Gamma, f)$ an integral varifold in U, then the **pushforward of V** is $\Phi_\# V = (\Phi(\Gamma), f \circ \Phi^{-1})$, which is itself an integral varifold in W.

Definition 3.3. [Lel12]
Given a vector field $X \in C^1_c(U, \mathbb{R}^n)$, the **one parameter family of diffeomorphisms generated by X** is $\Phi_t(x) := \Phi(x,t)$, where $\Phi : U \times \mathbb{R} \to U$ is the unique solution to

$$\begin{cases} \frac{\partial \Phi}{\partial t} &= X(\Phi) \\ \Phi(x,0) &= x. \end{cases}$$

Definition 3.4. [Lel12]
If V is a varifold in U and $X \in C^1_c(U, \mathbb{R}^n)$, then the **first variation of V along X** is defined by

$$\delta V(X) \;=\; \frac{d}{dt}\bigg|_{t=0} \mathcal{A}\big((\Phi_t)_\# V\big),$$

where Φ_t is the one-parameter family generated by X (and $\mathcal{A}$ is the area).

Discussion. The first variation is telling us how the area of our varifold initially changes as we move (smoothly) to nearby varifolds. Anytime we have a function with a critical point, as we initially move away from the critical point our function value doesn't change, so the variation of the function is zero.

Definition 3.5. [Lel12]
V is **stationary** if $\delta V(X) = 0$ for all $X \in C^1_c(U, \mathbb{R}^n)$.

Discussion. If $V = (\Gamma, f)$ is stationary, then it's a critical point of the varifold area functional. However, is Γ a classical minimal surface? Here's an example: suppose $U = \mathbb{R}^n$ and the multiplicity function $f \equiv 1$. Then,

$$\mathcal{A}(V) \;=\; \int_\Gamma f \, d\mathcal{H}^k \;=\; \int_\Gamma d\mathcal{H}^k \;=\; \mathcal{H}^k(\Gamma).$$

If Φ_t is a one-parameter family of diffeomorphisms generated by $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, then

$$\mathcal{A}\big((\Phi_t)_\# V\big) \;=\; \int_{\Phi_t(\Gamma)} f \circ \Phi_t^{-1} \, d\mathcal{H}^k \;=\; \int_{\Phi_t(\Gamma)} d\mathcal{H}^k \;=\; \mathcal{H}^k\big(\Phi_t(\Gamma)\big).$$

Since V is stationary,

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{A}\big((\Phi_t)_\# V\big) \;=\; \frac{d}{dt}\bigg|_{t=0} \mathcal{H}^k\big(\Phi_t(\Gamma)\big) \;=\; 0,$$

so Γ is also a critical point of the Hausdorff area functional. Hence, Γ is a minimal surface. In this way, we can see how varifolds generalize this basic setting by including multiplicity.

Definition 3.6. [Lel12]

We say that V has **bounded generalized mean curvature** if there exists a $C \geq 0$ such that

$$|\delta V(X)| \ \leq \ C \int_U |X| \, d\mu_V \qquad \text{for all } X \in C_c^1(U, \mathbb{R}^n).$$

Proposition 3.7. [Lel12]

If V is a varifold in U with bounded generalized mean curvature, then there is a bounded Borel map $H : U \to \mathbb{R}^n$ such that

$$\delta V(X) \ = \ - \int_U X \cdot H \, d\mu_V \qquad \textit{for all } X \in C_c^1(U, \mathbb{R}^n).$$

H is called the **generalized mean curvature** *of V and is defined μ_V-a.e.*

Discussion. If $V = (\Gamma, f)$ is stationary, then the proposition is telling us that $H \equiv 0$, so V has zero *generalized* mean curvature. Going back to our example with $U = \mathbb{R}^n$ and $f \equiv 1$, we can see that $\mu_V = \mathcal{H}^k\big|_{\Gamma}$. Then, for all $X \in C_c^1(U, \mathbb{R}^n)$, we have

$$\delta V(X) \ = \ - \int_{\mathbb{R}^n} X \cdot H \, d\mu_V \ = \ - \int_{\Gamma} X \cdot H \, d\mathcal{H}^k \ = \ 0.$$

Hence, Γ has zero mean curvature, so we see from a different perspective that Γ is a minimal surface.

We end this section by a basic compactness theorem:

Theorem 3.8. [All72]

Whenever κ is a real-valued function on the bounded open sets of $\mathbb{R}^n$, the set of m-dimensional varifolds in $\mathbb{R}^n$ satisfying $\mu_V(Z) \leq \kappa(Z)$ for all bounded open $Z \subset \mathbb{R}^n$ is compact.

4 Three geometric theorems

We discuss three famous geometric theorems that are of interest, but we first state a definition for completeness.

Definition 4.1. [Sim84]

M is an n-**dimensional** C^r **submanifold** of $\mathbb{R}^{n+k}$ if for all $y \in M$ there are

(1) open sets $U, V \subset \mathbb{R}^{n+k}$ such that $y \in U$ and $0 \in V$

(2) a C^r diffeomorphism $\Phi : U \to V$ such that $\Phi(y) = 0$ and $\Phi(M \cap U) = V \cap \mathbb{R}^n$.

Theorem 4.2. [Alm65]

Let M be a smooth compact n-dimensional Riemannian manifold. For each $0 < k < n$, there exists a stationary integral k-varifold in M.

Theorem 4.3. [All72]

If $V = (\Gamma, f)$ is a stationary integral k-varifold in a smooth compact n-dimensional Riemannian manifold M, $0 < k < n$, then there is an open dense subset of the support of V which is a smooth k-dimensional minimal submanifold of M (e.g. a critical point of the volume functional defined on M).

Discussion. These two theorems together give us the existence of minimal submanifolds of a given dimension within smooth compact Riemannian manifolds, *and* we know that the minimal submanifold is dense in the support of the varifold.

Theorem 4.4. Allard's regularity theorem [All72]

If $V = (\Gamma, f)$ is a k-dimensional integral varifold with bounded mean curvature in $U \subset \mathbb{R}^n$, then there exists an open set $W \subset U$ such that $\Gamma \cap W$ is a $C^{1,\alpha}$ submanifold of W without boundary and $\Gamma \cap W$ is dense in Γ. If in addition $f \equiv const$, μ_V a.e., then $\mu_V(\Gamma \setminus W) = 0$ (so $\mathcal{H}^k(\Gamma \setminus W) = 0$).

Discussion. This theorem gives us a handle on the singular set of a varifold. If the varifold has bounded mean curvature, then the theorem says that its singular set isn't too big so as to prevent the smooth part from being dense. And then, if the multiplicity function is constant, we know the k-dimensional Hausdorff measure of the singular set is 0.

5 General existence for Plateau's problem

Plateau's problem is stated as follows: given a closed curve and a function defined on the curve, find a minimal surface spanning the curve with the given boundary data. The idea is to model a soap film on a bent wire. We provide one theorem included in Almgren's book that was one of the preliminary results in providing a solution to the problem. We need two definitions first.

Definition 5.1. [Alm66]

Let V be an integral varifold of dimension k. We define the **support of** V, denoted by $\text{spt}(V)$, to be the smallest closed set F such that

$$\int_{\mathbb{R}^k} \phi \, d\mu_V \; = \; 0$$

for all $\phi \in C^1(\mathbb{R}^k)$ such that $\phi(x) = 0$ if $x \in F$.

Definition 5.2. [Alm66]

Let V be an integral varifold and C a closed curve in $\mathbb{R}^3$. We say that V **touches all of** C if $C \subset \overline{\text{spt}(V) \setminus C}$

Theorem 5.3. [Alm66]

Let m be a positive integer, and for each $i = 1, 2, ..., m$ let C_i be a simple closed curve in $\mathbb{R}^3$ such that: (1) C_i is a C^3 curve, and (2) $C_i \cap C_j = \emptyset$ if $i \neq j$. Then, there exists an integral varifold V of dimension 2 such that:

(a) $(V, ||C||)$ is stationary,

(b) V touches all of C,

(c) and, if V' is any other 2-dimensional integral varifold such that $(V', ||C||)$ is stationary and V' touches all of C, then $\mathcal{W}(V) \leq \mathcal{W}(V')$, where $\mathcal{W}$ is the continuous weight function defined on the space of all integral varifolds of dimension k.

Remark 4. Almgren defines $||C||$ as a varifold over the space of continuous differential k-forms through

$$||C||(\varphi) \;=\; \int_C ||\varphi||.$$

So, in the case above, we are saying that $(V, ||C||)$ is an element of a product space of varifolds.

What the theorem says, then, is that we can find an integral varifold V of dimension 2 that covers the curve C, such that V considered with the curve C is a critical point of an area functional *and* V's weight, defined by an objective weight function, is less than the weight of any other 2-d varifold covering C. Note that since Plateau's problem aims to model a film of soap, we require the dimension of our solution to be 2 in this case. So we find a solution that isn't necessarily smooth, as a soap film is, but it meets seemingly all other relevant physical criteria for a solution.

References

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