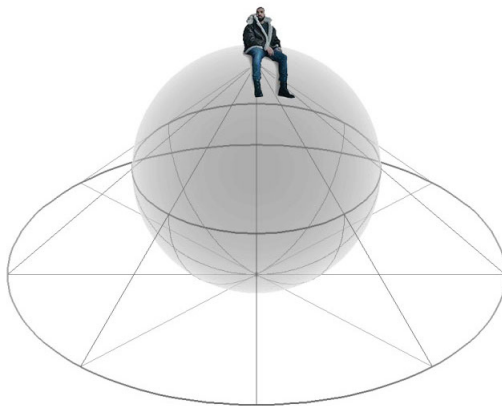


VIEWS from ∞



The extended complex plane $\hat{\mathbb{C}}$ is defined to be the complex plane with an additional point, ∞ , called the point at infinity. We visualize $\hat{\mathbb{C}}$ as the surface of a sphere via stereographic projection as follows. Think of the sphere as lying on top of the complex plane with Drake sitting on top at the north pole. If he looks at some point on \mathbb{C} then his line of sight will intersect the sphere at exactly one point. Conversely, if he looks at a point on the sphere besides the north pole, then his line of sight will intersect the complex plane at exactly one point. This gives a one-to-one correspondence between points on the sphere except the north pole and points on the complex plane. As we go further out on the complex plane, then the corresponding points on the sphere approach the north pole. Therefore we think of the north pole as infinity.

If a f is analytic except at singular points z_0 with $\lim_{z \rightarrow z_0} f(z) = \infty$ and $\lim_{z \rightarrow \infty} f(z)$ exists or is ∞ , then f extends to an analytic function $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. In particular, any rational function $\frac{p(z)}{q(z)}$ (p and q are polynomials) defines an analytic function $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ by

$$f(z) = \begin{cases} \frac{p(z)}{q(z)}; & q(z) \neq 0 \\ \lim_{w \rightarrow z} \frac{p(w)}{q(w)}; & q(z) = 0 \\ \lim_{w \rightarrow \infty} \frac{p(w)}{q(w)}; & z = \infty. \end{cases}$$

Note that the limits in the last two cases may be ∞ , which is fine since $\infty \in \hat{\mathbb{C}}$. In homework 11, we showed that these are actually all analytic functions $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. Something to think about: what are all of the analytic functions from $\hat{\mathbb{C}}$ whose images lie in $\mathbb{C} \subset \hat{\mathbb{C}}$?

The function $1/z$ is an analytic map $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ that interchanges 0 and ∞ . Using it we can define things at ∞ just as we did for points in \mathbb{C} . For example, $f : \mathbb{C} \rightarrow \mathbb{C}$ has a removable singularity/pole of order m /essential singularity at ∞ if the function $g(z) = f(1/z) : \mathbb{C} - \{0\} \rightarrow \mathbb{C}$ has a removable singularity/pole of order m /essential singularity at 0.

For the residue at infinity, we have to be a little more careful. Since residues have to do with integration, we should think of the residue at a point as something associated to $f(z)dz$ and not just $f(z)$ itself. Then if we change variables $z = 1/w$

$$f(z)dz \rightsquigarrow f(1/w)d(1/w) = -\frac{1}{w^2}f(1/w)dw.$$

Therefore we make the definition:

$$\text{Res}_\infty f(z) = \text{Res}_0 \left(-\frac{1}{z^2} f(1/z) \right).$$

Now the residue theorem can be generalized to:

Theorem 1. *Suppose f is analytic on \mathbb{C} except for finitely many singular points. Suppose C is a closed, positively oriented contour (and no singular points are on C by there may be some inside C). Then*

$$\begin{aligned} \oint_C f(z)dz &= 2\pi i \sum_{\text{singular points } p \text{ inside } C} \text{Res}_p f(z) \\ &= -2\pi i \sum_{\text{singular points } p \text{ outside } C} \text{Res}_p f(z) \end{aligned}$$

where ∞ is included as a singular point outside C and the negative sign in the second line comes from the fact that C is negatively oriented from the point of view of the points outside of C .

Exercise: compute $\oint_C z \cos(1/z)dz$, where C is the positively oriented unit circle centered at the origin, in two ways: by using the residues at singular points inside C and by using the residues at singular points outside C .