Elliptic involutive structures and generalized Higgs algebroids

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The overall theme of my thesis is on the module theory of certain Lie algebroids. This consists of three main parts:

- Higher direct images of Lie algebroid modules along a submersion.
- Modules over the distribution determined by a transversely holomorphic foliation.
- Modules over generalized Higgs algebroids.
  - Characteristic classes and a Grothendieck-Riemann-Roch theorem for Higgs bundles.

Motivation:

- Refined index theorems (e.g. the flat index theorem of Bismut and Lott).
- Generalized geometry.
- Physics (coisotropic branes and supersymmetric field theory).

Outline

Notation/basic objects

- M a smooth manifold.
- $A_M \xrightarrow{\rho} T_{\mathbb{C}}M$ a complex Lie algebroid over M.
- $\Omega^\bullet_{A_M}(M; E) = \Gamma(M; \Lambda^\bullet A_M^* \otimes E)$.
- $\nabla^{A_M;E} : \Omega^\bullet_{A_M}(M; E) \to \Omega^{\bullet+1}_{A_M}(M; E)$ an A_M-connection on a vector bundle $E \to M$.
- An A_M-module is a vector bundle $E \to M$ together with a flat A_M-connection.
- $H^\bullet_{A_M}(M; E)$ is the cohomology of the complex $(\Omega^\bullet_{A_M}(M; E), \nabla^{A_M;E})$.
- $\mathrm{Pic}_{A_M}(M)$ is the isomorphism classes of rank one A_M-modules.

Set-up:

- Smooth submersion $M \xrightarrow{\pi} B$ compatible with $A_M \to T_{\mathbb{C}}M, A_B \to T_{\mathbb{C}}B$:

$$\begin{tikzcd}
0 \arrow[r] & A_{M/B} \arrow[r] \arrow[d] & A_M \arrow[r, "\pi_A"] \arrow[d] & \pi^* A_B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{\mathbb{C}}(M/B) \arrow[r] & T_{\mathbb{C}}M \arrow[r, "\pi_*"] & \pi^* T_{\mathbb{C}}B \arrow[r] & 0.
\end{tikzcd}$$

- $E \to M$ an A_M-module.

Theorem

There is a canonical $\mathbb{Z}$-graded A_B-module $H^{\bullet}_{A_{M/B}}(M/B; E)$ obtained by taking the vertical Lie algebroid cohomology of E.

Properties

The connection is constructed from viewing $\nabla^{A_M;E}$ as an A_B-superconnection on the infinite rank bundle $\Gamma(M;\Lambda^\bullet A^*_{M/B}\otimes E)$ (Bismut construction).

Theorem (Projection formula)

If E is an A_M-module and F an A_B-module then

$$H^\bullet_{A_{M/B}}(M/B;\pi^*F\otimes E)\simeq F\otimes H^\bullet_{A_{M/B}}(M/B;E).$$

Leray-Hirsch theorem

Theorem (Twisted Leray-Hirsch)

Let E be an A_M-module and F an A_B-module. If there exist $\alpha_1, \dots, \alpha_d \in H^\bullet_{A_M}(M; E)$ that give a trivialization of $H^\bullet_{A_{M/B}}(M/B; E)$ then

$$H^\bullet_{A_M}(M; \pi^* F \otimes E) \simeq H^\bullet_{A_B}(B; F) \otimes H^\bullet_{A_{M/B}}(M/B; E).$$

Elliptic involutive structures

Definition

An elliptic involutive structure (EIS) on M is an elliptic complex Lie algebroid with injective anchor. It is determined by:

- *An involutive complex distribution $V \subset T_{\mathbb{C}}M$ with $V + \overline{V} = T_{\mathbb{C}}M$. Or*
- *A subbundle $V^{\perp} \subset T^*_{\mathbb{C}}M$ that generates a differential ideal and has $V^{\perp} \cap T^*M = 0$.*

Extreme cases:

- $V = T_{\mathbb{C}}M$. Modules are flat vector bundles.
- $V = T^{0,1}M$ for some complex structure on M. Modules are holomorphic vector bundles.

Generalizations of classical results [1]

Theorem (Newlander-Nirenberg)

If V is an EIS then locally there exist on M real coordinates $(t^1, \dots, t^d)$ and complex coordinates $(z^1, \dots, z^n)$ such that

$$V = \operatorname{span}\left\{\frac{\partial}{\partial t^i}, \frac{\partial}{\partial \overline{z}^j}\right\}.$$

Theorem (Poincaré lemma)

If $U \subset \mathbb{R}^d$ is open and convex and $W \subset \mathbb{C}^n$ open and pseudo-convex, then

$$H^k_{T_{\mathbb{C}}U \oplus T^{0,1}W}(U \times W) = \begin{cases} \mathbb{C}; & k = 0 \\ 0; & k \geq 1. \end{cases}$$

[1] see Trèves or Berhanu-Cordaro-Hounie

Corollary

$$H^\bullet_V(M) \simeq H^\bullet(\mathcal{O}_V),$$

where $\mathcal{O}_V$ *is the structure sheaf of* V*, i.e. the sheaf of germs of* C^∞ *functions annihilated by all vector fields in* V*.*

Modules over an EIS

Theorem

For V an EIS over M, There is an equivalence

$$
\begin{aligned}
(V\text{-}modules) &\longleftrightarrow (locally\ free\ sheaves\ of\ \mathcal{O}_V\text{-}modules) \\
(E, \nabla^{V;E}) &\longrightarrow \mathcal{O}_V(E) \\
C^\infty_M \otimes_{\mathcal{O}_V} \mathscr{E} &\longleftarrow \mathscr{E}.
\end{aligned}
$$

Corollary

For a V-module E, we have

$$H^\bullet_V(M; E) \simeq H^\bullet(\mathcal{O}(E)).$$

S^{2n+1} inherits an EIS V from the fibration $S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$ by

$$V^{\perp} = \pi^*(T^{1,0}\mathbb{C}P^n)^*.$$

- Leray-Hirsch applies and

$$H^{\bullet}_V(S^{2n+1}; \Lambda^{\bullet} V^{\perp}) \simeq H^{\bullet}(S^1; \mathbb{C}) \otimes H^{\bullet,\bullet}(\mathbb{C}P^n).$$

- $\mathrm{Pic}_V(S^{2n+1}) \simeq \mathbb{C}$ with $n \in \mathbb{Z} \subset \mathbb{C}$ corresponding to $\pi^*\mathcal{O}(n)$.

$$0 \to \mathrm{Pic}(\mathbb{C}P^n) \to \mathrm{Pic}_V(S^{2n+1}) \to \mathrm{Vect}^1_{flat}(S^1) \to 0.$$

- Even though the complex structure on $\mathbb{C}P^n$ is stable, the EIS on S^{2n+1} is not stable. Space of infinitesimal deformations is

$$H^1_V(S^{2n+1}; T_{\mathbb{C}}S^{2n+1}/V) \simeq H^{0,0}(\mathbb{C}P^n; T^{1,0}\mathbb{C}P^n) \simeq \mathfrak{sl}(n+1, \mathbb{C}).$$

A compact simply connected semi-simple Lie group inherits an EIS V from the fibration $G \xrightarrow{\pi} G/T$. V is the distribution determined by a Borel subalgebra.

- $H^\bullet_V(G;\Lambda^\bullet V^\perp) \simeq H^\bullet(T;\mathbb{C}) \otimes H^{\bullet,\bullet}(G/T)$.
- $\mathrm{Pic}_V(G) \simeq \mathfrak{t}^*_{\mathbb{C}}$, with integral weights corresponding to holomorphic line bundles on G/T.
- Again, the complex structure on G/T is stable, but the EIS has infinitesimal deformations given by

$$H^1_V(G;T_{\mathbb{C}}G/V) \simeq \mathfrak{t}^*_{\mathbb{C}} \otimes H^{0,0}(G/T;T^{1,0}G/T) \neq 0.$$

This EIS descends to an EIS on certain homogeneous spaces, e.g. $SU(n+1)/SU(n)(=S^{2n+1}), Spin(2n)/SU(n)$.

$E \xrightarrow{\pi} M$ flat complex vector bundle. The flat connection gives an integrable distribution $H \subset TE$. Have an EIS

$$V = T^{0,1} \ker \pi_* \oplus H_{\mathbb{C}}.$$

V induces an EIS on $\mathbb{P}(E)$.

Twisted generalized Higgs algebroids

Definition

A twisted generalized Higgs algebroid (TGHA) is an elliptic Lie algebroid $A \xrightarrow{\rho} T_{\mathbb{C}}M$ *such that* $K := \ker \rho$ *is abelian.*

Remarks:

- A determines an EIS $V := \rho(A)$.
- K is naturally a V-module and the obstruction to splitting

$$0 \to K \to A \to V \to 0$$

  is a class in $H^2_V(M; K)$. We call A an (untwisted) generalized Higgs algebroid (GHA) if this class vanishes.

Modules over Higgs algebroids

- Let A be a GHA with a splitting $A \simeq K \rtimes V$. An A-module is equivalent to the data of
  1. A V-module $E \to M$.
  2. A Higgs field $\theta \in H^0_V(M; \operatorname{End} E \otimes K^*)$ with $\theta \wedge \theta = 0$.
- If A is a TGHA, then by our results on EIS's, A is locally untwisted. Can work on the V-gerbe on K^* determined by the twisting class in $H^2_V(M; K)$.

Twisted generalized Higgs algebroids

Examples:

- The Lie algebroid determined by a generalized complex structure is a TGHA.
- Higgs bundles over a complex manifold X are modules over the GHA $T^{1,0}X \rtimes T^{0,1}X$. More generally, K-valued Higgs bundles are modules over $K \rtimes X$, with K a holomorphic vector bundle.

Now specialize to a Higgs bundle (E, θ) over a complex manifold X. We have the class $at(E) \in H^{1,1}(X; \operatorname{End} E)$ and $\theta \in H^{1,0}(X; \operatorname{End} E)$.

Definition

For $j \geq 0$, let

$$a_{2j+1}(E, \theta) = \operatorname{tr}(at(E)^j \theta) \in H^{j+1,j}(X).$$

Compatibility with the NAHT

If X is Kähler, the non-abelian Hodge theorem says that certain Higgs bundles correspond to certain flat vector bundles.
Flat vector bundles have characteristic classes in $c_{2j+1} \in H^{2j+1}(X;\mathbb{R})$, which vanish in degrees 3 and higher for Kähler manifolds (Reznikov's theorem).

Theorem

The classes a_{2j+1} and c_{2j+1} are compatible with the non-abelian Hodge theorem and the Hodge decomposition: if (E,θ) has corresponding flat connection ∇ then

- $\operatorname{Re} a_1(E,\theta) = c_1(E,\nabla) \in H^1(X;\mathbb{R})$.
- $a_{2j+1}(E,\theta) = 0$ *for* $j \geq 1$.

Using these classes, we have a secondary index theorem for our direct image construction along a projection:

Theorem

Suppose X is a complex manifold, Y is Kähler and (E, θ) is a Higgs bundle over $X \times Y$. Then

$$a_{2j+1}(\mathrm{ind}(\bar{\partial}_{X;E} + \theta)) = \int_Y e(TY) a_{2j+1}(E, \theta).$$

Thank you for your attention!