A hyperholomorphic line bundle on certain hyperkähler manifolds not admitting an S^1-action

Eric O. Korman
Department of Mathematics, University of Texas at Austin

Background

Hitchin [4] and Haydys [3] have shown that if a hyperkähler manifold M has an isometric S^1-action satisfying

$$\begin{gathered}\mathcal{L}_X\omega_I = 0, \quad \mathcal{L}_X\omega_J = -\omega_K, \quad \mathcal{L}_X\omega_K = \omega_J \\ \Leftrightarrow \\ d\alpha = 0, \quad d(J\alpha) = \omega_J, \quad d(K\alpha) = \omega_K,\end{gathered} \tag{1}$$

where X is the vector field generating the action and $\alpha = i_X\omega_I$, then

$$\omega_I - d(I\alpha)$$

is of type (1,1) in each complex structure. Thus if (M, ω_I) is prequantizable, M admits a hyperholomorphic line bundle with the above 2-form as its curvature.

There is a one-to-one correspondence between hyperholomorphic line bundles on M and hyperholomorphic line bundles on the twistor space Z of M. If ω_I is not integral, we instead get a holomorphic Lie algebroid extension of $\mathcal{T}_Z$ by $\mathcal{O}_Z$. Hitchin [4] gives Cěch cocycles for this Lie algebroid.

This line bundle makes an appearance in physics [9] in the case where M is the moduli space of Higgs bundles over a Riemann surface.

Generalization

In [6] we generalize this to

Theorem 1 [6]

Suppose M has a 1-form α satisfying

$$d\alpha = 0, \quad d(J\alpha) = \omega_J + F_1, \quad d(K\alpha) = \omega_K + F_2, \tag{2}$$

where F_1, F_2 are of type (1,1) in each complex structure. Then

$$\omega_I - d(I\alpha)$$

is of type (1,1) in each complex structure.

We use the differential form description since in the infinite dimensional setting X may not exist.

The proof uses the vanishing of the Nijenhuis tensor for each complex structure, and is very different from the proof in [4] in the case that $F_1 = 0 = F_2$ and X is Killing (i.e. $\nabla(I\alpha)$ is skew).

Holomorphic line bundle on twistor space

The twistor space $Z \to \mathbb{C}P^1$ has a meromorphic vertical symplectic (2,0) form

$$\omega = \frac{1}{i\zeta}(\omega_J + i\omega_K) + 2i\omega_I + \frac{1}{i}\zeta(\omega_J - i\omega_K).$$

and vector field $Y = X + i\zeta\frac{\partial}{\partial\zeta}$.

The holomorphic Lie algebroid on Z is determined by

Cěch description

Following Hitchin, find singular $(1,0)$-forms $\phi_j \in \mathscr{A}^{1,0}(U_j)$ ($\{U_j\}$ a cover of Z) so that

- $d\phi_j = i_Y\left(\frac{d\zeta}{\zeta} \wedge \omega\right) + \underbrace{\frac{1}{\zeta}F + \zeta\overline{F}}_{\text{type (1,1)}}$ (so $\bar\partial\phi_j \neq 0$, unlike in the case of an S^1-action).
- $\phi_k - \phi_j$ non-singular so gives a class in $H^1(d\mathcal{O}_Z)$, the group that classifies holomorphic Lie algebroid extensions of $\mathcal{T}_Z$ by $\mathcal{O}_Z$.
- Chern class of the Lie algebroid is $2i\omega_I - 2id(I\alpha) \in H^{1,1}(Z) \simeq H^1(\Omega^1_Z)$.

Hyperkähler reduction

Examples of hyperkähler manifolds with 1-forms satisfying (2) come from hyperkähler reduction. Suppose

- M is a hyperkähler manifold with a hamiltonian G-action with moment map $\mu_G : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$.
- α is a G-basic 1-form satisfying (1).
- The automatically locally constant functions $J\alpha(Y^*), K\alpha(Y^*)$ on $\mu_G^{-1}(0)$ are constant, where Y^* denotes the action vector field of $Y \in \mathfrak{g}$.
- $(J\alpha)_\mathfrak{g}, (K\alpha)_\mathfrak{g} \in \mathfrak{g}^*$ the corresponding linear functions on $\mathfrak{g}$.
- $\Omega \in \mathscr{A}^2(\mu_G^{-1}(0); \mathfrak{g})$ is the curvature of the canonical connection on the principal G-bundle $\mu_G^{-1}(0) \to M//G$.

Theorem 2: Reduction of eq. (1)

If $\hat\alpha$ denotes the induced 1-form on $M//G = \mu_G^{-1}(0)/G$ we have

$$d\hat\alpha = 0, \quad d(J\hat\alpha) = \omega_J + (J\alpha)_\mathfrak{g} \circ \Omega, \quad d(K\hat\alpha) = \omega_K + (K\alpha)_\mathfrak{g} \circ \Omega.$$

Ω is of type (1,1) in each complex structure [2] so $\hat\alpha \in \mathscr{A}^1(M//G)$ satisfies eq. (2).

Example 1: Moduli space of parabolic Higgs bundles [5, 8]

Fix:

- Closed Riemann surface Σ and a divisor $D = p_1 + \cdots + p_n$.
- Rank r vector bundle $E \to \Sigma$ with trivial determinant and hermitian metric singular at each puncture p_j.
- Flag at each puncture p_j with parabolic weights.
- Prescribed eigenvalues $\underline{\lambda} = \{\lambda_k^{(j)}, k = 1, \ldots, r, j = 1, \ldots, n\} \subset \mathbb{C}$ for residues of Higgs fields at the punctures.

The moduli space of parabolic Higgs fields is obtained by hyperkähler reduction of the infinite dimensional affine space

$$\mathcal{C} = \{\text{singular } \mathfrak{su}(E)\text{-connections}\} \times \mathscr{A}^{1,0}_{\underline{\lambda}}(\mathrm{Par}\,\mathfrak{sl}(E)(D)).$$

by the action of the gauge group $\mathcal{G} = \mathscr{A}^0(\mathrm{Par}\,SU(E))$.

Have a 1-form α on $\mathcal{C}$ defined by

$$\begin{gathered}\alpha_{(A,\theta)}(a,b) = -i\int_\Sigma \mathrm{tr}(\theta \wedge b^* + b \wedge \theta^*), \\ (a,b) \in \mathscr{A}^{0,1}(\Sigma; \mathfrak{sl}(E)) \times \mathscr{A}^{1,0}(\mathrm{SPar}\,\mathfrak{sl}(E)(D)) \simeq T_{(A,\theta)}\mathcal{C}.\end{gathered}$$

If all $\lambda_k^{(j)} = 0$ or there are no punctures, α is the exterior derivative of the L^2-norm on Higgs fields. In the general case though, the L^2-norm does not converge but the integral defining α does.

α is $\mathcal{G}$-basic and satisfies eq. (1) and

$$(J\alpha + iK\alpha)_{\mathrm{Lie}(\mathcal{G})}(Y) = -2\sum_{j=1}^n \mathrm{tr}(\mathrm{diag}(\lambda_1^{(j)}, \ldots, \lambda_r^{(j)})Y_{p_j}), \quad Y \in \mathscr{A}^0(\mathrm{Par}\,\mathfrak{su}(E)).$$

Thus by theorem 2, it descends to a 1-form on the moduli space satisfying (2). Therefore the moduli space of parabolic Higgs bundles has a natural hyperholomorphic line bundle, generalizing the one on the moduli space of non-singular Higgs bundles (or singular ones with nilpotent residues).

Example 2: Moduli space of solutions to Nahm's equations [7, 1]

Fix:

- A compact Lie group G with Lie algebra $\mathfrak{g}$ and Ad-invariant inner product.
- $\tau_1, \tau_2, \tau_3 \in \mathfrak{g}$ such that the intersection of the centralizers is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_\mathbb{C}$.

Define

$$\mathcal{A}_{\tau_1,\tau_2,\tau_3} = \{T_0 + iT_1 + jT_2 + kT_3 : [0,\infty) \to \mathfrak{g} \otimes \mathbb{H} \mid {}^{T_0 \to 0,}_{T_i \to \tau_i, i=1,\cdots 3}\}$$

This is an infinite dimensional hyperkähler affine space modeled on $\mathcal{A}_{0,0,0}$. The group

$$\mathcal{G} = \{g : [0,\infty) \to G \mid g(0) = e, \lim_{s\to\infty} g(s) \in \exp\mathfrak{h}\}$$

acts with moment map given by Nahm's equations. If the τ_i are regular then the hyperkähler quotient is biholomorphic (in complex structure I) to the complex adjoint orbit of $\tau_2 + i\tau_3$.

The 1-form α on $\mathcal{A}_{\tau_1,\tau_2,\tau_3}$ defined by

$$\begin{gathered}\alpha_{(T_1,T_2,T_3,T_4)}(t_1,t_2,t_3,t_4) = -\int_0^\infty (\langle T_2(s), t_2(s)\rangle + \langle T_3(s), t_3(s)\rangle)\, ds, \\ (T_1,T_2,T_3,T_4) \in \mathcal{A}_{\tau_1,\tau_2,\tau_3}, \quad (t_1,t_2,t_3,t_4) \in \mathcal{A}_{0,0,0}.\end{gathered}$$

is $\mathcal{G}$ basic and satisfies eq. (1) and

$$(J\alpha)_{\mathrm{Lie}(\mathcal{G})}(Y) = -\langle\tau_2, Y(\infty)\rangle, \quad (K\alpha)_{\mathrm{Lie}(\mathcal{G})}(Y) = -\langle\tau_3, Y(\infty)\rangle.$$

Thus by theorem 2, it descends to a 1-form on the moduli space satisfying (2). Therefore the moduli space of solutions of Nahm's equations has a natural hyperholomorphic line bundle, generalizing the one on the cotangent bundle of the flag variety in the case of $\tau_2 = 0 = \tau_3$.

References

[1] R. Bielawski. Lie groups, Nahm's equations and hyperkähler manifolds. *Algebraic Groups*, 1-17, 2007.

[2] T. Gocho, & H. Nakazima. Einstein-Hermitian connections on hyper-Kähler quotients. *Journal of the Mathematical Society of Japan*, 44(1), 43-51, 1992.

[3] A. Haydys. HyperKähler and quaternionic Kähler manifolds with S^1-symmetries. *Journal of Geometry and Physics*, 58(3), 293-306, 2008.

[4] N. Hitchin. On the hyperkähler/quaternion Kähler correspondence. *Communications in Mathematical Physics*, 324(1), 77-106, 2013.

[5] H. Konno. Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface. *Journal of the Mathematical Society of Japan*, 45(2), 253-276, 1993.

[6] E. Korman. A hyperholomorphic line bundle on certain hyperkähler manifolds not admitting an S^1-symmetry *arXiv:1507.05951*, 2015.

[7] P. Kronheimer. A hyper-Kählerian structure on coadjoint orbits of a semisimple complex group. *Journal of the London Mathematical Society 2.2*, 193-208, 1990.

[8] H. Nakajima. Hyper-Kähler structures on moduli spaces of parabolic Higgs bundles on Riemann surfaces. *Lecture notes in pure and applied mathematics*, 199-208, 1996.

[9] A. Neitzke. On a hyperholomorphic line bundle over the Coulomb branch. *arXiv:1110.1619*, 2011.