Handle Diagram and Heegard Diagram Problems

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Problem 1. (Alg. Top Sanity Check) Prove computationally that handle slides and cancelling pairs don't change H_* or π_1.

Problem 2. Build a handle diagram for a closed M^3 with $H_*(M) = H_*(S^3)$ and such that you feel in your heart that $M \ncong S^3$. The *Poincare Conjecture* (1900) asserted that if $H_*(M) = H_*(S^3)$ then $M \cong S^3$. Use your example and π_1 to disprove this Poincare conjecture.

Problem 3. Let H be a handle decomposition of a closed M^3 with one 0-handle, one 3-handle, and g 1-handles. How many 2-handles does H have?

Problem 4. Prove or disprove: Let $f : \#_n(S^1 \times S_1) \to \#_n(S^1 \times S^1)$ be a homeomorphism. Then there exists $F : \natural_n(S^1 \times D^2) \to \natural_n(S^1 \times D^2)$ with $F|_\partial = f$. (Fact: the analog of this claim where $S^1 \times S^1$ is replaced by $S^1 \times S^2$ and $S^1 \times D^2$ with $S^1 \times D^3$ is *true*, and fundamentally important to how we use handle calculus for four manifolds. We will rely on it next lecture.)

Problem 5. Draw Heegard diagrams for: (a) $S^1 \times S^2$ (b) $I \times T^2$ (c) $S^1 \times S^1 \times S^1$ (d) $\mathbb{R}P^3$ (e) $S^3_2(U)$

Problem 6. (Bouns) Let $f : \Sigma \to \Sigma$ be the homeomorphism given by Dehn twisting around γ in the associated picture. Draw a a handle decomposition for the mapping torus of f.

Problem 7. A knot $K \hookrightarrow S^3 = \partial B^4$ is said to be *ribbon* if $K = \partial D^2 \overset{j}{\hookrightarrow} B^4$ and such that with respect to the Morse function on B^4 giving the standard handle diagram j can be isotoped to have only index 0 and 1 critical points.

(a) Show that $B^4 \backslash \nu(j(D))$ admits a handle decomposition with only 0, 1, and 2 handles (do this at the conclusion of the seminar).

(b) Show that $i_* : \pi_1(\partial(B^4 \backslash \nu(j(D))) \to \pi_1(B^4 \backslash \nu(j(D)))$ is surjective (do this now).

Problem 8. Identify the manifold in the associated picture.