Gauge Theory and 4-Manifolds

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M392C - Spring 2018

These are lecture notes following Tim Perutz's Gauge Theory course M392C taught Spring 2018 at UT Austin. The reader should be comfortable with differential and algebraic topology as well as bundle theory. Homotopy theory and differential geometry won't hurt either. Exercises are included, especially in the first few sections. Solutions to some of these exercises can be found in the Appendix. Special thanks to Riccardo Pedrotti for some of these solutions. Please send any corrections to `gdavtor@math.utexas.edu`. Concurrent notes can be found on Tim's web page: `math.utexas.edu/users/perutz/GaugeTheory`. These notes and Tim's are meant to complement one another; Tim's notes sometimes go into more depth on certain proofs whereas these notes offer more explanation and worked examples.

1. Introduction: Classification of Manifolds

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A more accurate title for this course is "Seiberg-Witten Theory and 4 Manifold Topology," as the goal of this course is to understand a particular instatiation of Gauge Theory, namely Seiberg-Witten (S-W) theory in dimension 4. Most of the results we will get to arose between the years 1994 and 2004. The essential motivation behind S-W theory comes from using partial differential equations and gauge theory to classify 4 dimensional manifolds. In this section, we will state some of the current knowledge on the general problem of classifying manifolds of a fixed dimension up to diffeomorphism.

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Let M be a smooth, connected, boundary-less n manifold (assume these adjectives henceforth unless stated otherwise). We call two manifolds equivalent if they are diffeomorphic. The "ideal solution" to the classification of n manifolds up to diffeomorphism consists of answers to the following four questions for M in some class of manifolds (for example, compact manifolds):

1. What is a standard set of manifolds $\{X_i\}_{i \in I}$ such that each manifold in te class is diffeomorphic to some X_i? The index set I need not be countable.
2. Given a description of M in some finite manner, how do we compute invariants of M to decide for which $i \in I$ $M \cong X_i$? A particularly nice answer would be an algorithm for doing so.
3. Given M, M' in our class of manifolds, how do we compute invariants of M and M' so that we can determine if $M \cong M'$? Again, an algorithm is a good answer to this.
4. How can we understand families (fiber budles) of manifolds diffeomorphic to M. For example, what is the homotopy type of $\mathrm{Diff}(M) = \{\phi : M \to M \text{ smooth}\}$?

In low dimensions, we can answer all four questions in a satisfactory answer. In higher dimensions ($n \geq 4$), the story is more complicated and there are certain obstructions that don't occur in dimension 3 or less.

1.1 Dimensions 1,2, and 3 ❖

The classification of 1 dimensional compact manifolds is trivial and is a standard exercise in introductory differential topology. As such, the above four questions have easy answers. In the case of surfaces, they also have complete answers. For example, the class of compact orientable surfaces:

1. The surfaces Σ_g of genus $g \in \mathbb{Z}_{\geq 0}$ given by connect summing the torus g times.

2./3. These are answered by the Euler characteristic $\chi(S) = 2 - 2g$, which is a complete invariant that is computable through various methods.

4. Let $\mathrm{Diff}^+(S)$ be the group of orientation preserving self-diffeomorphisms of S and $\mathrm{Diff}^+(S)$ be the identity component of $\mathrm{Diff}^+(S)$. Some results due to Earle and Eels are that there are homotopy equivalences $SO(3) \hookrightarrow \mathrm{Diff}^+(S^2)$ and $T^2 \hookrightarrow \mathrm{Diff}^+(T^2)_0$, where T^2 is the 2 torus. Moreover, $\pi_0(\mathrm{Diff}^+(T^2)) \cong SL_2(\mathbb{Z})$. This covers the answer to 4) for genus $0, 1$. For $g > 1$, $\mathrm{Diff}^+(\Sigma_g)$ is contractible (i.e. is homotpy type of $\{*\}$). The component group $\pi_0(\mathrm{Diff}^+(\Sigma_g))$ (the "mapping class group") is infinite but finitely presented. It acts with finite stabilizers on a certain contractible manifold called Teichmüller space.

In dimension 3, the classification is more intricate (once again for the compact orientable class of manifolds). Thurston's vision, which was realized by Hamilton and Perelman in recent years (see Ricci flow), led to a solution that is almost as complete as the one we have for surfaces. Moreover, the fundamental group $\pi_1(M)$ is very nearly a complete invariant (the main exception being with Lens spaces).

1.2 Higher Dimensions

For $n \geq 4$, this theory gets less optimistic. One initial warning to have is that, in general, the index set I for $\{X_i\}$ doesn't generally need to be countable. Thus, we can't really hope for a list or moduli space as an answer to 1). However, it will be countable if we consider the class of compact n manifolds.

If you give M an n-handlebody decomposition, you get a presentation for $\pi_1(M)$. If M is compact, this is a finite presentation. In fact, for $n \geq 4$, all finite presentations of groups arise from n-handlebodies in this way. However, a result due to Markov says that there is no algorithm to tell if $\langle g_1, ...g_k \mid r_1, ..., r_\ell \rangle$ is a trivial group. As a result, there is no algorithm to decide whether a given n-handlebody is simply connected for $n \geq 4$.

Another obstruction to classification theory in these dimensions is that we can't use powerful Riemannian metric techniques that were critical to lower dimensions. This is because for $n \geq 5$, there isn't a unique "optimal" Riemannian metric on M for Ricci flow or something equivalent for any definition of "optimal." See [11] for a detailed discussion.

In light of these obstructions, there are some revisions we have to make in order to make classification answerable for $n \geq 5$. The first is that we restrict to considering compact, simply connected manifolds so that I is countable and there are no decidability problems with the fundamental group. In some cases, these assumptions are good enough to answer questions 1)-3). In a wider range of cases, we have conceptual answers to these questions as well using surgery theory. In particular, surgery theory gives answers to two questions:

a) Give a finite n-dimensional CW complex ($n \geq 5$), when is it the homotopy type of a compact n manifold?

b) Given a simply-connected compact manifold of dimension $n \geq 5$, what are the diffeomorphism types of manifolds within its homotopy equivalence class?

1.2.1 Dimension 4

The focus of this class is on dimension $n = 4$, for which most of the above methods fail. In this case, we will still restrict to simply connected compact manifolds to avoid the obstructions mentioned above. The basic invariant of 4 dimensional Poincaré spaces (spaces for which Poincaré duality holds) is the intersection form $Q_P : H^2(P;\mathbb{Z}) \times H^2(P;\mathbb{Z}) \to \mathbb{Z}$. This can be cast as a unimodular matrix over $\mathbb{Z}$[0] modulo a $\mathbb{Z}$ equivalence. This equivalence is $Q \sim Q'$ if and only if $Q' = M^T Q M$ for $M \in GL_n(\mathbb{Z})$. Thus to every Poincaré space we get a unimodular matrix (modulo a $\mathbb{Z}$ relation). Two important classification results related to this are:

Theorem 1.1 (Milnor)**.** *The above correspondence:*

$$\{4\text{-}\textit{dim s.c. compact Poincaré spaces}\}/\textit{homotpy equiv.} \to \{\textit{unimodular matrices over } \mathbb{Z}\}/\mathbb{Z}$$

is a bijection.

Theorem 1.2 (Feedman, ∼1980)**.** *There is a bijection:*

$$\{\textit{Compact,simply connected topological } 4\textit{-manifolds}\}/\textit{homeomorphism} \to \{\textit{unimodular matrices over } \mathbb{Z}\}/\mathbb{Z}$$

In the smooth case, there is Rokhlin's theorem: if X is smooth of dimension 4 and Q_X has even diagonal entries, then the signature of Q_X is divisible by 16. It is at this point that Gauge theory comes in to prove sharper and stronger results. The genesis of this was Donaldson's diagonalizability theorem, which states that if X is compact and simply connected and if Q_X is positive definite, then $Q_X \sim I$. In subsequent years, he divised invariants distinguishing infinitely many diffeomorphism types in one homotopy class. Starting in 1994 came a flood of new proofs using Seiberg-Witten theory with sharper and more general results.

[0] A unimodular matrix is one which is symmetric, has entries in $\mathbb{Z}$, and has unit determinant

2. 4-Manifold Theory I: Intersection Forms and Vector Bundles

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This section and the next are dedicated to reviewing/introducing the important notions and theorems currently known about 4-manifolds. In this section, we will review the algebraic aspects; namely, the theory of intersection forms, vector bundles, and characteristic classes. We will apply these at the end of the section to show the equivalence of Rokhlin's theorem and $\pi_8(S^5) \cong \mathbb{Z}/24$ as well as to prove Whitehead's theorem on 4-dimensional homotopy types.

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2.1 Review of Cohomology and Cup Products ❖

In this subsection, we will review the various points of view for cohomology that will be useful to us (cellular, Čech, de Rham) and how the cup product manifests in each of these settings. Recall that if X and Y are CW complexes, then $X \times Y$ is a CW complex and that for any CW complex X, there is an associated chain complex $C_*(X) = \mathbb{Z}^{\{\text{cells}\}}$ and cochain complex $C^*(X) = \hom(C_*(X), \mathbb{Z})$.

The Künneth theorem for CW complexes states that $C^*(X \times Y) \cong C^*(X) \otimes C^*(Y)$, since the cells of $X \times Y$ can be identified with the cells of X cross the cells of Y. Let $\delta : X \to X \times X$ be the diagonal map; this is not a cellular map (i.e. it doesn't preserve skeleta). However, it is homotopic to a cellular map δ, by the cellular approximation theorem. Then we can define the cup product $\smile : C^*(X) \otimes C^*(X) \to C^*(X)$ by the composition:

$$\begin{tikzcd} C^*(X) \otimes C^*(X) \arrow[r, "\cong"] \arrow[rr, bend right, "\smile"'] & C^*(X \times X) \arrow[r, "\delta^*"] & C^*(X) \end{tikzcd}$$

This descends to the cup product on $H^*(X)$ and is associative, graded, unital, and graded-commutative.

Remark 2.1. The construction of the cup product here was very easy, since the Künneth theorem was straightforward. However, this definition of a cup product isn't easy computationally, since it requires us to find δ. In other settings, it does have an explicit formula, such as in de Rham cohomology and Čech cohomology.

2.1.1 Čech Cohomology

Definition 2.2. An open cover $\{U_i\}_{i \in I}$ of a smooth manifold M is called *good* if it is locally finite and for every nonemtpy $J \subset I$, the set $U_J := \bigcap_{j \in J} U_j$ is either empty or connected.

Exercise 2.3. Show that any manifold admits a good cover, either by using a Riemannian metric or by embedding it into Euclidean space.

An important fact that we wont prove is that any two good covers of M have a unique, good, common refinement. Given any good cover $\mathcal{U}$ of M, define:

$$S_k = \{k \text{ simplices of } \{U_i\}\} = \{\sigma : \{0, 1, ..., k\} \hookrightarrow I \mid U_{\{\sigma(0), \ldots, \sigma(k)\}} \text{ nonempty}\}$$

Further, define the map $\partial_i : S_k \to S_{k-1}$ by $\sigma \mapsto \sigma|_{\{0, \ldots, \hat{\imath}, \ldots, k\}}$, which omits the ith index. Fix a ring A and define the chain complex:

$$\check{\mathrm{C}}^k(M; \mathcal{U}; A) = \prod_{S_k} A$$

$$\check{\mathrm{C}}^*(M; \mathcal{U}; A) = \bigoplus_{k \geq 0} \check{\mathrm{C}}^k(M; \mathcal{U}; A)$$

For an element $\eta \in \check{C}^k(M;\mathcal{U};A)$, we denote the $\sigma \in S_k$ component by $\eta(\sigma)$. The differential on this complex is defined by:

$$\delta : \check{C}^k \to \check{C}^{k+1}$$
$$\eta \mapsto (\delta\eta)(\sigma) = \sum_{i=0}^{k+1} (-1)^{i+1}\eta(\partial_i\sigma)$$

It can be shown that $\delta^2 = 0$, so we can then define the Čech cohomology:

$$\check{H}^*(X;\mathcal{U};A) = \ker\delta / \operatorname{im}\delta$$

This is well-defined, since we can refine any two covers to a common good cover uniquely.

The cup product on Čech cohomology is given by $(\alpha \smile \beta)(\sigma) = \alpha(\text{beginning of } \sigma)\beta(\text{end of } \sigma)$. The degree commutativity is not clear in this case, but still holds.

2.1.2 de Rham Cohomology

Let M be a smooth manifold. The de Rham complex is:

$$\Omega^*(M) = \bigoplus_{k \geq 0} \Omega^k(M)$$

where $\Omega^k(M)$ is the vector space of differential k forms. The differential operator $d : \Omega^*(M) \to \Omega^{*+1}(M)$ is the exterior derivative. As usual $d^2 = 0$, and the de Rham cohomology $H^*_{dR}(M)$ is the cohomology of this complex. In this setting, the cup product manifests itself through the wedge operation $\wedge : \Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M)$.

The $\mathbb{R}$-Čech cohomology and de Rham cohomology are related through the double complex $\check{C}^*(M, \Omega^\bullet(M))$. This is the Čech cohomology with coefficients taken in the ring $\Omega^\bullet(M)$. Specifically, the totalization complex D^* of this double complex yields a pair of canonical maps:

$$\check{C}^*(M;\mathbb{R}) \to D^* \leftarrow \Omega^\bullet(M)$$

which are quasi-isomorphisms respecting the cup products. This means we have an $\mathbb{R}$ algebra isomorphism $\check{H}^*(M;\mathbb{R}) \cong H^*_{dR}(M)$, which is canonical and natural with respect to smooth maps. For details see [2].

2.1.3 Poincaré Duality and Intersection

Recall that if X is smooth and of dimension n, then $H_k(X) = 0$ for all $k > n$. For $k = n$, the top homology group $H_n(X) \cong \mathbb{Z}$ if X is compact and orientable, and vanishes otherwise. A choice of orientation of X determines a generator $[X] \in H_n(X)$, which is called the fundamental class. This has the property that, given an orientation-preserving diffeomorphism $f : X \to Y$, we have $f_*([X]) = [Y]$. If it reverses orientation, then $f_*([X]) = -[Y]$. The fundamental class determines a trace map:

$$H^n(X;A) \to A$$
$$c \mapsto \operatorname{eval}(c, [X])$$

In the case where $A = \mathbb{R}$ and we use de Rham cohomology, this is the integration map $\alpha \mapsto \int_X \alpha$.

Poincaré duality uses the fundamental class to establish an isomorphism $H^k(X) \cong H_{n-k}(X)$. This is done via the cap product $\frown : H^k(X) \otimes H_j(X) \to H_{j-k}(X)$, which gives H_{-*} the structure of a graded module over the graded ring $H^*(X)$. In the case of cellular homology, it can be defined in a simlar way as the cup product:

$$\frown : C^*(X) \otimes C_*(X) \xrightarrow{\operatorname{id}\otimes\delta_*} C^*(X) \otimes C_*(X) \otimes C_*(X) \xrightarrow{\operatorname{eval}\otimes\operatorname{id}} C_*(X)$$

where δ is a cellular homotopy representative of the diagonal map δ as before. This descends to a map on (co)homology, which we call the cap product. There are analogous ways to define the cap product in the other cohomology theories we have discussed as well.

Example 2.4. If $f : X^n \to Y^m$ is a smooth map of manifolds, then we have a map:

$$(-) \frown f_*([M]) : H^n_{dR}(Y) \to H^0_{dR}(Y) \cong \mathbb{R}$$

$$\eta \mapsto \eta \frown f_*([M]) = \int_X f^*\eta$$

The Poincaré Duality theorem says that the map $(-) \frown [X] : H^*(X) \to H_{n-*}(X)$ is an isomorphism. We will denote this map D_X and its inverse D^X. Poincaré duality gives a geometric interpretation of the cup product. Let X^n, Y^{n-p} and Z^{n-q} be closed, oriented manifolds and $f : Y \to X, g : Z \to X$ be smooth maps. Let $c_Y = D^X f_*[Y]$ and $c_Z = D^X g_*[X]$. By a choice of homotopy, we can assume f and g are transverse so that $P = X \times_{f,g} Z := \{(y,z) \in Y \times Z \mid f(y) = g(z)\}$ is an oriented submanifold of codimension $p+q$. If $\phi = (f,g) : P \to X$, then define $c_P = D^X \phi_*[P]$. Finally:

$$c_P = c_Y \smile c_Z$$

In the special case where f and g are inclusions of submanifolds, P is the transverse intersection of Y and Z. This shows that the cup product is Poincaré dual to the intersection of submanifolds representing the classes.

This leads to a natural question: can all homology classes be represented by the class of a submanifold? The answer is no, and there are known counterexamples. However, it does hold when considering classes of codimension 1 and 2. The idea to prove this is to use the Eilenberg-MacLane spaces $K(\mathbb{Z},1)$ and $K(\mathbb{Z},2)$ and the isomorphism $H^n(X) \cong [X, K(\mathbb{Z},n)]$ (as sets).

Codimension 1: By Poincaré duality, $H_{n-1}(X) \cong H^1(X)$. Moreover, since $K(\mathbb{Z},1) = S^1$, we have $H^1(X) \cong [X, S^1]$. Explicitly, this isomorphism sends a homotopy class of $f : X \to S^1$ to $f^*\omega$, where ω is a generator of $H^1(S^1) = \mathbb{Z}$. Pick $c \in H_{n-1}(X)$ and let $f : X \to S^1$ be a smooth representative of the corresponding class in $[X, S^1]$. For a regular value $t \in S^1$, let $H_t = f^{-1}(t) \subset X$. This is a submanifold and inherits an orientation, so we have $[H_t] = D_X(f^*\omega) = c$ as desired.

Codimension 2: In this case, we use $K(\mathbb{Z},2) \cong \mathbb{C}P^\infty$. Once again we get $H_{n-2}(X) \cong H^2(X) \cong [X, \mathbb{C}P^\infty]$ (as sets). The second bijection is $[f] \mapsto f^*\omega$, where ω is a generator of $H^2(\mathbb{C}P^\infty) \cong \mathbb{Z}$. For $c \in H_{n-2}(X)$, pick $f : X \to \mathbb{C}P^\infty$ corresponding to c in this bijection. This map is homotopic to a smooth map $g : X \to \mathbb{C}P^N$ for some N. If $D \subset \mathbb{C}P^N$ is a hyperplane transverse to g, then the class of $g^{-1}(D)$ is Poincaré dual to $g^*\omega$. Therefore $[g^{-1}(D)] = c$.

In higher codimensions, there are classes which are not realizable as submanifolds, as shown by Thom in 1954. For example, let $X = \mathrm{Sp}(2)$, the compact symplectic group of dimension 10. The cohomology ring is $H^*(\mathrm{Sp}(2)) \cong \Lambda[x_3, x_7]$, the exterior algebra on generators of degrees 3 and 7. The class x_3 is not representable by an embedded submanifold. See [1].

2.2 Unimodular Forms and Signatures ❖

To define the intersection form on middle cohomology on a even dimensional manifold, we will need to understand the algebra of unimodular forms. Let M be a closed, oriented, $2n$ dimensional manifold. Then $H^n(M)$ carries a bilinear form:

$$(\cdot) : H^n(M) \times H^n(M) \to \mathbb{Z}$$

$$(x,y) \mapsto x \cdot y := \mathrm{eval}(x \smile y, [M])$$

This form is symmetric if n is even and skew symmetric if n is odd, which follows from the graded commutativity of cup products. There are two other ways to understand this. Both of which pre-compose with Poincaré duality on the input:

$$H^n(M) \times H_n(M) \to \mathbb{Z}$$

$$H_n(M) \times H_n(M) \to \mathbb{Z}$$

The third is the intersection form of cycles $(h,k) \mapsto h \frown k$. The second is the evaluation pairing, namely:

$$x \cdot y = \mathrm{eval}(x, D_M y)$$

This follows from the fact that:

$$x \cdot y = (x \smile y) \frown [M] = x \frown (y \frown [M]) = x \frown D_M y$$

For an abelian group A, denote $A' = A/\text{torsion}$. Then the form $(\cdot)$ necessarily descends to a form on $H^n(M)'$. We usually use that version of the pairing.

Remark 2.5. For the case of $\dim M \leq 4$, we have $H^2(M) \cong H^2(M)' \oplus$ torsion (non-canonically) by the Universal Coefficient Theorem, where "torsion" is the torsion part of $H_1(M)$. In particular, if $H_1(M) = 0$, then $H^2(M)$ is torsion-free.

If we choose an $\mathbb{Z}$ basis $\{e_i\}$ of $H^n(M)'$, we get a matrix Q with $Q_{ij} = e_i \cdot e_j$. This matrix is symmetric if n is even and skew symmetric if n is odd. Note that $H^n(M)' \cong \hom(H_n(M), \mathbb{Z})$ by the Universal Coefficient Theorem. From this, and the fact that $(\cdot)$ is dual to evaluation, we get:

Proposition 2.6. *The form $(\cdot)$ is nondegenerate on $H^n(M)'$, i.e. the map $H^n(M)' \to \hom(H^n(M)', \mathbb{Z})$ given by $x \mapsto (y \mapsto x \cdot y)$ is an isomorphism of abelian groups.*

Exercise 2.7. Show that the nondegeneracy of Q_M is equivalend to the unimodularity condition $\det Q = \pm 1$.

Let us consider the case of manifolds that arise as boundaries, i.e. let $M^{2n} = \partial N^{2n+1}$ for some manifold N. Let $i : M \hookrightarrow N$ be the inclusion map. In the following proposition, we use $\mathbb{R}$ coefficients without denoting it explicitly.

Proposition 2.8. *Let $L = \operatorname{im}(i^*) \subset H^n(M)$. Then:*

i. L is isotropic, i.e. $x \cdot y = 0$ for all $x, y \in L$.

ii. $\dim L = \frac{1}{2} \dim H^n(M)$.

In other words, L is a Lagrangian subspace.

Proof:

For the first part, consider:

$$\begin{aligned} i^*u \cdot i^*v &= \operatorname{eval}(i^*u \smile i^*v, [M]) \\ &= \operatorname{eval}(i^*(u \smile v), [M]) \\ &= \operatorname{eval}(u \smile v, i_*[M]) = 0 \end{aligned}$$

Because $i_*[M] = 0$, as M is a boundary. For $ii.$, observe the following diagram with exact rows:

$$\begin{tikzcd} {} \arrow[r] & H^n(N) \arrow[r, "i^*"] \arrow[d, "D_M"] & H^n(M) \arrow[r, "\delta"] \arrow[d, "D_M"] & H^{n+1}(N, M) \arrow[r, "q"] \arrow[d, "D"] & H^{n+1}(N) \arrow[r] \arrow[d, "D"] & {} \\ {} \arrow[r] & H_{n+1}(N, M) \arrow[r, "\partial"] & H_n(M) \arrow[r, "i_*"] & H_n(N) \arrow[r, "p"] & H_n(N, M) \arrow[r] & {} \end{tikzcd}$$

The vertical isomorphisms D_M are Poincaré-Lefschetz duality. Now we chase this diagram. Fix a complement K to L in $H^n(M)$. We want $\dim(K) = \dim(L)$. By exactness, $L = \ker \delta$. Therefore $K \cong \operatorname{im} \delta = \ker q = \ker p$. But $\ker p = \operatorname{im} i_* \subset H_n(N)$. Since i_* is dual to i^*, algebra tells us that i^* and i_* have the same rank so that $\dim K = \dim L$.

□

Definition 2.9. A *unimodular lattice* (Λ, σ) is a free abelian group Λ of finite rank with a non-degenerate symmetric bilinear form $\sigma : \Lambda \times \Lambda \to \mathbb{Z}$.

So, a closed oriented $4m$ manifold determines a unimodular lattice $(H^{2m}(M), (\cdot))$.

2.2.1 Signatures

Recall that for (V, σ), a real symmetric bilinear form on a finite dimensional vector space V, there exists an orthogonal decomposition $V = R \oplus V^+ \oplus V^-$, where R is the radical (orthogonal to everything), V^+ is positive definite subspace and V^- is a negative definite subspace. This isn't a canonical decomposition, but the dimensions are invariant. Moreover, the dimensions of R, V^+, V^- fuly determine (V, σ) up to isomorphism (Sylvester's theorem).

Definition 2.10. For (V, σ) as above, the signature is $\tau = \dim V^+ - \dim V^-$.

The signature of a lattice (Λ, σ) is defined to be $\tau(\Lambda \otimes \mathbb{R}, \sigma)$. Moreover, we define $\tau(M^{4m}) := \tau(H^{2m}(M)', (\cdot))$. A simple observation is that M^{4m} can admit an orientation reversing self-diffeomorphsm if and only if $\tau(M) = 0$ because such a diffeomorphism reverses the sign of τ.

Theorem 2.11 (Thom). *Fix two closed oriented 4-manifolds X_1 and X_2. Then $\tau(X_1) = \tau(X_2)$ if and only if there is an oriented cobordism Y of X_1 and X_2.*

Proof:

The forward direction is easier. Proposition 2.8 says $H^2(-X_1 \coprod X_2) \otimes \mathbb{R}$ admits a middle dimensional isotropic subspace, hence (by algebra) has signature zero. But then $\tau(-X_1 \coprod X_2) = \tau(-H^2(X_1) \otimes \mathbb{R} \oplus H^2(X_2) \otimes \mathbb{R}) = \tau(-X_1) + \tau(X_2) = -\tau(X_1) + \tau(X_2) = 0$. So $\tau(X_1) = \tau(X_2)$. The other direction, due to Thom, uses the oriented cobordism group Ω_4.

$\square$

2.2.2 Unimodular lattices mod 2

Definition 2.12. A *characteristic vector* c for a unimodular lattice (Λ, σ) is a vector $c \in \Lambda$ such that $\sigma(c, x) \equiv \sigma(x, x)$ mod 2 for all $x \in \Lambda$.

Lemma 2.13. *The characteristic vectors form a coset of 2Λ in Λ.*

Proof:

Set $\lambda = \Lambda/2\Lambda$, a $\mathbb{Z}/2$ vector space. There is a map $\lambda \to \mathbb{Z}/2$ sending $[x] \mapsto \sigma(x, x)$ mod 2. This is linear over $\mathbb{Z}/2$. The symmetric bilinear form $\overline{\sigma}$ on λ induced by σ has determinant 1, and hence is non-degenerate. Then there exists a unique $\overline{c} \in \lambda$ such that $\overline{\sigma}(x, x) = \overline{\sigma}(\overline{c}, x)$ for all $x \in \lambda$. The characteristic vectors c are exactly the lifts of $\overline{c}$ to Λ.

$\square$

Remark 2.14. In the case of a simply connected 4-manifold M, the element $\overline{c} \in H^2(M)/2H^2(M) = H^2(M; \mathbb{Z}/2)$ is the second Stiefel-Whitney class of TM, namely $w_2(TM)$. This is an instance of the Wu formula. Moreover, the corresponding characteristic vectors $c \in H^2(M; Z)$ are exactly the first Chern classes of spin^c structures on M. More on these things later.

Lemma 2.15. *If c, c' are characteristic vectors of (Λ, σ), then $\sigma(c, c) \equiv \sigma(c', c')$ mod 8.*

Proof:

By the previous Lemma we can write $c' - c = 2x$ for $x \in \Lambda$. Thus:

$$\sigma(c', c') = \sigma(c + 2x, c + 2x) = \sigma(c, c) + 4(\sigma(c, x) + \sigma(x, x))$$

The term in parentheses is even by defintion of characteristic vectors. Therefore we get mod 8 congruence.

$\square$

Definition 2.16. The *type* of a lattice (Λ, σ) is zero if Λ is even (i.e. $\sigma(x, x)$ is even for all $x \in \Lambda$). The type is one otherwise. Sometimes the time is referred to as even or odd, respectively.

Theorem 2.17 (Hasse-Minkowski)**.** *An indefinite (i.e. neither positive definite or negative definite) unimodular form (Λ, σ) is determined up to isomorphism by three invariants: its rank $\dim_{\mathbb{R}}(\Lambda \otimes \mathbb{R})$, its signature $\tau \in \mathbb{Z}$, and its type (as an element of $\mathbb{Z}/2$).*

A reference for the proof is [10]. The idea is to solve the quadratic equation $\sigma(x, x) = 0$ for $x \neq 0 \in \Lambda \otimes \mathbb{Q}$. You do this by a local-to-global principle, which says that it suffices to find solutions $x_\infty \in \Lambda \otimes \mathbb{R}$ and $x_p \in \Lambda \otimes \mathbb{Q}_p$ for all primes p.

Example 2.18. Let I_+ denote $(\mathbb{Z}, 1)$ and I_- denote $(\mathbb{Z}, -1)$. The forms explicitly are $\sigma(x, y) = xy$ and $\sigma(x, y) = -xy$, respectively. Then for any integers m, n, $mI_+ \oplus nI_-$ has rank $m + n$, signature $m - n$, and odd type.

Example 2.19. Let $U = (\mathbb{Z}^2, \rho)$ where $\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is rank 2, signature 0 and even.

Corollary 2.20. *For all unimodular lattices Λ and all characteristic vectors c, we have $c \cdot c \equiv \tau$ mod 8. In particular, if Λ is even then $\tau \equiv 0$ mod 8.*

Proof:

Either $\Lambda \oplus I_+$ or $\Lambda \oplus I_-$ is indefinite. Make such a choice. The result is odd because $I_\pm$ are odd, and $\tau(\Lambda \oplus I_\pm) = \tau(\Lambda) \pm 1$. By Hasse-Minkowski, $\Lambda \oplus I_\pm \cong mI_+ \oplus nI_-$, so $c \cdot c \pm 1 \equiv \tau(\Lambda) \pm 1 \bmod 8$. So, in particular, if Λ is even, positive-definite and unimodular, its rank is $8k$.

□

Example 2.21. An important unimodular lattice is the E_8 lattice. This is characterized as the unique positive-definite, even, unimodular lattice of dimension 8. Explicitly, it is given in $\mathbb{R}^8$ by:

$$E_8 = \left\{ (x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + 1/2)^8 \;\middle|\; \sum_i x_i \equiv 0 \bmod 2 \right\}$$

Exercise 2.22. Let Λ be a lattice in $\mathbb{R}^n$ (with inner product inherited from $\mathbb{R}^n$) and $\Lambda' \subset \Lambda$ a sub-lattice of finite index $[\Lambda \colon \Lambda']$.

1. Show that the determinants of the matrices representing these lattices are related by

$$\det \Lambda = [\Lambda \colon \Lambda'] \det \Lambda'$$

2. Show that $[\mathbb{Z}^8 \colon \Gamma] = 2$ and $[E_8 \colon \Gamma] = 2$

3. Deduce that $\det E_8 = 1$

2.3 The intersection form and Characteristic Classes ❖

Recall that, for a closed oriented four manifold X, the homology and cohomology groups are:

$$H^4(X) \cong H_0(X) = \mathbb{Z}[pt]$$

$$H^3(X) \cong H_1(X) = \pi_1(X)^{\text{ab}}$$

$$H^2(X) \cong \hom(H_2(X), \mathbb{Z}) \oplus H_1(X)_{\text{torsion}}$$

$$H^1(X) \cong H_3(X) = \hom(\pi_1(X), \mathbb{Z})$$

$$H^0(X) \cong H_4(X) = \mathbb{Z} \cdot 1$$

Additively, the homology and cohomology are determined by $\pi_1(X)$ and $H_2(X)$ up to isomorphism. We also have an intersection form Q_X on $H^2(X)' = H^2(X)/\text{torsion}$. However, there could be further structure, e.g. the cup product $H^1 \otimes H^2 \to H^3$.

In the simply connected case, however, we have $H^0(X) = \mathbb{Z} \cdot 1, H^1(X) = 0, H^2(X) = \hom(H_2, \mathbb{Z}), H^3(X) = 0, H^4(X) = \mathbb{Z}$. Therefore Q_X on H^2 determines the graded ring $H^*(X)$ and module $H_*(X)$. Moreover, all mod p cohomology classes are reductions of $\mathbb{Z}$ classes (by UCT) and the Hurewicz map $\pi_2(X) \to H_2(X)$ is an isomorphism. The conclusion from all of this is that Q_X determines all cohomological information.

2.3.1 Characteristic Classes

The next candidate invariant comes from the tangent bundle $TX \to X$, a distinguished rank 4 vector bundle on X. Specifically, the characteristic classes of TX. As invariants, they're useless: they are determined by Q_X; however, they are useful tools for computing Q_X.

Stiefel-Whitney Classes

Recall that for any finite rank vector bundle $V \to X$ over any space X, the Stiefel-Whitney classes $w_i \in H^i(X, \mathbb{Z}/2)$ are defined for $i \geq 0$ with $w_0 = 1$ and $w_i = 0$ if i exceeds the rank of V. The total class is $w(V) = w_0(V) + ... + w_k(V) \in H^*(X, \mathbb{Z}/2)$. These are uniquely characterized by the following axioms:

1. If $f : X \to Y$ is a map, then $w_i(f^*V) = f^*w_i(V)$.
2. $w_i(V) = 0$ if i exceeds the rank of V.
3. $w(U \oplus V) = w(U) \smile w(V)$ for any two bundles U, V over X.
4. For the tautological bundle $L \to \mathbb{RP}^1$, the class $w_1(L) \in H^1(\mathbb{RP}^1, \mathbb{Z}/2) \cong \mathbb{Z}/2$ is nonzero.

When X is path connected, the first Stiefel-Whiteny class $w_1(V)$ vanishes when V is orientable. If X is a closed n-manifold and $V \to X$ is a rank r vector bundle, then the top class $w_r(V) \in H^r(M, \mathbb{Z}/2)$ is the Euler class (mod 2).

Exercise 2.23. Show that $w(T\mathbb{RP}^n) = (1 + H)^{n+1}$, where $H \in H^1(\mathbb{RP}^n, \mathbb{Z}/2) \cong \mathbb{Z}/2$ is the nonzero element.

Theorem 2.24 (Wu)**.** *If X is dimension 4 and closed, then $\omega := (w_1^2 + w_2)(TX)$ is the characteristic element of $H^2(X, \mathbb{Z}/2)$, i.e. $\omega \smile u = u \smile u$ for all $u \in H^2(X, \mathbb{Z}/2)$.*

If X is simply connected, then $w_1(X) = 0$ because the first cohomology group vanishes. By above, $w_2(TX) \smile u = u \smile u$ for all $u \in H^2(X, \mathbb{Z}/2) \cong H^2(X)/2H^2(X)$.[1] Therefore w_2 is the mod 2 reduction of any characteristic vector for Q_X. By our discussion on unimodular lattices, it then follows that if Q_X is even, $w_2(TM) = 0$. The top class $w_4(TX)$ evaluated on $[X]$ is the number of zeros of a generic vector field mod 2, i.e. $\chi(X)$ mod 2. Even in the non-simply connected case, $w_3(X)$ vanishes:

Theorem 2.25 (Hirzebruch-Hopf)**.** *For all closed, oriented X^4 one has $w_3(TX) = 0$.*

These facts show that Stiefel-Whitney classes don't bring any new information to the table.

Remark 2.26. A more general form of Wu's formula says that $w_3(TX) = \mathrm{Sq}^1 w_2(TX) = \beta w_2(TX)$, meaning liftability to $\mathbb{Z}$ coefficients. Therefore $w_3(TX) = 0 \iff w_2(TX)$ has a $\mathbb{Z}$ lift. Such lifts are the first Chern classes of spin^C strutures; more on this later.

Chern Classes

In the complex vector bundle case, there are Chern classes. For all complex vector bundles $E \to X$, the Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$ are elements satisfying $c_0(E) = 1, c_i(E) = 0$ for i sufficiently large. They are determined by the following axioms:

[1] This equality comes from UCT and the fact that $\hom(A, \mathbb{Z}/2) \cong \hom(A, \mathbb{Z})/2\hom(A, \mathbb{Z})$ for any abelian group A

1. $c_i(E) = 0$ if i exceeds the rank (over $\mathbb{C}$ of E).
2. $c(E \oplus F) = c(E) \smile C(F)$.
3. If $L \to \mathbb{C}P^1$ is the tautological bundle, then $D_{\mathbb{C}P^1}(c_1(L)) = -1 \in H_0(\mathbb{C}P^1) = \mathbb{Z}[pt]$.

where $c = c_1 + ... + c_r$ is the total Chern class.

Example 2.27. $c(T\mathbb{C}P^n) = (1 + H)^{n+1}$, where H is Poincaré dual to a Hyperplane in $\mathbb{C}P^n$.

The top Chern class $c_r(E)$ for $E \to X$ for X closed, oriented and smooth is once again the Euler class. Recall that the Picard group $\mathrm{Pic}(X)$ is a group of line bundles on X. Then we have a map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$. This is a homomorphism:

$$c_1(L \otimes L') = c_1(L) + c_1(L')$$

This can be shown using the previous fact that the top class is the Euler class. In fact, this is an isomorphism.

Theorem 2.28. *For a complex vector bundle E, the Stiefel-Whiteny classes on the E thought of as a real vector bundle satisfy $w_{2i}(E_{\mathbb{R}}) = c_1(E)$ mod 2 and $w_{2i+1}(E_{\mathbb{R}}) = 0$. In particular, $w_2 = c_1$ mod 2.*

Definition 2.29. An *almost complex* structure on a manifold M^{2r} is $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{id}$. Such a J makes TM a complex vector bundle, where $i = \sqrt{-1}$ acts by J. Then we have $c_i(TM, J) \in H^2(M, \mathbb{Z})$.

Example 2.30. Let $X_d \subset \mathbb{C}P^n$ be the zero set of a degree d homogeneous polynomial F in the variables $x_0, ..., x_n$. Also assume that X_d is smooth. It can be shown that the associated line bundle $O_{\mathbb{P}^n}(X_d)$ is isomorphic to $(L^*)^{\otimes d}$ and $c_i(O_{\mathbb{P}^n}(X_d)) = dH$, where L is the tautological line bundle and $H = c_i(L^*)$. There is a recursive definition of the Chern classes:

$$c_j(TX_d) + dhc_{j-1}(TX_d) = \binom{n+1}{j} h^j$$

Which means $c_1(TX_d) = (n + 1 - d)h$, and the remaining can be found using the recursion. Then one can show that:

$$\mathrm{eval}(c_2(TX_d), [X_d]) = d(d^2 - 4d + 6)$$

This integer is the Euler characteristic of X_d. Since $\chi(X_d) = 1+0+b_2(X)+0+1$, we find $b_2(X_d) = d(d^2-4d+6)-2$. (Many details were left out in this example, see Tim's online notes for full details).

Pontryagin Classes

If $V \to X$ is a real vector bundle, define the *Pontryagin* classes $p_i(V) := (-1)^i c_{2i}(V \otimes \mathbb{C}) \in H^{4i}(X, \mathbb{Z})$. For a closed, oriented 4 manifold X, TX has only $p_1(TX) \in H^4(X, \mathbb{Z})$.

Lemma 2.31. *For X^4 closed and oriented, the integer $p_1(TX)[X]$ only depends on the cobordism class of X.*

Lemma 2.32. *For $V \to X$ a complex vector bundle, $p_1(V_{\mathbb{R}}) = c_1(V)^2 - 2c_2(V)$.*

Proof:

Let V be a complex vector bundle and let $\overline{V}$ be its conjugate (i.e. if J is the complex structure on V, then $\overline{V}$ is the same space with complex structure $-J$). Then if we denote $V_{\mathbb{R}}$ to be the underlying real vector bundle, we have the following:

$$V_{\mathbb{R}} \otimes \mathbb{C} \cong V \oplus \overline{V}$$

Then the first Pontryagin number is:

$$p_1(V_{\mathbb{R}}) = -c_2(V_{\mathbb{R}} \times \mathbb{C}) = -c_2(V \oplus \overline{V})$$

By the second axiom of Chern classes, this is:

$$p_1(V_{\mathbb{R}}) = -c_2(V) - c_2(\overline{V}) - c_1(V)c_1(\overline{V})$$

Suppose for the moment that $V = L \oplus L \oplus \ldots \oplus L$ for line bundles L. Then $\overline{V} = L^* \oplus L^* \oplus \ldots \oplus L^*$. Similarly, $c_1(L^*) = -c_1(L)$ because $c_1(L) + c_1(L^*) = c_1(L \otimes L^*) = c_1(\mathbb{C}) = 0$. Then denoting $\ell := c_1(L)$, we have:

$$c_1(V) = \prod(1+\ell), \quad c_1(\overline{V}) = \prod(1-\ell)$$

from which it follows that $c_i(\overline{V} = (-1)^i c_i(V)$. By the splitting principle and naturality of c_1, this still remains true for an arbitrary vector bundle V. Taking $V = TX$, we finally get:

$$p_1(TX_{\mathbb{R}}) = c_1(TX_{\mathbb{R}})^2 - 2c_2(TX_{\mathbb{R}})$$

□

Example 2.33. An important example that we will use later is the degree 4 hypersurface X_4 in $\mathbb{CP}^5$. We have shown in the previous example that $c_1(TX_d) = (5+1-4)h = 2h$ and that $c_2(TX_d)[X_d] = 4(4^2 - 4 \cdot 4 + 6) = 24$. Therefore by the formula we just proved:

$$p_1(TX_4)[X_4] = c_1(TX_4)[X_4] - 2c_2(TX_4)[X_4] = -48$$

The content of Roklhin's theorem which we will prove later is that all 4-manifolds with $c_1(TX) \equiv 0 \bmod 2$ must have $p_1(TX)[X_4]$ divisible by 48.

2.4 Tangent bundles of 4-manifolds ❖

2.4.1 Obstruction Theory and Stiefel Whitney Classes

Obstruction theory addresses the following question: suppose $E \to X$ is a fiber bundle. When is there a section $s : X \to E$? Assume X is a simply connected CW complex and let F be the fiber. The strategy to construct s is to inductively construct sections s^k on the k-skeleta of X. If s^k is a section on X^k, when does it extend to a section x^{k+1}? Let $\Phi : (D^{k+1}, \partial D^{k+1}) \to (X^{k+1}, X^k)$ be the inclusion of a $(k+1)$-cell, and let $\phi = \Phi|_{\partial D^{k+1}}$ be the attaching map. We have a section $\phi^*(s^k) = s^k \circ \phi$ of $\phi^* E \to S^k$. Since this a sub-bundle of $\Phi^* E \to D^{k+1}$, a trivial bundle (because D^{k+1} is contractible). So we can think of $\phi^* s^k$ as a map $S^k \to E_{\Phi(0)} \cong F$, which is an element of $\pi_k(F)$. Thus we have a map:

$$\{(k+1)\text{-cells}\} \to \pi_k(F)$$

which is a cellular cochain $o^{k+1}(E, s^k) \in C^{k+1}(X; \pi_k(F))$. A few facts about this which are proven in [6] are:

1. o^{k+1} is a cocycle.
2. Its class $O^{k+1} = [o^{k+1}] \in H^{k+1}(X; \pi_k(F))$ depends only on the homotopy class of s^k.
3. If $O^{k+1} = 0$, then s^k extends to s^{k+1}.
4. If $\pi_i(F) = 0$ for $i < k$, then the primary obstruction $O^{k+1} \in H^{k+1}(X; \pi_k(F))$ is an invariant $O^{k+1}(E)$ of the bundle.

The above claims are valid as well when $\pi_1(X)$ is nontrivial but still acts trivially in $\pi_i(F)$ for $i \leq k$.

Historically, Stiefel-Whitney classes were discovered as obstructions to a certain class of bundles, which we describe here. If $E \to X$ is a real vector bundle of rank n with a Euclidean metric, there is an associated fiber bundle $V_k(E) \to X$. The fibers of this bundle $V_k(E)_x = V_k(E_X)$ are the space of orthonormal k-frames for E_x. A section of this bundle is a k-tuple of orthonormal (hence linearly independent) sections of E. The first nonzero homotopy group of the typical fiber $V_k(\mathbb{R}^n)$ is π_{n-k} and is:

$$\pi_{n-k} V_k(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \text{if } n-k \text{ is even or } k=1 \\ \mathbb{Z}/2 & \text{otherwise} \end{cases}$$

Then there are primary obstructions for $V_k(E)$ which are elements $O_n^k(E)$:

$$O_n^k(E) \in \begin{cases} H^{n-k+1}(X;\mathbb{Z}) & \text{if } n-k \text{ is even or } k=1 \\ H^{n-k+1}(X;\mathbb{Z}/2) & \text{otherwise} \end{cases}$$

In either case, we get a mod 2 class $\overline{O}_n^k \in H^{n-k+1}(X;\mathbb{Z}/2)$ by reducing mod 2 in the $\mathbb{Z}$ case.

Theorem 2.34. *$\overline{O}_k^n(E) = w_{n-k+1}(E)$, the Stiefel-Whitney class.*

See [5] for more details.

2.4.2 Vector Bundles over a 4-manifold

Let X be a 4-manifold and $E \to X$ be a vector bundle of rank 4. By the theorem we just stated, $w_2(E)$ is the obstruction to finding 3 linearly independent sections of E over X^2. These three sections are complemented by a line bundle $\ell \subset E$, which is trivial if and only if $w_1(E) = 0$.

Corollary 2.35. *Let X be a closed, oriented 4-manifold. Then $w_2(TX)$ is the obstruction to trivializing TX over $X - \{pt\}$.*

Proof:

Choose a metric g in TX and set $Y = X - \{pt\}$. Choose a CW structure for Y. Over the 2-skeleton Y^2, we can find orthonormal vector fields (v_1, v_2, v_3) by the Theorem, and hence $v_4 \in (v_1, v_2, v_3)^\perp$ because $w_1(TX) = 0$. Let $P \to Y$ be the principal $SO(4)$ bundle of oriented, orthonormal frames of T_yY. Then (v_1, v_2, v_3, v_4) is a section of $P|_{Y^2} \to Y^2$. To extend to Y^3, the obstruction lies in $H^3(Y;\pi_2 SO(4))$. A fact we will not prove is $\pi_2(SO(4)) = 0$, so there is no obstruction. This means $(v_1, ..., v_4)$ extend to Y^3. For higher cells, we get obstructions in $H^{i+1}(Y;\pi_i SO(4))$ for $i+1 \geq 4$, but for punctured 4-manifolds the top cohomology group vanishes anyway, so we can extend to Y^4, and hence Y.

□

Theorem 2.36. *Let X^4 be a closed oriented manifold, and suppose $T, T' \to X$ are two rank 4 oriented vector bundles. Assume that $w_2(T) = 0 = w_2(T')$. Then $T \oplus \underline{\mathbb{R}} \cong T' \oplus \underline{\mathbb{R}}$ if and only if $p_1(T) = p_1(T')$. Moreover, $T \cong T'$ if and only if $p_1(T) = p_1(T')$ and $e(T) = e(T')$, where $e(-)$ is the Euler class.*

Lemma 2.37. *There exists a rank 4 vector bundle $E \to S^4$ with $p_1(E)[S^4] = -2$ and $e(E)[S^4] = 1$.*

Proof:

S^4 is diffeomorphic to projective quaternion space $\mathbb{HP}^1$. Let $\Lambda \to \mathbb{HP}^1$ be the tautological bundle on $\mathbb{HP}^1$ and let $E = \hom_{\mathbb{H}}(\Lambda, \mathbb{H})$ be its dual. Since E can be viewed as a rank 2 complex vector bundle, it suffices to find a section of E and compute its zero locus in order to determine $e(E)$. Homogenous coordinates $[X : Y]$ on E are exactly sections of E. The zero locus of X (a point) gives $e(E)[S^4] = 1$. Moreover, by Lemma 2.32:

$$p_1(E)[S^4] = (c_1(E)^2 - 2c_2(E))[S^4] = -2c_2(E)[S^4] = -2e(E)[S^4] = -2$$

where use used that $c_2(E) = e(E)$ and $c_1(E) = 0$.

□

Proof of Theorem 2.36:

We will construct $f : X \to S^4$ which is smooth near $x \in X$ and has degree 1 with $D_x f$ an isomorphism and $f^{-1}(f(x)) = \{x\}$, which is known as the Pontryagin-Thom collapse. Since $w_2(T) = 0$, we have that T is trivial over $X - B$ for some ball B containing x. Then $T \cong f^*U$ for some $U \to S^4$. Furthermore $f^*p_1(U) = p_1(f^*U) = p_1(T)$ and $f^*e(U) = e(T)$. Thus it suffices to prove the statements for the bundles U, U' over S^4. Since S^4 is a union of two disks over S^3, U is trivial over each of these disks. Over the intersection, S^3, the trivializations differ by a map $S^3 \to SO(4)$, which is an element of $\pi_3 SO(4)$ (called the clutching function). Conversely, elements of $\pi_3 SO(4)$ can be used to construct rank 4 vector bundles over S^4. Likewise, rank 5 vector bundles over S^4 correspond to elments of $\pi_3 SO(5)$.

To prove the first statement, $U \oplus \underline{\mathbb{R}} \cong U' \oplus \underline{\mathbb{R}}$ if and only if the have the same clutching function. Since $\pi_3 SO(5) \cong \mathbb{Z}$, there is a homomorphism $\pi_3 SO(5) \to \mathbb{Z}$ given by $[\gamma] \mapsto p_1(E_\gamma)[S^4]$, where E_γ is the bundle over S^4 constructed from the clutching function γ. We invoke the previous lemma to show that this homomorphism is nonzero, and hence injective. This means the Pontryagin number determines the bundle.

For the second statement, we use $\pi_3 SO(4) = \pi_3(S^3 \times S^3) = \mathbb{Z} \times \mathbb{Z}$. Then define a map as before $(p_1, e) : \mathbb{Z}^2 \to \mathbb{Z}^2$. This map can be computed to be $\begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix}$, which is nonsingular. Therefore it is an injection, and hence (p_1, e) determines the bundle.

□

Corollary 2.38. *Suppose that X, X' are closed oriented simply connected four-manifolds and suppose $f : X' \to X$ is a homotopy equivalence of degree $+1$. Then $f^*TX \cong TX'$ as oriented vector bundles.*

Proof:

The intersection form Q_X determines $p_1 = 3\tau$ and $e = \chi$.

□

2.5 Rokhlin's Theorem ❖

In this section, we will show the logical equivalence between Rokhlin's theorem and $\pi_3(S^4) \cong \mathbb{Z}/24$. We will do this via the Thom-Pontryagin construction and the J-homomorphism.

Theorem 2.39 (Rokhlin). *If X is a closed oriented manifold of dimension four and $w_2(TX) = 0$, then 16 divides the signature $\tau(X)$.*

The vanishing of $w_2(TX)$ ensures that the intersection form Q_X is even because the characteristic element $\omega = w_1^2 + w_2$ vanishes. Therefore, by Corollary 2.20, the signature $\tau(X)$ must be divisible by 8. This theorem says that actually it is divisible by an extra factor of two than expected. It should be noted that this divisibility is sharp; a quartic surface in $\mathbb{CP}^3$ has signature -16. By the Hirzebruch signature theorem, we have $p_1(X) := p_1(TX)[X] = 3\tau(X)$, and hence Rokhlin's theorem is equivalent to $48 \mid p_1(X)$.

Theorem 2.40. *The following are equivalent:*

- *If X is a closed, oriented 4-manifold with $w_2(TX) = 0$, then 48 divides $p_1(X)$.*
- *$\pi_8(S^5) \cong \mathbb{Z}/24$.*

The first statement can be proved using differential geometry via the Dirac operator, and the second can be proven using homotopy theory and K-theory (see [6]).

Remark 2.41. The group $\pi_8(S^5)$ is a *stable* homotopy group, which means that it is isomorphic to $\pi_{3+k}(S^k)$ for $k \geq 5$ by the Freudenthal suspension theorem.

2.5.1 The Thom-Pontryagin Construction

Let M^k be a closed manifold embedded in $\mathbb{R}^{m+k}$ for $m > k+1$. The Thom-Pontryagin homomorphism is a map from $\pi_{k+m}(S^m)$ to Ω_k^{framed}, the framed cobordism group. To define the latter:

Definition 2.42. A *normal framing* of M is a choice of trivialization $\phi : N_M \to \underline{\mathbb{R}}^m$ of the normal bundle N_M.

Definition 2.43. A *framed cobordism* between two normally framed manifolds (M_0, ϕ_0) and (M_1, ϕ_1) is compact a manifold P with $\partial P = M_0 \sqcup M_1$ and an embedding $j : P \to \mathbb{R}^{k+m} \times [0,1]$ transverse to the boundary such that $j^{-1}(\mathbb{R}^{k+m} \times \{i\}) = M_i$ for $i \in \{0, 1\}$. Moreover, there must be a normal framing Φ of P that agrees with ϕ_i on the respective boundary components.

The set of equivalence classes of framed cobordisms Ω_k^{framed} has the structure of an abelian group, where the group law is disjoint union. With this, we can define the Thom-Pontryagin homomorphism:

$$PT : \pi_{k+m}(S^m) \to \Omega_k^{framed}$$

Given $[f] \in \pi_{k+m}(S^m)$, we can choose a regular value $x \in S^m = \mathbb{R}^m \cup \{\infty\}$. Consider the k dimensional submanifold $f^{-1}(x) \subset S^{k+m} = \mathbb{R}^{n+m} \cup \{\infty\}$. Without loss of generality, assume $f^{-1}(x) \subset \mathbb{R}^{n+m}$. The standard basis for $\mathbb{R}^m$ pulls back via f to define a normal framing ϕ for $f^{-1}(x)$. Then we define $PT([f]) = [(f^{-1}(x), \phi)]$. One can check that this is well defined since we can connect any two regular values by a path of regular values in S^m, which defines a cobordism of the resulting manifolds.

Theorem 2.44. *$PT : \pi_{k+m}(S^m) \to \Omega_k^{framed}$ is an isomorphism.*

The actual Thom-Pontryagin construction is the inverse of PT. Consult [8] for details.

2.5.2 The J-homomorphism

The second ingredient for our proof of Rokhlin's theorem will be the J-homomorphism, which we construct here. For each m, k the J-homomorphism is a map:

$$J_k^m : \pi_k SO(m) \to \pi_{k+m}(S^m)$$

Let $\theta : (S^k, *) \to (SO(m), I)$ be a representative element of $\pi_k SO(m)$. We want to define $J_k^m(\theta) : (S^{k+m}, *) \to (S^m, *)$; to do this, we regard S^{k+m} and S^m as:

$$S^{k+m} \cong \partial(D^{k+1} \times D^m) = (S^k \times D^m) \cup (D^{k+1} \times S^{m-1})$$

$$S^m \cong D^m/\partial D^m$$

For each $x \in S^k$, we have an element $\theta(x) \in SO(m)$, which is an isometry of S^m. Equivalently, it is a map $\theta(x) : (D^m, \partial D^m) \to (D^m, \partial D^m)$. Letting x vary over S^k, we can thus regard θ as a map $S^k \times D^m \to D^m/\partial D^m$ sending the boundary $S^k \times S^{m-1}$ to the basepoint ∂D^m. Combining this with the collapsing map $(D^{k+1} \times S^{m-1}) \to *$, this gives us the desired map $J_k^m(\theta)$.

Consider the range $k < m-1$, which is called the stable range. For these values, the suspension theorem gives an isomorphism $\pi_{k+m}(S^m) \cong \pi_{k+m+1}(S^{m+1})$. Moreover, the homotopy long exact sequence for the fibration $SO(m) \to SO(m+1) \to S^m$ gives isomorphisms $\pi_k SO(m) \cong \pi_k SO(m+1)$. For fixed k, we then get the following diagram:

$$\begin{tikzcd}
\pi_k SO(m) \arrow[r, "J_k^m"] \arrow[d, "\cong"] & \pi_{k+m} S^m \arrow[d, "\cong"] \\
\pi_k SO(m+1) \arrow[r, "J_k^{m+1}"] \arrow[d, "\cong"] & \pi_{k+m+1} S^{m+1} \arrow[d, "\cong"] \\
\pi_k SO(m+2) \arrow[r, "J_k^{m+2}"] \arrow[d, "\cong"] & \pi_{k+m+2} S^{m+2} \arrow[d, "\cong"] \\
\vdots & \vdots
\end{tikzcd}$$

for $m > k+1$. This intertwines all J_k^* homomorphisms in this range. We call the stable homomorphism J_k to be the limit of these homomorphisms for $m \to \infty$. There is a geometric interpretation of J_k, which comes by noticing that we can identify:

$$\{\text{framings of } S^k \subset \mathbb{R}^{k+m}\} = \pi_k SO(m)$$

Namely, any such framing determines over every point $x \in S^k$ a an orientation preserving isometry of the normal space at x to $\mathbb{R}^m$, which is an element of $SO(m)$. Running over all $x \in S^k$ gives a map $S^k \to SO(m)$. Then we can consider the composition of the J homomorphism and the Thom-Pontryagin homomorphism:

$$\pi_k SO(m) = \{\text{framings of } S^k \subset \mathbb{R}^{k+m}\} \xrightarrow{J_k^m} \pi_{k+m} S^m \xrightarrow{PT} \Omega_k^{framed}$$

Then we claim that, under this composition, a framing $\phi : N_{S^k} \to \underline{\mathbb{R}}^m$ is sent to its class $[S^k, \phi] \in \Omega_k^{framed}$.

Verify this claim.

We now specialize to the case where $k = 3$ and $m = 5$. For simplicity, we denote the composition $PT \circ J_3^5 \equiv J_3$.

Proposition 2.46. *$J_3 : \pi_3 SO(5) \to \Omega_k^{framed}$ is a surjection.*

For a sketch of a proof of this, see Tim's online notes (lecture 6). Since PT is a bijection, we must have that $J_3^5 : \pi_3 SO(5) \to \pi_8 S^5$ is a surjection. By stable homotopy theory, one can compute that $\pi_3 SO(5) \cong \mathbb{Z}$. Therefore $\pi_8 S^5$ must be a quotient of $\mathbb{Z}$. Now we have the tools to prove Theorem 2.40:

Proof of Theorem 2.40:

($\Rightarrow$) First we assume $\pi_8 S^5 \cong \mathbb{Z}/24$; then we wish to conclude that $p_1(X)$ is divisible by 48 for any closed oriented 4-manifold X with $w_2(TX) = 0$. By Corollary 2.35, we can trivialize the tangent bundle to X in $X - B$ for some open 4-ball B, and therefore we can trivialize the normal bundle ν along $X - B$ as well. Let Φ be such a trivialization. There is an obstruction $O(\nu, \Phi) \in H^4(X, \pi_3 SO(5)) = H^4(X, \mathbb{Z}) = \mathbb{Z}$ to extending Φ to all of X. By our construction of $O^4(E, \cdot)$ in the previous section, $O(\nu, \Phi)$ is the homotopy class of $\Phi_{\partial B}$ when viewed as an element of $\pi_3 SO(5)$. We have shown that J_3 sends $O(\nu, \Phi)$ to its framed cobordism class $[S^3, \Phi_{\partial B}]$. However, we have just demonstrated a cobordism of S^3 empty set via $X - B$; therefore $[S^3, \Phi_{\partial B}] = 0$ and $O(\nu, \Phi) \in \ker J_3$. Since we are assuming $\pi_8 S^5 \cong \mathbb{Z}/24$, we have $\ker J_3 = 24\mathbb{Z}$ and so $O(\nu, \Phi)$ is a multiple of 24. Now consider the map we described in the proof of Theorem 2.36 which associated to every element $\gamma \in \pi_3 SO(5)$ the class $p_1(E_\gamma)$, where E_γ is the bundle over S^4 with clutching function γ. We showed that this map was nonzero from Lemma 2.37, which gave a nonzero element of the image. From that example we can also conclude that this map is multiplication by ± 2, since its Euler class evaluated to 1, which is indivisible. Therefore $p_1(TX) = \pm 2 O(\nu, \Phi)$, and hence $p_1(X)$ is divisible by 48.

($\Leftarrow$) Now assume Rokhlin's theorem. Let $a \in \ker J_3$ be a class, which can be thought of as a normally framed 3-sphere (S^3, ϕ). As an element of $\ker J_3$, there is a framed 4-manifold (P, Φ) with boundary (S^3, ϕ). Let $X = P \cup_\phi D^4$. Since $w_2(P) = 0$, we have $w_2(X) = 0$ because adding D^4 does not change the second cohomology group. By Rokhlin's theorem, we then have $p_1(X) = 48r$ for some $r \in \mathbb{Z}$. However, by the same reasoning as above, $p_1(X) = \pm 2a$. Thus $a = 24r$ and so $\ker J_3 \subset 24\mathbb{Z}$. We have shown an example of a quartic surface with $p_1 = -48$, so we have equality $\ker J_3 = 24\mathbb{Z}$ and hence $\pi_3 S^8 \cong \mathbb{Z}/24\mathbb{Z}$.

□

3. 4-Manifold Theory II: Differential Forms and Hodge Theory

❖

Continuing with our review of 4-manifold theory, this section is dedicated to the more differential aspects. It will cover the Hodge theory of 2-forms and harmonic forms as well as covariant derivatives and their relation to gauge transformations. It will conclude with a brief discussion of abelian instanton theory.

❖

3.1 Self-duality of 2-forms ❖

The middle cohomology group of differential 2 forms $H^2_{dR}(X)$ has a duality induced by the Hodge star operator. In this subsection, we will first define this operator in the case of a vector space and discuss eigenspaces of this operator. Then we will apply this to the tangent bundle of a 4-manifold and introduce harmonic forms and the Hodge theorem.

3.1.1 The Hodge Star

Let (V, g) be a an oriented, finite dimensional real vector spae with g a positive definite bilinear form. Let $\{v_1, ..., v_n\}$ be an oriented basis for V. The spaces $\Lambda^k V$ inherit $O(V)$-invariant inner products. Under this, an orthonormal basis for $\Lambda^k V$ is $\{v_{i_1} \wedge ... \wedge v_{i_k} \mid i_1 < ... < i_k\}$. We write $\mathrm{vol} = v_1 \wedge ... \wedge v_n \in \det V = \Lambda^n V$, where $n = \dim(V)$. The wedge product defines a nondegenerate bilinear form:

$$\wedge : \Lambda^k V \times \Lambda^{n-k} V \to \det V$$

It is then valid to define a linear map, the *Hodge star*, by:

$$* : \Lambda^k V \to \Lambda^{n-k} V$$

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \mathrm{vol} \in \det V$$

where $\langle \cdot, \cdot \rangle$ is the inner product induced by g.

A property that follows from the symmetry of this inner product is:

$$\alpha \wedge *\beta = (-1)^{k(n-k)} * \alpha \wedge \beta$$

Let $I \in \binom{n}{k}$ be a k-element subset of $\{1, ..., n\}$. Write I as $\{i_1, ..., i_k\}$ where $i_1 < ... < i_k$. Set $v_I = v_{i_1} \wedge ... \wedge v_{i_k}$. Then $*v_I = \sigma(I, I^c) v_{I^C}$, where I^C is the complement of I in $\{1, ..., n\}$, also written in increasing order. Because of this $*$ is an isometry. Moreover, $** = (-1)^{k(n-k)}\,\mathrm{id}$. If you conformally rescale the inner product, i.e. replace $\langle \cdot, \cdot \rangle$ by $\lambda \langle \cdot, \cdot \rangle$, then $*$ changes to $\lambda^\alpha *$ for some α.

In four dimensions, we are consider the case $k = 2$ of 2-forms. We have $* : \Lambda^2 V \to \Lambda^2 V$ with $** = \mathrm{id}$ and $* \in O(\Lambda^2 V)$. The (± 1) eigenspaces of $*$ define a splitting $\Lambda^2 = \Lambda^+ V \oplus \Lambda^- V$, where $* = 1$ on the first component and $* = -1$ on the second. Concretely, this writes $\omega \in \Lambda^2 V$ as $\omega^+ + \omega^-$, where $\omega^+ = \frac{1}{2}(\mathrm{id} + *)\omega$ and $\omega^- = \frac{1}{2}(\mathrm{id} - *)\omega$. The components $\Lambda^\pm V$ only depend on the conformal class of $\langle \cdot, \cdot \rangle$. In this casee $*$ is conformally invariant, i.e. it is invariant under scaling the inner product. Since $\Lambda^\pm V$ is three dimensional, we write a basis:

$$(\omega_1^\pm, \omega_2^\pm, \omega_3^\pm) = ((v_1 \wedge v_2)^\pm, (v_1 \wedge v_3)^\pm, (v_1 \wedge v_4)^\pm)$$

Applying the operators $\frac{1}{2}(\mathrm{id} \pm *)$:

$$\left(\frac{1}{2}(v_1 \wedge v_2 \pm v_3 \wedge v_4), \frac{1}{2}(v_1 \wedge v_3 \pm v_4 \wedge v_2), \frac{1}{2}(v_1 \wedge v_4 \pm v_2 \wedge v_3) \right)$$

By permuting the v_i, we permute the $\omega_i^\pm$ via a standard homomorphism $\theta : S_4 \to S_3$ whose kernel is the Klein four group $K_4 \subset S_4$. Since θ preserves signs, $\Lambda^\pm V$ acquire orientations from that of V.

Exercise 3.1. Determine $\theta : S_4 \to S_3$ as above and show that its kernel is K_4.

The metric on $\Lambda^{\pm}V$ induced by $\Lambda^2 V$ then gives us $|\omega_i^{\pm}| = \sqrt{2}$. The action of $O(V)$ on V induces an action of $O(V)$ on $\Lambda^2 V$ preserving/permuting $\Lambda^{\pm}V$ according to the sign of the permutation. We then get homomorphisms $\lambda^{\pm} : SO(V) \to SO(\Lambda^{\pm}V)$.

Proposition 3.2. *There is a short exact sequence of Lie groups:*

$$1 \longrightarrow \{\pm\,\mathrm{id}\} \longrightarrow SO(V) \xrightarrow{(\lambda^+,\lambda^-)} SO(\Lambda^+V) \times SO(\Lambda^-) \longrightarrow 1$$

So $SO(4)/\{\pm\,\mathrm{id}\} = SO(3) \times SO(3)$.

Lemma 3.3. *Unit length decomposable 2 forms $u \wedge v$ correspond bijectively to oriented 2-planes $P \subset V$ by sending P to its volume form.*

Proof of Proposition 3.2:

Take $A \in \ker(\lambda^+, \lambda^-)$, so A acts trivially on Λ^+V and Λ^-V and therefore also on Λ^2V. By the Lemma above, A preserves every 2-plane, and hence every line (which can be expressed as the intersection of two planes). Therefore A is scalar, and $\{\text{scalars}\} \cap SO(V) = \{\pm\,\mathrm{id}\}$. The dimension of $SO(V)$ is $\binom{4}{2} = 6$, while $\dim(SO(\Lambda^+V)) \times SO(\lambda^-V)) = 2\dim SO(3) = 6$. Then the map of Lie algebras $D_I(\lambda^+, \lambda^-)$ has kernel 0 (since (λ^+, λ^-) has discrete kernel), so it is an isomorphism. It follows that $\mathrm{im}(\lambda^+, \lambda^-)$ contains the identity component of $SO(\Lambda^+V) \times SO(\Lambda^-V)$, which is path connected.

□

3.1.2 Equivalence of conformal structures with maximal positive definite subspaces

When $\dim V = 4$, Λ^2V has an intrinsic quadratic form $q : \Lambda^2V \to \det V$ given by $q(\eta) = \eta \wedge \eta$. The signature of q is zero. A choice of conformal structure $[g] = \mathbb{R}_+ \cdot g$ determines maximal positive and negative definite subspaces $\Lambda^+ = \Lambda^+_g$ and $\Lambda^- = \Lambda^-_g$. Moreover, $\Lambda^- = (\Lambda^+)^\perp$ with respect to $\wedge$. Consider $\sigma : \mathrm{Conf}(V) \to \mathrm{Gr}_3(\Lambda^2V)^+$ sending the conformal structure $[g]$ to $\Lambda^+_{[g]}$.

Proposition 3.4. *The map σ described is bijective.*

Proof:

Fix an inner product g. Note that $SL(V)$ acts transitively on $\mathrm{Conf}(V)$. Then $\mathrm{Conf}(V) \cong SL(V)/SO(V,g)$. Now for the Grassmannian $\mathrm{Gr}_3(\Lambda^2V)^+$, note that the identity component of $SO(\Lambda^2V, q)$ acts transitively on $\mathrm{Gr}_3(\Lambda^2V)^+$. Therefore $\mathrm{Gr}_3(\Lambda^2V)^+ = SO(\Lambda^2V, g)_o/SO(\Lambda^+) \times SO(\Lambda^-)$.

Now the action of $SL(V)$ on Λ^2V defines a representation $\rho : SL(V) \to SO(\Lambda^2V, q)_o$ and $\ker\rho = \pm I$, as before. Since $\dim SL(V) = 4^2 - 1 = 15$ and $\dim SO(\Lambda^2V, q)_o = \dim_{\mathbb{C}} SO(\Lambda^2V_{\mathbb{C}}) = \binom{6}{2} = 15$. Therefore $D\rho$ is an isomorphism of Lie algebras and ρ is surjective to the identity component. Thus we have a short exact sequence:

$$1 \to \{\pm I\} \to SL(V) \xrightarrow{\rho} SO(\Lambda^2V, q) \to 1$$

We also know that ρ carries $SO(V,g)$ to $SO(\Lambda^+) \times SO(\Lambda^-)$. Therefore ρ induces a diffeomorphism $SL(V)/SO(V,g) \to SO(\Lambda^2V, g)_o/SO(\Lambda^+) \times SO(\Lambda^-)$. This map is exactly σ.

□

3.1.3 Conformal structures as maps $\Lambda^- \to \Lambda^+$

Fix a reference inner product g_0 which defines the splitting $\Lambda^2V = \Lambda_0^+ \oplus \Lambda_0^-$. Any other negative definite 3-dimensional subspace of Λ^2V can be written as a graph Γ_m for $m \in \hom(\Lambda_0^-, \Lambda_0^+)$. Take $\alpha \neq 0 \in \Lambda_0^-$. Then $\alpha + m\alpha \in \Lambda^-$. Then:

$$0 > q(\alpha + m\alpha) = q(\alpha) + q(m\alpha) = |\alpha|^2 - |m\alpha|^2$$

which means m has operator norm < 1. Therefore, given g_0, we have $\mathrm{Conf}(V) \cong \hom(\Lambda_0^-, \Lambda_0^+)_{<1}$, which is a ball in a 9 dimensional vector space. One can check that if $\Lambda^- = \Gamma_m$, then $\Lambda^+ = (\Lambda^-)^\perp = \Gamma_{m^*}$, where m^* is the adjoint of m. Note that there explicit formulas for the ocmponents of $\alpha = \alpha_0^+ + \alpha_0^-$ with respect to the new splitting $\Lambda^2 V = \Gamma_m \oplus \Gamma_{m^*}$, and these can be found in Tim's online notes.

Let $N_V \subset \Lambda^2 V$ be the *null cone* of V, namely $\eta \in N_V \iff \eta \wedge \eta = 0$. One can check that $N_V = \{\text{decomposable 2-forms}\}$ and that $N_V/\mathbb{R}_+ = \{\text{decomposable 2-forms of square length 2}\}$, which is the same as oriented 2-planes. Now suppose $\omega \in \Lambda^2 V$ with $|\omega|^2 = 2$; then ω is decomposable if and only if $|\omega^+| = |\omega^-| = 1$. The upshot of this is that the Grassmannian of oriented 2-planes $\widetilde{\mathrm{Gr}}_2(V)$ is identified with $S(\Lambda^+) \times S(\Lambda^-)$ by $\omega \mapsto (\omega^+, \omega^-)$. In other words:

$$\widetilde{\mathrm{Gr}}_2(\mathbb{R}^4) \cong S^2 \times S^2$$

3.1.4 The Hodge Theorem and Harmonic Forms

Let (M, g) be an n-dimensional oriented Riemannian manifold without boundary. Recall the exterior derivative is $d : \Omega_M^k \to \Omega_M^{k+1}$. We will define the co-exterior derivative $d^* : \Omega_M^{k+1} \to \Omega_M^k$. Recall we have the Hodge star:

$$* : \Lambda^k(T_x^* M) \to \Lambda^{n-k}(T_x^* M)$$

This defines a global bundle map $\Lambda^k T^* M \to \Lambda^{n-k} T^* M$. Then define:

$$d^* = (-1)^{k+1} (*)^{-1} d *$$

Since $(*)^{-1} = (-1)^{k(n-k)} *$, we have:

$$d^* = (-1)^{nk+1} * d *$$

Since $d^2 = 0$ and $** = \pm\,\mathrm{id}$, then $(d^*)^2 = 0$. This is all local so far and doesn't require compactness. However, if M is compact, then we have an L^2 inner product on Ω_M^k

$$\langle \alpha, \beta \rangle_{L^2} = \int_M g(\alpha, \beta) \mathrm{vol}_g$$

Recall that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$. Stokes's theorem then gives:

$$\int_M d\alpha \wedge \beta = -(-1)^{|\alpha|} \int_M \alpha \wedge d\beta$$

Taking $\beta = *\gamma$, we get:

$$\int_M g(d\alpha, \gamma) \mathrm{vol}_g = (-1)^{k+1} \int_M g(\alpha, (*)^{-1} d * \gamma) \mathrm{vol}_g$$
$$\iff \langle d\alpha, \gamma \rangle_{L^2} = \langle \alpha, d^* \gamma \rangle_{L^2}$$

That is, d^* is the formal adjoint to d.

Harmonic Forms

On any oriented Riemannian manifold (M^n, g), set $\Delta = dd^* + d^* d : \Omega_M^k \to \Omega_M^k$. Notice that $\Delta = (d + d^*)^2$. Then define $\mathcal{H}_g^k = \ker \Delta$ to be the vector space of harmonic k-forms. Clearly, $\ker(d + d^*) \subset \mathcal{H}_g^k$; under the assumption that M is compact, this is actually an equality. To see this, consider:

$$\langle \alpha, \Delta\alpha \rangle_{L^2} = \langle d\alpha, d\alpha \rangle_{L^2} + \langle d^*\alpha, d^*\alpha \rangle_{L^2} = ||d\alpha||_{L^2}^2 + ||d^*\alpha||_{L^2}^2$$

Given a harmonic form (i.e. α such that $\Delta\alpha = 0$), then it must follow that $d\alpha = 0$ and $d^*\alpha = 0$. Therefore $\mathcal{H}_g^k \subset \ker(d + d^*)$.

There is also a variational characterization. Suppose $\eta \in \mathcal{H}_g^k$; we have shown this implies $d\eta = 0$. Therefore it represents a class $[\eta] \in H_{dR}^k(M)$.

Lemma 3.5. *A harmonic form η strictly mininmizes $|| \cdot ||_{L^2}^2$ of closed k-forms representing $[\eta]$.*

Proof:

Consider another cohomology representative $\eta + d\gamma$. Then:

$$\begin{aligned} ||\eta + d\gamma||_{L^2}^2 - ||\eta||_{L^2}^2 &= 2\langle \eta, d\gamma \rangle_{L^2} + ||d\gamma||_{L^2}^2 \\ &= 2\langle d^*\eta, \gamma \rangle_{L^2} + ||d\gamma||_{L^2}^2 \\ &= ||d\gamma||_{L^2}^2 \end{aligned}$$

Where we used $d^*\eta = 0$ because η is harmonic. This difference is positive if $d\gamma \neq 0$. Hence, $[\eta]$ contains at most one harmonic representative.

□

Lemma 3.6. *If η is closed and minimizes $||\cdot||_{L^2}^2$ in $[\eta]$, then η is harmonic.*

Proof:

Let t be a linear parameter and consider $\eta + td\gamma$ for some γ. Since η minimizes the norm, we have:

$$\frac{d}{dt}\Big|_{t=0} ||\eta + td\gamma||_{L^2}^2 = 0$$

$$\Rightarrow 0 = 2\langle \eta, d\gamma \rangle_{L^2}$$

This is true for all $\gamma \in \Omega_M^{k-1}$. In particular, taking $\gamma = d^*\eta$, we have $0 = \langle \eta, dd^*\eta \rangle_{L^2} = ||d^*\eta||_{L^2}^2$. Therefore $d^*\eta = 0$, hence it is harmonic because $d\eta = 0$.

□

An observation is that $\ker d \perp \operatorname{im} d^*$ in Ω_M^k and $\ker d = (\operatorname{im} d^*)^\perp$, which is evident from the adjunction of d and d^*. A stronger statement of this is:

Theorem 3.7 (Hodge Theorem)**.** *Let (M^n, g) be compact and oriented. There exist L^2 orthogonal decompositions:*

$$\Omega_M^k = \ker d \oplus \operatorname{im} d^*$$

$$\ker d = \mathcal{H}_g^k \oplus \operatorname{im} d$$

Threrefore the quotient map $\mathcal{H}_g^k \to H_{dR}^k(M)$ is an isomorphism.

Proof (sketch):

The key point is showing the existence of harmonic representatives of cohomology classes. This involves two steps: finding weak solutions to $\Delta\eta = 0$ (which lie in a Sobolev space L_ℓ^2) and elliptic regularity showing that $\eta \in \bigcap_\ell L_\ell^2 = C^\infty$. Both steps require elliptic estimates showing that Δ is a bounded operator between Sobolev spaces.

□

Remark 3.8. Because of the decomposition $\ker d = \mathcal{H}_g^k \oplus \operatorname{im} d$, we can identify $\mathcal{H}_g^k$ with $H_{dR}^k(M)$ in a unique way.

3.1.5 (Anti)-self dual harmonic 2-forms

Let (X^4, g) be closed and oriented. Recall we had a decomposition $\Lambda^2 = \Lambda_{[g]}^+ \oplus \Lambda_{[g]}^-$, where $* = 1$ on the first subspace and $* = -1$ on the second. Let $\Omega_{[g]}^\pm := C^\infty(X, \Lambda_{[g]}^\pm)$. All of the discussion we had in the case of linear spaces applies now pointwise to the cotangent bundle. For example, $*$ only depends on the conformal class of g.

Definition 3.9. A *conformal structure* on X is a Riemannian metric up to scaling by positive functions.

By our discussion before, a 2-plane field $P \subset T^*X$ is equivalently gien by a pair of forms (ω^+, ω^-) with $\omega^+, \omega^- \in \Omega^\pm_g$ of unit length. In this dimension, we have $d^* = - * d* : \Omega^2_X \to \Omega^2_X$. For a 2-form η, it follows that $\eta \in \ker(d + d^*) \iff *\eta \in \ker(d + d^*)$. Hence, if $\eta \in \mathcal{H}^k_g$, its components $\eta^\pm = \frac{1}{2}(1 \pm *)\eta$ are again harmonic. Therefore we have a decomposition:

$$\mathcal{H}^2_g = \mathcal{H}^+_g \oplus H^-_g$$

where $\mathcal{H}^\pm_g = \mathcal{H}^2_g \cap \Omega^\pm_g$. For $\eta \in \mathcal{H}^+_g$ nonzero, notice that the wedge square quadratic form is always positive:

$$\int_g \eta \wedge \eta = \int_X \eta \wedge *\eta = \int |\eta|^2 \mathrm{vol}_g > 0$$

While for $\omega \in \mathcal{H}^-_g$ the same calculation gives a negative value. In other words, the conformal structure $[g]$ determines maximal positive and negative definite subspaces of the the wedge square quadratic form on $H^2_{dR}(X) = \mathcal{H}^2_g$, namely $\mathcal{H}^+_{[g]}, \mathcal{H}^-_{[g]}$. Moreover $\dim \mathcal{H}^\pm_{[g]} = b^\pm(X)$, where $\tau(X) = b^+ - b^-$.

Remark 3.10. An (anti)self dual form is harmonic if and only if it is closed, which follows from the formula $d^* = - * d*$.

Definition 3.11. The signature complex is:

$$0 \longrightarrow \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d^+} \Omega^+_g \longrightarrow 0$$

where $d^+ = \frac{1}{2}(1 + *)d$. We denote this complex $(\mathcal{E}^*, D)$.

Theorem 3.12. *The cohomology of the signature complex is $H^0(\mathcal{E}) = H^0_{dR}(X), H^1(\mathcal{E}) = H^1_{dR}(X)$ and $H^2(\mathcal{E}) = \mathcal{H}^+_g(X)$.*

Proof:

The first cohomology is a consequence of the definition of $H^0_{dR}(X)$. To show the second, we will show that $\alpha \in \ker d^+$ is closed.

$$d\alpha = d^+\alpha + d^-\alpha$$

where $d^\pm = \frac{1}{2}(1 \pm *)d$. Then:

$$\int_X d\alpha \wedge d\alpha = \int_X |d^*\alpha|^2 \mathrm{vol}_g - \int_X |d^-\alpha|^2 \mathrm{vol}_g$$

By Stokes, however, this must be zero because $d\alpha \wedge d\alpha = d(\alpha \wedge d\alpha)$. Therefore $||d^+\alpha||_{L^2} = ||d^-\alpha||_{L^2}$. Therefore if $d^+\alpha = 0$, then $d^-\alpha = 0$, hence $d\alpha = 0$.

Finally, to show the second cohomology, we need to show that $\Omega^+/\operatorname{im} d^+ \cong \mathcal{H}^+_g$. More precisely, we mus show that the composition:

$$H^+_g \hookrightarrow \Omega^+_g \to \Omega^+/\operatorname{im} d^+$$

is an isomorphism. By the Hodge theorem, any $\omega \in \Omega^+$ has a unique decomposition into three parts $\omega = \omega_{harm} + d\alpha + *d\beta$, where $\omega_{harm} \in \mathcal{H}^k_g$ is a harmonic form. Notice that since $*\omega = \omega$, from uniqueness it must be that $*\omega_{harm} = \omega_{harm}$ and $d\beta = d\alpha$. Thus:

$$\omega = \omega_{harm} + d\alpha + *d\alpha = \omega_{harm} + 2d^+\alpha$$

Therefore we can decompose ω uniquely as a sum of a harmonic form in $\mathcal{H}^+_g$ and something in the image of d^+. Then the result follows.

□

3.2 The Period Map ❖

Let Conf(X) be the conformal structures on X of class C^r for some $r \geq 3$, i.e. C^r Riemannian metrics modulo C^r positive functions. Fix some reference conformal structure $[g_0] \in$ Conf(X), and let $\Lambda^\pm := \Lambda^\pm_{[g_0]}$. Then:

$$\text{Conf}(X) \leftrightarrow \{C^r \text{ bundle maps } m : \Lambda^- \to \Lambda^+ \mid \text{ operator norm of } m \text{ is everywhere} < 1\}$$

This correspondence is given by writing $\Lambda^-_{[g]} = \Gamma_m$ for some m, as in our previous discussion. Therefore Conf(X) is an open set in a banach space, hence it is a Banach manifold. The tangent space $T_{[g_0]}\text{Conf}(X) = C^\infty(X, \hom(\Lambda^-, \Lambda^+))$.

The period map $P : \text{Conf}(X) \to \text{Gr} = \text{Gr}_{b^-(X)} H^2_{dR}(X)$ is defined by $P[g] = H^-_{[g]} \subset H^2_{[g]} = H^2_{dR}(X)$. The derivative of P is of interest to us. The point $[g_0] \in$ Conf(X) has a neighborhood which is a neighborhood of 0 in $C^r(X, \hom(\Lambda^-, \Lambda^+))$. On the other hand, Gr near $\mathcal{H}^-_{[g_0]}$ is identified with a neighborhood of zero in $\hom(\mathcal{H}^-_{[g_0]}, \mathcal{H}^+_{[g_0]})$ by taking graphs. Therefore, at $[g_0]$, the derivative $D_{[g_0]}P$ is a map $C^r(X, \hom(\Lambda^-, \Lambda^+)) \to \hom(\mathcal{H}^-, \mathcal{H}^+)$.

Proposition 3.13. *The derivative acts as $(D_{[g_0]}P)(m)(\alpha^-) = (m(\alpha^-))_{harm}$, which is the L^2 harmonic projection of $m(\alpha^-)$.*

Since the harmonic projection is onto, we have the immediate corollary:

Corollary 3.14. *P is a submersion.*

3.3 Covariant Derivatives ❖

References for this section: [7], [3]. In what follows let $E \to M$ be a complex vector bundle and let $\Gamma(M, E)$ be the space of C^∞ sections of $E \to M$.

Definition 3.15. Let $(\cdot, \cdot)$ be a Hermitian inner product in E. A *covariant derivative* (also known as a connection) on E is a $\mathbb{C}$-linear map:

$$\nabla : \Gamma(M, E) \to \Omega^1_M(E)$$

where $\Omega^1_M(E) := \Gamma(M, T^*M \otimes_{\mathbb{R}} E)$. It must also obey the Leibniz rule:

$$\nabla(f \cdot s) = df \otimes s + f \nabla s \tag{L}$$

where s is a section and $f \in C^\infty(M)$. ∇ is called unitary if:

$$d(s_1, s_2) = (\nabla s_1, s_2) + (s_1, \nabla s_2)$$

Lemma 3.16. *Covariant derivatives are local operators: $(\nabla s)(x)$ depends only on the germ of s near x.*

Proof:

Say $s_1 = s_2$ on a neighborhood U of x. Let χ be the smooth cutoff function which is supported in U and is 1 near x. Then $\chi s_1 = \chi s_2$. Then apply the Leibniz rule (L):

$$\nabla(\chi s_i)(x) = (\nabla s_i)(x)$$

so $(\nabla s_0)(x) = (\nabla s_1)(x)$.

□

Example 3.17. Let $\underline{\mathbb{C}} \to M$ be a trivial line bundle. A covariant derivative amounts to a map $C^\infty(M, \mathbb{C}) \to C^\infty(M, T^*M \otimes \mathbb{C})$ satisfying (L). The exterior derivative d is one such map. This is the "trivial connection." Note that this still works in higher rank trivial bundles. Let $\underline{V} = V \times M$ be trivial for some complex vector space V. Then $d = d \otimes \text{id}_V : \Gamma(M, V) \to \Gamma(M, T^*M \otimes V)$. The trivial connection is unitary with resepct to a trivialized hermitian inner product.

As an immediate consequence of (L), if ∇, ∇' are connections, then $\nabla - \nabla'$ is a $C^\infty(M)$-linear map. In fact, this is an arbitrary $C^\infty(M)$-linear map; that is, if we write $\mathcal{C}(E)$ to be the space of covariant derivatives, then given a reference $\nabla \in \mathcal{C}(E)$ we have a vector space:

$$\mathcal{C}(E) = \nabla + \Omega^1_M(\text{End}(E))$$

Moreover, if ∇ is unitary, then:

$$\mathcal{C}^{\text{unitary}}(E) = \nabla + \Omega^1_M(U(E))$$

where $U(E)$ is the space of skew hermitian endomorphisms of E_x for $x \in M$.

Example 3.18. Returning to the trivial bundle $\underline{V} \to M$. Then taking our reference to be the trivial connection d, we have $C(\underline{V}) = d + \Omega^1_M(\text{End}(V))$. Thus any connection ∇ takes the form $\nabla = d + A$, where A is a 1-form value in $\text{End}_{\mathbb{C}} V$.

An observation is that covariant derivatives are first order operators in the sense that $(\nabla s)(x)$ depends only on $s(x)$ and $D_x s$. To see this, we may assume that E is a trivial bundle because ∇ is a local operator. Then by the example above, $\nabla = d + A$ which is first order because d is first order and A is zero-th order.

Exercise 3.19. Show that given a vector bundle $E \to M$ with connection ∇ and a smooth map $f : M' \to M$, there is a natural notion of a pullback connection $f^*\nabla$ on f^*E. Moreover:

$$(g \circ f)^*\nabla = f^*(g^*\nabla)$$

Exercise 3.20. Show that $\mathcal{C}(E)$ is nonempty by using a partitition of unity subordinate to a trivializing cover of M.

3.3.1 Curvature

Let $\nabla \in \mathcal{C}(E)$ be a connection on $E \to M$. This defines a "coupled exterior derivative":

$$d_\nabla : \Omega^k_M(E) \to \Omega^{k+1}_M(E)$$

$$d_\nabla(\eta \otimes s) = (-1)^k \eta \wedge \nabla s + d\eta \otimes s \quad \text{(and extend linearly)}$$

where η is a k-form on M and s is a section. Note that it obeys the Leibniz rule:

$$d_\nabla(f\eta \otimes s) = df \wedge \eta \otimes s + f d_\nabla(\eta \otimes s)$$

By construction, for $k = 0$ we have $d_\nabla = \nabla$.

Lemma 3.21. *$d_\nabla \circ d_\nabla : \Omega^*_M(E) \to \Omega^{*+2}_M(E)$ is linear over $C^\infty(M)$ and in fact over Ω^*_M.*

Proof:

Let $\eta \in \Omega^k(M)$ and $s \in \Gamma(E)$. Then:

$$\begin{aligned} d_\nabla \circ d_\nabla(\eta \otimes s) &= d_\nabla((-1)^k \eta \wedge \nabla s + d\eta \otimes s) \\ &= (-1)^{k+k} \eta \wedge d_\nabla \circ d_\nabla s + (-1)^k d\eta \wedge \nabla s + d^2\eta \otimes s + (-1)^{k+1} d\eta \wedge \nabla s \\ &= \eta \wedge d_\nabla \circ d_\nabla s \end{aligned}$$

The same applies if s is an E-valued higher form.

□

Definition 3.22. The *curvature* $F_\nabla \in \Omega^2_M(\text{End}(E))$ is defined by the relation:

$$d_\nabla \circ d_\nabla s = F_\nabla \wedge s$$

In the case of unitary connections, we have $F_\nabla \in \Omega^2_M(U(E))$.

Definition 3.23. A *flat connection* is one for which $F_\nabla = 0$.

Since we defined the curvature intrinsically, it should be natural: $F_{f^*\nabla} = f^* F_\nabla$. Moreover, curvature is defined to be the obstruction to $d_\nabla^2 = 0$. Note that a connection on E induces a covariant derivative (and coupled exterior derivatives) on End$(E) \to M$. These are defined in the following way:

$$d_\nabla : \Omega^k_M(\text{End}(E)) \to \Omega^{k+1}_M(\text{End}(E))$$

$$d_\nabla \alpha = [d_\nabla, \alpha]$$

where $[\cdot,\cdot]$ is the commutator. Explicitly, this means $(d_\nabla \alpha)s = d_\nabla(\alpha s) - \alpha(d_\nabla s)$. As a consequence, we have:

$$\boxed{F_{\nabla + A} = F_\nabla + d_\nabla A + A \wedge A}$$

Example 3.24. In the case of a trivial bundle, we have $F_{d+A} = dA + A \wedge A$ because d is flat.

Example 3.25. If $L \to M$ is complex line bundle, then since endomorphisms of a line bundle are $\Omega^1_M(\mathbb{C})$, we find $A \wedge A = 0$ for any $A \in \Omega^1_M(\mathbb{C}) = \Omega^1_M(\text{End}(L))$. If L is hermitian, then $A \in \Omega^1_M(U(1)) = \Omega^1_M(i\mathbb{R}) = i\Omega^1_M$.

If v is a vector field on M, we denote $\nabla_v : \Gamma(E) \to \Gamma(E)$ given by contracting ∇ with v. Therefore $\nabla_{fv} = f\nabla_v$ for any function $f \in C^\infty(M)$.

Coordinate Perspective

Let $(x_1, ..., x_n)$ be coordinates on M and $\partial_i \equiv \frac{\partial}{\partial x_i}$ be the coordinate vector fields. Let $\nabla_i = \nabla_{\partial i}$. Then in a trivialization of E, we write $\nabla = d + A = d + \sum_k A_k \otimes dx_k$, where $A_k(x) \in \text{End}(\mathbb{C}^r)$. Similarly, the curvature can be written as:

$$F_{d+A} = \sum_{i,j} F_{ij} dx_i \wedge dx_j$$

where $F_{ij}(x)$ is a matrix in End$(\mathbb{C}^r)$. Then given $F_{d+A} = dA + A \wedge A$, we have:

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j]$$

Moreover, we find:

$$[\nabla_i, \nabla_j] = [\partial/\partial x_i + A_i, \partial/\partial x_j + A_j] = F_{ij}$$

because partial derivatives commute. This shows that the curvature components measure the failure of commutativity of ∇_i in different coordinate directions.

Lemma 3.26. *For any vector fields u, v, the matrix valued section $F_\nabla(u,v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]}$.*

Proof:

If $u = \partial_i$ and $v = \partial_j$, we have already shown this because $[u,v] = 0$ in this case. We can expand an arbitrary u and v in terms of $\{\partial_i\}$:

$$u = \sum_i u_i \partial_i$$

$$v = \sum_i v_i \partial_i$$

Therefore it suffies to check that the RHS is $C^\infty(M)$-bilinear.

$$[\nabla_{fu}, \nabla_v] = f[\nabla_u, \nabla_v] + df(v)\nabla_u$$

$$\nabla_{[fu,v]} = f\nabla_{[u,v]} - df(v)\nabla_u$$

thus the $df(v)\nabla_u$ terms cancel and bilinearity follows.

□

Proposition 3.27 (Bianchi identity)**.** *The curvature is covariantly closed:* $d_\nabla F_\nabla = 0$*.*

3.4 Gauge Transformations ❖

Definition 3.28. A *gauge transformation* of a vector bundle $E \to M$ is a (unitary) bundle automorphism $u : E \to E$. That is, a family of (unitary) automorphisms $u_x : E_x \to E_x$ for $x \in M$.

The set of gauge transformations form a group $\mathcal{G}_E$ and we can identify $\mathcal{G}_E = \Gamma(M, GL(E))$, where $GL(E)$ is the bundle of groups with fibers $GL(E_x)$. Note that this is *not* a principal bundle. Gauge transformations act on connections by:

$$u \cdot \nabla := u^* \nabla$$

Explicitly, this means:

$$(u^*\nabla)(s) = u\nabla(u^{-1}s)$$

Lemma 3.29. *Given* $\nabla \in \mathcal{C}(E)$ *and* $u \in \mathcal{G}_E$*, we have:*

1. $F_{u\cdot\nabla} = u^* F_\nabla = uF_\nabla u^{-1}$*.*
2. $u^*\nabla - \nabla = -(d_\nabla u)u^{-1}$*.*
3. *For* E *trivial,* $u^*(d + A) - d = -(du)u^{-1} + uAu^{-1}$*.*

Theorem 3.30. *If* $F_\nabla = 0$*, then near any point in* M *there exists a trivialization of* E *in which* ∇ *is the trivial connection* d*.*

This follows immediately from the following:

Proposition 3.31. *Let* $H = (-1, 1)^n \subset \mathbb{R}^n$ *and* $\underline{\mathbb{C}}^r \to H$ *be the trivial bundle. If* ∇ *is a flat (unitary) connection in* $\underline{\mathbb{C}}^r$*, then there exists a gauge transformation* u *such that* $u^*\nabla$ *is trivial.*

Proof:

Let $\nabla = d + A = d + \sum_k A_k dx_k$. Each A_k is a matrix valued function on H (which is skew hermitian in the unitary case). Flatness says that $\partial_i A_j - \partial j A_i + [A_i, A_j] = 0$. Inductively, asssume we've gauge transformed so that $A_i = 0$ for $i = 1, ..., p$ and start at $p = 0$ (which is vacuous). We want to find u such that $u^*\nabla$ has $A_i = 0$ for $i = 1, ..., p + 1$. But:

$$u^*\nabla = d + \sum B_k dx_k$$

where $B_k = -(\partial_k u)u^{-1} + uA_k u^{-1}$. We want to solve $\partial_i(u) = 0$ for $i = 1, ..., p$ and $\partial_{p+1}u + uA_{p+1} = 0$. This is a system of linear ODE's. The last equation is a linear ODE in x_{p+1} with coefficients depending smoothly on $(x_{p+1}, ..., x_n)$. It is independent of $(x_1, ..., x_p)$ because $\partial_i A_{p+1} = 0$, $i \leq p$ by flatness.. There is a unique solution such that $u(x_1, ..., x_p, 0, x_{p+2}, ..., x_n) = I$, which is smooth in x and independent of $x_1, ..., x_p$. This completes the inductive step.

□

A covariant derivative ∇ on $E \to M$ of rank r can be encoded in a collection A of local data:

- $M = \bigcup U_\alpha$ an open cover, where E is trivialized over each U_α, which gives transformations $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$.
- Over U_α, $\nabla = d + A_\alpha$ for some $A_\alpha \in \Omega^1_{U_\alpha}(\mathrm{End}(\mathbb{C}^r))$. This gives a collection $\{A_\alpha\}$ of matrix valued 1-forms.
- These satisfy the rule $A_\beta = \tau_{\alpha\beta} A_\alpha \tau_{\beta\alpha} - (d\tau_{\alpha\beta})\tau_{\beta\alpha}$.

In the case of a flat connection, we just found that the family $A_\alpha = 0$. Then the third condition reduces to $d\tau_{\alpha\beta} = 0$, i.e. the transition functions $\tau_{\alpha\beta}$ are locally constant. Therefore a vector bundle with a flat connection determines (and is determined by) a vector bundle with locally constant transition functions (also known as a local system).

3.5 $U(1)$ connections ❖

Let $L \to M$ be a hermitian line bundle. Write $\mathcal{A}_L = \{\text{unitary covariant derivatives in } L\}$. Given $\nabla \in \mathcal{A}_L$, we can identify this space as $\mathcal{A}_L = \nabla + \Omega^1_M(i\mathbb{R})$ because the unitary endomorphisms of a line are $U(1) = i\mathbb{R}$. Then $\mathcal{G}_L = \Gamma(M, U(L)) = C^\infty(M, U(1))$ because $U(L) \cong \underline{U(1)}$. Moreover, $\mathcal{A}_L$ is a topological vector space (a Fréchet space) and $\mathcal{G}_L$ has a topoogy inherited from the space $\Gamma(M, \text{End}(L))$. The action of $\mathcal{G}_L$ on $\mathcal{A}_L$ induces an orbit space:

$$B_L = \mathcal{A}_L/\mathcal{G}_L = \{\text{gauge orbits of connections}\}$$

It is not obvious this is a reasonable space, for example it isn't clear that it is even Hausdorff. We will later get a concrete description of B_L as a torus cross a Fréchet space.

3.5.1 Chern-Weil Theory

Let $\nabla \in \mathcal{A}_L$ be a unitary connection. Then $iF_\nabla \in \Omega^2_M$ is closed because locally $\nabla = d + A$ and $F_\nabla = dA$ (hence it is locally exact which is the same as closed). The class $c_L = [iF_\nabla] \in H^2_{dR}(M)$ is independent of ∇. Indeed, any other connection is $\nabla + ia$ for $a \in \Omega^1_M$ and $F_{\nabla+ia} = F_\nabla + ida$.

We claim there is a universal $\lambda \in \mathbb{R}$ such that $c_L = \lambda c_1(L)$ because the curvature class transforms naturally just as the Chern class does. All line bundles over manifolds are pulled back via a smooth map $f : M \to \mathbb{CP}^N$ for some N (i.e. $L \cong f^*\Lambda_N$ where $\Lambda_N \to \mathbb{CP}^N$ is the tautological bundle).[2] It then suffices to show $c_{\Lambda_N} = \lambda c_1(\Lambda_N)$. Since $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^N$ induces an isomorphism on H^2 and $i^*\Lambda_N = \Lambda_1$, it actually suffices to show that $c_{\Lambda_1} = \lambda c_1(\Lambda_1)$. This is clear since $H^2(\mathbb{CP}^1) = \mathbb{R}$ and $c_1(\Lambda_1) \neq 0$. In fact:

$$\frac{1}{2\pi} c_L = c_1(L)$$

This is found by doing an example over S^2. See [2].

3.5.2 Structure of B_L

Recall that the action of $\mathcal{G}_L$ on $\mathcal{A}_L$ is $u \cdot \nabla = u^*\nabla = \nabla - (du)u^{-1}$. The action of $\mathcal{G}_L$ is semi-free, which means that the action of $\mathcal{G}_L/U(1)$ is free, where the subgroup $U(1) \subset \mathcal{G}_L$ of constant gauge transformations acts trivially. Notice that:

$$\pi_0\mathcal{G}_L = \{\text{homotopy classes of maps } M \to S^1\} \cong H^1(M, \mathbb{Z})$$

The identity component $\mathcal{G}^0_L$ is $\{u = e^{i\xi} \mid \xi \in \mathbb{C}^\infty(M, \mathbb{R})\}$. For $u \in \mathcal{G}^0_L$, we have $u^*\nabla = \nabla - id\xi$. Hence, after choosing a reference ∇, we get an identification of $\mathcal{A}_L/\mathcal{G}^0_L = i\left(\Omega^1_M/d\Omega^0_M\right)$.

Definition 3.32. The *Coulomb gauge slice* is the set $S = \{\nabla + ia \mid d^*a = 0\}$ with respect to a reference connection $\nabla \in \mathcal{A}_L$.

By the Hodge theorem, we have $\Omega^1_M/d\Omega^0_M = \ker d^*$. So we have a homeomorphism $S \to A_L/\mathcal{G}^0_L$. The identity component $\pi_0\mathcal{G}_L = H^1(M, \mathbb{Z})$ acts on $\mathcal{A}_L/\mathcal{G}^0_L = S$ in the following way. Given $u \in \mathcal{G}_L$, notice:

$$u^*\nabla = \nabla - (du)u^{-1} = \nabla - \underbrace{d(\log u)}_{\text{cloed 1-form with } 2\pi i\mathbb{Z} \text{ period}}$$

[2] This is because $\mathbb{CP}^\infty = BU(1)$.

There exists a unique cohomologous form $d(\log u) + d\xi$ with $\nabla d(\log u) + d\xi \in S$. Then $d(\log u) + d\xi \in \ker d \cap \ker d^* = \mathcal{H}^1_g$. Since $\ker d^* = \mathcal{H}^1_g \oplus \operatorname{im} d^*$, we have:

$$B_L = S/\pi_0\mathcal{G}_L = \mathcal{H}^1_g/2\pi(\mathbb{Z}\text{ lattice}) \times \operatorname{im} d^* = \underbrace{\left(\frac{H^1(X,\mathbb{R})}{2\pi H^1(X,\mathbb{Z})}\right)}_{\text{Picard torus } (P)} \times \operatorname{im} d^*$$

3.6 $U(1)$ instantons ❖

In this section let X be a compact 4-manifold with a conformal structure $[g]$.

Definition 3.33. Let $E \to X$ be a vector bundle with a hermitian metric. An *instanton* (also known as an anti-self-dual connection) is a unitary connection ∇ such that:

$$(F_\nabla)^+ := \frac{1}{2}(1+*)(F_\nabla) = 0.$$

If $u \in \mathcal{G}_E$, then $(F_{u^*\nabla})^+ = (uF_\nabla u^{-1})^+ = uF^+_\nabla u^{-1}$. Thus $\mathcal{G}_E$ preserves instantons. Note also the equation $(F_{\nabla+A})^+ = 0$ amounts to a first order PDE in A.

Donaldson theory is the study of instantons, chiefly in rank 2 bundles. We will talk about the rank 1 case, the $U(1)$ instantons. The criterion for existence of an instanton in a line bundle $L \to X$ is $c_1(L) \in \mathcal{H}^-_{[g]}(\mathbb{Z}) := \mathcal{H}^-_{[g]} \cap H^2(X,\mathbb{Z})'$.[3] This is because we found that $[i/2\pi F_\nabla] = c_1(L) \in H^2(X,\mathbb{Z})'$. If $F^+_\nabla = 0$, then $i/2\pi F_\nabla$ is anti-self-dual. Being closed and anti-self-dual implies that it is harmonic. This condition requires the "random" subspace $\mathcal{H}^-_{[g]}$ to intersect the lattice $H^2(X,\mathbb{Z})$ nontrivially.

Uniqueness of Instantons

Suppose ∇ is an instanton. Then $\nabla + ia$ is an instanton if and only if $(F_\nabla + ida)^+ = 0 \iff d^+a = 0$, where $d^+ = \frac{1}{2}(1+*)d$. Recall the signature complex $(\mathcal{E}^*, d)$:

$$0 \longrightarrow \Omega^0 \xrightarrow{\ d\ } \Omega^1 \xrightarrow{\ d^+\ } \Omega^+_{[g]} \longrightarrow 0$$

We computed $H^1(\mathcal{E}^*) = H^1_{dR}(X)$ and $H^2(\mathcal{E}^*) \cong \mathcal{H}^+_{[g]}$. There is an identification $\mathcal{A}_L/\mathcal{G}^0_L \cong \Omega^1/d\Omega^0$ via $[\nabla + ia] \mapsto [a]$. Let $\mathcal{I}_L \subset \mathcal{A}_L$ be the subspace of instantons. The gauge group $\mathcal{G}_L$ preserves $\mathcal{I}_L$. Then $\mathcal{I}_L/\mathcal{G}^0_L \cong \ker d^+/d\Omega^0 = H^1(\mathcal{E}^*) = H^1_{dR}(X)$. Since $\pi_0\mathcal{G}_L = H^1(X,\mathbb{Z})$ acts on this space, we have $\mathcal{I}_L/\mathcal{G}_L = H^1_{dR}(X)/2\pi H^1(X,\mathbb{Z}) = P$, the Picard torus.

Why did $\mathcal{I}_L/\mathcal{G}_L$ work out to be a finite dimensional manifold? Lets work in the gauge slice $S = \nabla + i\ker d^+ \cong \mathcal{A}_L/\mathcal{G}^0_L$. Then $\mathcal{I}_L \cap S = \nabla + i\ker(d^* \oplus d^+)$. The operator $d^* \oplus d^+$ maps $\Omega^1 \to \Omega^0 \oplus \Omega^+_{[g]}$. The Hodge decomposition and the signature complex give us the following decompositions:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0 \arrow[r, "d"] \arrow[d, equal] & \Omega^1 \arrow[r, "d^+"] \arrow[d, equal] & \Omega^+_{[g]} \arrow[r] \arrow[d, equal] & 0 \\
& \mathcal{H}^0_g & \mathcal{H}^1_g & \mathcal{H}^+_g & \\
& \oplus & \oplus & \oplus & \\
& d^*\Omega^1 \arrow[r, bend left, "d"] & d\Omega^1 \arrow[l, bend left, "d^*"] & d^+\Omega^1 & \\
& & \oplus & & \\
& & d^*\Omega^2 \arrow[ruu, "d^+"] & &
\end{tikzcd}$$

Here, Ω^0_0 denotes the space of functions f with mean zero: $\int_X f\text{vol} = 0$. Then d^+ maps the component $d^*\Omega^2 \to d^+\Omega^1$ isomorphically, and d^* maps $d\Omega^0$ isomorphically to $d^*\Omega^1$. Therefore we can regard $d^* \oplus d^+$ as a

[3] $H^2(X,\mathbb{Z})' \subset H^2_{dR}(X)$ denotes the integer lattice of integer classes, which is isomorphic to $H^2(X,\mathbb{Z})/\text{torsion}$

map $\Omega^1 \to \Omega^0_0 \oplus \Omega^+_{[g]}$ having kernel $\mathcal{H}^1_g$ and cokernel $\mathcal{H}^+_{[g]}$. So, the instanton moduli space $\mathcal{I}_L/\mathcal{G}_L$ is cut out by a function whose derivative $d^* \oplus d^+$ is not surjective but has cokernel of constnat rank ($b^+(X)$). Thus it is cut out "cleanly" (not transversely) by its defining equation. Technically, we need a Banach space set-up to apply the inverse function theorem and show that it is a manifold, so this discussion is incomplete.

Generic Nonexistence

Now we return to the existence question of instantons, which we described in terms of intersecting a subspace with a lattice above. In order for them to exist, we must have $\mathcal{H}^-_{[g]}(\mathbb{Z}) := \mathcal{H}^-_{[g]} \cap H^2(X,\mathbb{Z})'$ be nonempty.

Theorem 3.34. *For $k < b^+(X)$ and for any C^r family of conformal structures $[g_t]_{t\in T}$, where T is a manifold of dimension k, there exist perturbations $\hat{g}_t$ arbitrarily close to $[g_t]$ in a C^r norm induced by Riemannian metrics on T and on X such that $\mathcal{H}^-_{[\hat{g}_t]}(\mathbb{Z}) = 0$ for all $t \in T$.*

Proof:

Recall from section 3.2 that the set of conformal structures of class C^r (call it $\mathcal{C}_X$) is identified with an open set in $\Gamma(X, \hom(\Lambda^-, \Lambda^+))$, where $\Lambda^\pm = \Lambda^\pm_{[g_0]}$ and g_0 is a reference metric. We also defined the period map $P : \mathcal{C}_X \to \mathrm{Gr}^-$ defined by $P[g] = \mathcal{H}^-_{[g]}$. The derivative we showed can be seen as a map:

$$D_{[g]}P : \Gamma(X, \hom(\Lambda^-, \Lambda^+)) \to \hom(\mathcal{H}^-_{g_0}, H^+_{g_0})$$

The formula for this derivative (which we didn't prove) is $(DP)(m)(\alpha_-) = m(\alpha_-)_{\text{harm}}$, where $m : X \to \hom(\Lambda^-, \Lambda^+)$ and $\alpha_- \in \mathcal{H}^-_{g_0}$.

Fix a nonzero class $c \in H^2_{dR}(X)$ and define $S_c = \{H \in \mathrm{Gr}^- \mid c \in H\} \subset \mathrm{Gr}^-$. Pick a splitting $H^2_{dR} = H \oplus H'$ and suppose $H \in S_c$. Then given a map $\mu : H \to H'$, we have the condition $c \in \Gamma_\mu \iff \mu(c) = 0$, where Γ_μ is the graph of μ. This presents S_c as a submanifold with tangent space $T_H S_c = \{\mu \in \hom(H, H') \mid \mu(c) = 0\}$. THe normal space $N_H S_c$ maps isomorphically to H' via $[\mu] \mapsto \mu(c)$. Therefore the codimension of S_c is $b^+(X)$.

Now we claim that the period map P is transverse to S_c. To show this, we need to show that if $P[g] \in S_c$, then $\mathrm{im}(D_{[g]}P)$ spans the normal space $N_{P[g]}S_c$. Unravelling this using our formula for $D_{[g]}P$, this amounts to showing that if $\alpha^- \in \mathcal{H}^-_{[g]}$ representing c, then for all $\alpha^+ \in \mathcal{H}^+_g$ there exists $m : \Lambda^- \to \Lambda^+$ such that $m(\alpha^-)_{\text{harm}} = \alpha^+$. We will argue this by contradiction. If this were false, then there exists $\alpha^+ \in \mathcal{H}^+[g]$ which cannnot be represented as such, and so it can be taken to be L^2 orthogonal to $m(\alpha^-)$ for all m. In other words:

$$0 = \langle \alpha^+, m(\alpha^-)_{\text{harm}} \rangle_{L^2} = \langle \alpha^+, m(\alpha^-) \rangle_{L^2}$$

Now choose $x \in X$ where $\alpha^+(x) \neq 0$ and $\alpha^-(x) \neq 0$. This point exists because harmonic forms like $\alpha^\pm$ are nonvanishing on an open dense subset. Take a geodesic ball $B \ni x$ and choose $m_0 : \Lambda^-_x \to \Lambda^+_x$ such that $m_0(\alpha^-)_x = \alpha^+_x$. Take a cutoff function χ supported in $\frac{1}{2}B$ which is 1 near x. Then in B we can extend m_0 to m such that $\langle m(\alpha_-), \alpha_+ \rangle \neq 0$. Then:

$$0 \neq \langle \chi m, \alpha_+ \rangle = \int_B \chi \langle m(\alpha_-), \alpha_+ \rangle \, d\mathrm{vol}$$

which is a contradiction, so our claim is proved.

By this claim, we have that for all nonzero $c \in H^2_{dR}(X)$ we have $P^{-1}(S_c) \subset \mathcal{C}_X$ is a submanifold of codimension $b^+(X)$ (by the inverse function theorem for Banach spaces). The family $\{[g_t]\}_{t\in T}$ defines a smooth map $G : T \to \mathcal{C}_X$ by $t \mapsto [g_t]$. Consider the space of all such maps:

$$\mathcal{G} := \{C^r \text{ maps } T \to \mathcal{C}_X \text{ lying within a fixed distance from } G \text{ with resepect to a } C^r \text{ norm}\}$$

Then for all nonzero $c \in H^2_{dR}(X)$, there exists an open dense subset $U_c \subset \mathcal{G}$ of maps $\hat{G}$ transverse to $P^{-1}(S_c)$. To say that $\hat{G}$ is transverse to $P^{-1}(S_c)$ means that $\hat{G}$ misses S_c because the codimension of $P^{-1}(S_c)$ is b^+. Then $\bigcap_{c \in H^2(X,\mathbb{Z})'} U_c$ is a countable intersection of open dense subsets and is therefore dense by the Baire category theorem. The $\hat{G}$ in this intersection then do the job: $\mathcal{H}^-_{\hat{G}(t)}(\mathbb{Z}) = 0$.

$\square$

4. Differential Operators and Spin Geometry

❖

Let M be a manifold and lent $E, F \to M$ be two real vector bundles over M. Then first forder linear differential operators D are maps $\Gamma : \Gamma(E) \to \Gamma(F)$. Moreover, in local trivialization coordinates, the operator should look like:

$$(Ds)_\alpha = \sum_{i,\beta} P^i_{\alpha,\beta}(x) \frac{\partial}{\partial x_i} s_\beta + \sum_\beta Q_{\alpha,\beta}(x) s_\beta(x) \tag{$*$}$$

where $s = (s_1, ..., s_p)$ is a section of E expressed locally and P, Q are smooth functions.[4]

Definition 4.1. A *zeroth order differential operator* is a $C^\infty(M)$-linear map $L : \Gamma(M, E) \to \Gamma(M, F)$.

Definition 4.2. A *first order differential operator* is an $\mathbb{R}$-linear map $D : \Gamma(M, E) \to \Gamma(M, F)$ such that for all $f \in C^\infty(M)$, the commutator $[f, D]$ is a zeroth order differential operator.

The zeroth order differential operators form a vector space $D_0(E, F)$ and the first order differential operators form a vector space $D_1(E, F) \supset D_0(E, F)$. Define the map $\sigma^0 : D_0(E, F) \to \Gamma(M, \hom(E, F))$ by:

$$\sigma^0(L)(x) : E_x \to F_x$$

$$e_x \mapsto (Ls)(x)$$

where $x \in M$, $e_x \in E_x$ and $s \in \Gamma(M, E)$ is any section such that $s(x) = e$. This is an isomorphism, called the symbol operator.

Exercise 4.3. Show that σ^0 is well-defined, i.e. it doesn't depend on the choice of s, and is an isomorphism.

Example 4.4. A covariant derivative $\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$ is a first order differential operator between E and $F = T^*M \otimes E$ because $[f, D]s = df \otimes s$.

These definitions can be recasted in terms of jet bundles. Define $J^1E \to M$ to be the vector bundle of 1-jets of sections of E. The fiber $(J^1E)_{x_0}$ is the set of points $(x_0, [s])$, where $[s]$ is an equivalence classes of sections $s \in \Gamma(E)$, where $s \sim s'$ if $s(x_0) = s'(x_0)$ and $s - s'$ is tangent to 0 at x (i.e. $D_{x_0}(s - s') : T_{x_0}M \to T_{(x_0,0)}E$ maps to the tangent space of the zero section). Another way to say this is that s and s' have the same first order Taylor expansion at x_0.

Exercise 4.5. Show that there is a short exact sequence:

$$0 \longrightarrow (T^*M) \otimes E \longrightarrow J^1E \xrightarrow{ev} E \longrightarrow 0$$

where $ev : J^1E \to E$ is evaluation.

A section $s \in \Gamma(E)$ determines a section $j^1(s) \in \Gamma(J^1E)$ defined by $j^1_x(s) = (x, [s])$. Notice that the difference $j^1(fs) - f \cdot j^1(s)$ evaluates to zero, and so it must be of the form $\xi \otimes \theta$ for $\xi \in T^*_xM$ and $\theta \in E_x$. One can check that $\xi = df_x$ and $\theta = s(x)$ (see Appendix exercises), which gives us the following Leibniz rule:

$$j^1(fs) = f \cdot j^1(s) + df_x \otimes s(x)$$

Definition 4.6. A *first order jet operator* is a map $D : \Gamma(E) \to \Gamma(F)$ of the form:

$$Ds = L(j^1s)$$

where $L \in \Gamma(\hom(J^1E, F))$. In other words, it can be written as the composition of a zeroth order operator from $J^1E \to F$ with j^1.

[4]Note that in formula $(*)$ we are using Roman indices for coordinates on M and Greek indices for coordinates on the bundles.

The set of such operators forms a vector space $D_1(E,F)_{\text{jet}} \cong \Gamma(\hom(J^1E,F))$.

Theorem 4.7. $D_1(E,F) = D_1(E,F)_{jet}$.

Lemma 4.8. *A jet operator is a differential operator:* $D_1(E,F)_{jet} \subset D_1(E,F)$.

Proof:

Let $D = L \circ j^1$ be a jet operator. This is $\mathbb{R}$-linear. Then $[D,f]s = L(df \otimes s)$. We wish to check that $[D,f]$ is $\mathbb{C}^\infty$ linear, which means $[[D,f],g] = 0$ for any $g \in \mathbb{C}^\infty(M)$.

$$\begin{aligned}[D,f](gs) &= L \circ j^1(fgs) - fL(j^1(gs)) \\ &= L \circ (df \otimes gs + f \circ j^1(gs)) - fL(j^1(gs))) \\ &= gL(df \otimes s) = g[D,f]s\end{aligned}$$

so it is $\mathbb{C}^\infty$ linear.

$\square$

Taking hom of the above short exact sequence and then global sections gives another short exact sequence:

$$0 \longrightarrow \Gamma(\hom(E,F)) \longrightarrow \Gamma(\hom(J^1E,F)) \xrightarrow{symb} \Gamma(\hom(T^*M \otimes E, F)) \longrightarrow 0$$

This remains exact because we are working with projective objects and flabby sheaves (or something to that effect). The map $symb$ is called the principal symbol and is simply restriction to $T^*M \otimes E \subset J^1E$. For any first order differential operator D, define $\sigma^1_D(f) = \sigma^0_{[D,f]} \in \Gamma(\hom(E,F))$. One can check that:

$$\sigma^1_D(fg) = f\sigma^1_D(g) + g\sigma^1_D(f)$$

Lemma 4.9. *If $D \in D_1(E,F)$ and $f(x) = 0, df_x = 0$, then $\sigma^1_D(f)_x = 0$.*

Moreover, if f is a constant function c near x, then $\sigma^1_D(c) = 0$ since $[D,c] = 0$. This means that $\sigma^1_D(f)$ only depends on $df_x \in T^*_xM$. Therefore, for $\xi \in T^*_xM$, we can write $\sigma^1_D(\xi) := \sigma^1_D(f)$ for any f with $df_x = \xi$. This constitutes a map $\sigma^1 : D_1(E,F) \to \Gamma(\hom(T^*M \otimes E,F))$. From the definitions, we now have for any $D \in D_1(E,F)$ the difference $D - \sigma^1_D \circ j^1$ lies in $D_0(E,F)$, which means $D \in D_1(E,F)_{\text{jet}}$. This proves that $D_1(E,F) = D_1(E,F)_{\text{jet}}$.

4.0.1 Higher Order Operators

Higher order differential operators can be defined recursively:

Definition 4.10. An *nth order differential operator* is a map $D : \Gamma(E) \to \Gamma(F)$ such that $[D,f] \in D_{n-1}(E,F)$ for all smooth functions f.

Similarly, the jet bundle $J^n(E) \to M$ is defined in a similar way to J^1, except using the nth order term in the Taylor series of a section. There we get the same short exact sequences as before:

$$0 \longrightarrow \text{Sym}^n(T^*M) \otimes E \longrightarrow J^nE \longrightarrow J^{n-1}E \longrightarrow 0$$

$$0 \longrightarrow \Gamma(\hom(J^{n-1}E,F)) \longrightarrow \Gamma(\hom(J^nE,F)) \longrightarrow \Gamma(\hom(\text{Sym}^n(T^*M) \otimes E,F)) \longrightarrow 0$$

Letting $D_n(E,F)_{\text{jet}} = \{L \circ j^n \mid L \in \Gamma(\hom(J^nE,F))\} \cong \Gamma(\hom(J^nE,F))$, we again have the equality $D_n(E,F) = D_n(E,F)_{\text{jet}}$. Moreover, there is a symbol isomorphism induced by the second exact sequence:

$$\sigma^n : \frac{D_n(E,F)}{D_{n-1}(E,F)} \to \Gamma(\hom(\text{Sym}^n(T^*M) \otimes E,F))$$

This isomorphism is compatible with compositions: if $D : \Gamma(E) \to \Gamma(F)$ is an order m differential operator and $D' : \Gamma(F) \to \Gamma(K)$ is an order n differential operator, then $\sigma^{n+m}_{D \circ D'} = \sigma^n_{D'} \circ \sigma^m_D$.

4.0.2 Examples of Symbols

Example 4.11. Let $d : \Omega^k_M \to \Omega^{k+1}_M$ be the exterior derivative. Since $[d, f] = df \wedge (-)$, this is a first order differential operator. The symbol $\sigma^1_d : T^*M \to \hom(\Lambda^k_M, \Lambda^{k+1}_M)$ is defined by $\sigma^1_d(\xi)(\alpha) = \xi \wedge \alpha$. Thus the symbol of d is $\wedge$.

Example 4.12. Let ∇ be a covariant derivative on E and consider the coupled exterior derivative $d_\nabla : \Omega^k_M(E) \to \Omega^{k+1}_M(E)$. Once again, $[d_\nabla, f] = df \wedge (-)$. The symbol $\sigma^1_{d_\nabla}$ is again the wedge.

Example 4.13 (Formal Adjoints)**.** Assume that M be compact and let E, F have Euclidean metrics. Let $D \in D_1(E, F)$ and $D^* \in D_1(F, E)$. Then D^* is formally adjoint to D if $\langle s', Ds\rangle_{L^2(F)} = \langle D^*s', s\rangle_{L^2(E)}$. An easy thing to check is $[D^*, f] = -[D, f]^*$. It then follows that $\sigma^1_{D^*}(\xi) = -\sigma^1_D(\xi)^*$.

Example 4.14. If (M, g) is oriented and Riemannian, then we have an operator $d^* = \pm * d*$. Then $\sigma^1_{d^*}(\xi) = -\iota_\xi$, where ι denotes contraction. A similar result hold for $(d_\nabla)^* = \pm * d_\nabla *$.

Example 4.15. The Hodge laplacian $\Delta = dd^* + d^*d = (d + d^*)^2$ is a map $\Omega^k_M \to \Omega^k_M$. We can compute the symbol by the previous example:

$$\sigma_\Delta(\xi, \xi) = \sigma_d(\xi) \circ \sigma_{d^*}(\xi) + \sigma_{d^*}(\xi) \circ \sigma_d(\xi)$$
$$\alpha \mapsto -\xi \wedge \iota_\xi \alpha - \iota_\xi(\xi \wedge \alpha) = -|\xi|^2 \alpha$$

So the symbol $\sigma_\Delta(\xi, \xi)$ is $-|\xi|^2 \,\mathrm{id}$.

4.0.3 Elliptic Operators

Definition 4.16. An *elliptic operator* is a linear differential operator $D \in D_n(E, F)$ such that for all $x \in M$ and for all nonzero $\xi \in T^*_x M$ the map $\sigma^n_D(\xi, ..., \xi) \in \hom(E_x, F_x)$ is an isomorphism.

More explicitly, this map is:

$$\sigma^n_D(\xi, ..., \xi) = \frac{1}{n!}[\cdots [[D, f], f] \cdots , f]$$

where f is such that $\xi = df(x)$.

Definition 4.17. A *generalized laplacian* is a second order operator $\Delta \in D_2(E, E)$ such that $\sigma^2_\Delta(\xi, \xi) = -|\xi|^2 \,\mathrm{id}_E$.

Definition 4.18. A *Dirac operator* is a first order operator $D \in D_1(E, E)$ with the property that D^2 is a generalized Laplacian. Namely, $\sigma_{D^2}(\xi, \xi) = \sigma_D(\xi)^2 = -|\xi|^2 \,\mathrm{id}_E$.

Example 4.19. Let (X^4, g) be an oriented Riemannian manifold. Recall the operator $d^* \oplus d^+ : \Omega^1_X \to \Omega^0_X \oplus \Omega^+_g$. The symbol is $\sigma^1(\xi)(a) = -\iota_\xi a + (\xi \wedge a)^+$. One can show that this is an isomorphism.

Example 4.20. The map $d^* \oplus d : \Omega^\bullet_M \to \Omega^\bullet_M$ squares to the Hodge laplacian, so $d^* \oplus d$ is a Dirac operator.

Remark 4.21. If D is a Dirac operator, then $\sigma^2_D(\xi)^2 = -|\xi|^2 \,\mathrm{id}$. Abbreviating $\sigma \equiv \sigma^1_D$, the universal property of the Clifford algebras says that this map extends the Clifford map $\sigma : \mathrm{Cl}(T^*_x X) \to \mathrm{End}(E_x)$. We will expand on the notion of Clifford algebras in the next section.

4.1 Clifford Algebras, Spinors, and Spin Groups ❖

The suggested reference for this section is [4]. We begin by recalling a few things about Clifford algebras. Let K be a commutative ring with $2^{-1} \in K$ (usually, $K = \mathbb{R}, \mathbb{C}$) and (V, q) a K module with quadratic form q. This defines a bilinear map:

$$\langle u, v\rangle = \frac{1}{2}(q(u + v) - q(u) - q(v))$$

so that $\langle v, v\rangle = q(v)$.

Definition 4.22. The *Clifford algebra* $\mathrm{Cl}(V, q)$ is the associative unital K algebra generated by V subject to the relations $v \cdot v = -q(v) \cdot 1$ for all $v \in V$.

The Clifford algebra has the following universal property. If A is a unital associative algebra with a K linear map $f : V \to A$ such that $f(v)^2 = -q(v)1_A$, then there is a unique map $\tilde{f} : \mathrm{Cl}(V) \to A$ such that:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, hook] & A \\ \mathrm{Cl}(V) \arrow[ur, dashed, "\tilde{f}"] & \end{tikzcd}$$

Notice also that:

$$v^2 = -q(v)1 \iff uv + vu = -2\langle u, v\rangle 1 \quad \forall u, v \in V$$

A few obeservations about Clifford algebras:

- The formalism of Clifford algebras is compatible with extensions of scalars $k \to K$, i.e. if (V, q) is a quatratic k module, then $\mathrm{Cl}(V \otimes_k K, q) \cong \mathrm{Cl}(V, q) \otimes_k K$.
- With K fixed, it is also functorial in (V, q). In particular, the action $O(v, q) \to \mathrm{Aut}(V)$ extends to $O(V, q) \to \mathrm{Aut}(\mathrm{Cl}(V, q))$ via monomials: $G(v_1 \cdots v_\ell) := G(v_1) \cdots G(v_\ell)$ for all $G \in O(V, q)$.
- Given a monomial $v_1 \cdots v_\ell \in \mathrm{Cl}(V, q)$, the length ℓ is well-defined mod 2. As a consequence, $\mathrm{Cl}(V, q) = \mathrm{Cl}^0(V, q) \oplus \mathrm{Cl}^1(V, q)$, where the first summand is the subalgebra generated by even length elements and the second is the algebra generated by odd length elements. This shows that $\mathrm{Cl}(V, q)$ is a $\mathbb{Z}/2$ graded algebra (aka a superalgebra). The two summands are the ± 1 eigenspaces of the action of $-\mathrm{id}_V \in O(V, q)$ acting on $\mathrm{Cl}(V, q)$.
- If $\mathrm{Cl}(V, q)^{op}$ denotes the opposite algebra (the same as a K module, but the product is reversed), there is a unique algebra isomorphism $\beta : \mathrm{Cl}(V, q) \to \mathrm{Cl}(V, q)^{op}$ extending id_V. The map β is called the principal anti-automorphism of $\mathrm{Cl}(V, q)$. It is given by $\beta(v_1 \cdots v_\ell) = v_\ell \cdots v_1$.
- Let $\mathrm{Cl}(V, q)^{op}_{super}$ denote the opposite super algebra. This is the same algebra but the product is given by $a \cdot^{op}_{super} b = (-1)^{|a||b|} b \cdot a$. Then $\mathrm{Cl}(V, q)^{op}_{super} = \mathrm{Cl}(V, -q)$.
- The length of elements defines an increasing filtration $F^\ell \mathrm{Cl}(V, q) = \{\text{span of monomials of length} \le \ell\}$. Then there is an associated graded algebra $\mathrm{gr}^\bullet \mathrm{Cl}(V, q) := \bigoplus_\ell F^\ell / F^{\ell-1}$ and a map $i : \Lambda^\bullet V \to \mathrm{gr}^\bullet \mathrm{Cl}(V, q)$ sending $v_1 \wedge .... \wedge v_k \to [v_1 \cdots v_k]$. This map is an isomorphism of graded algebras (in the case where V is a free module). In particular, if V is free of rank r, then $\mathrm{Cl}(V, q)$ is free of rank 2^r, so this makes sense.
- There is a cononical superalgebra isomorphism:

  $$\mathrm{Cl}(V_1 \oplus V_2, q_1 \oplus q_2) \cong \mathrm{Cl}(V_1, q_1) \otimes^s \mathrm{Cl}(V_2, q_2)$$

  where $\otimes^s$ is the tensor product of super algebras, which gives the same K module but with product:

  $$(a \otimes b) \cdot_s (a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb'$$

4.1.1 Spinors

As a point of notation, suppose we have a super K-module (i.e. a $\mathbb{Z}/2$ graded K module) $U = U_0 \oplus U_1$. We write $\mathrm{sEnd}(U)$ to be the superalgebra $\mathrm{End}(U)$, where $\mathrm{sEnd}(U) = \mathrm{sEnd}^0(U) \oplus \mathrm{sEnd}^1(U)$ with the former summand being the parity preserving endomorphisms and the latter being the parity reversing endomorphisms.

Definition 4.23. Suppose k is a field with extension K and suppose (V, q) is a quadratic k module that is even dimensional and nondegenerate. The *spinor module* (defined over K) is a $\mathrm{Cl}(V, q)$ supermodule $S = S^+ \oplus S^-$ and a map $\rho : \mathrm{Cl}(V, q) \to \mathrm{sEnd}_K(S)$ whose K-linear extension $\rho_K : \mathrm{Cl}(V, q) \otimes K \to \mathrm{sEnd}_K(S)$ is an isomorphism of superalgebras.

Definition 4.24. Suppose (U, Q) is a quadratic K-module with Q nondegenerate of even dimension. A *polarization* for (U, Q) is a pair of subspaces $P = (L, L')$ with $U = L \oplus L'$ and such that $Q|_L = 0$ an $Q|_{L'} = 0$.

Given a polarization, one has $L' \cong L^*$ via Q and $(U, Q) \cong (L \oplus L^*, \mathrm{ev})$ where ev is evaluation. Over $\mathbb{C}$, a polarization always exists; over $\mathbb{R}$, a polarization exists if and only if the signature is zero. We claim a polarization P gives us a spinor module S_p in the following way. Define:

$$S = S_p = \Lambda^\bullet L^* = \Lambda^{even} L^* \oplus \Lambda^{odd} L^*$$

which is evidently a superalgebra. If $\mu \in L^*$, define a "creation operator" $c(\mu) = \mu \wedge (-) \in \mathrm{sEnd}^1(S)$. For $\lambda \in L$, define an "anhilation operator" $a(\lambda) = -\iota_\lambda \in \mathrm{sEnd}^1(S)$ given by contraction. These satisfy anticommutator relations[5] :

$$\{c(\mu), c(\mu')\} = 0, \quad \{a(\lambda), a(\lambda')\} = 0, \quad \{c(\mu), a(\lambda)\} = \mu(\lambda)$$

where $\{\cdot, \cdot\}$ is the anticommutator. Therefore $\mathrm{Cl}(V, q)$ acts on S by:

$$\rho \, \mathrm{Cl}(L \oplus L^*, \mathrm{ev}) \to \mathrm{sEnd}(S)$$

$$\rho(\lambda, \mu) \mapsto c(\mu) - a(\lambda)$$

Proposition 4.25. (S, ρ) *defined above is a spinor module.*

Proof:

Write $L = L_1 \oplus ... \oplus L_d$, where L_i are one dimensional. Then $L^* = L_1^* \oplus ... \oplus L_d^*$ and:

$$(L \oplus L^*, \mathrm{ev}) = \oplus_i (L_i \oplus L_i^*, ev)$$

Then:

$$S = \bigotimes_i \underbrace{\Lambda^\bullet L_i^*}_{S_i}$$

and therefore:

$$\mathrm{Cl}(L \oplus L^*, \mathrm{ev}) = \widetilde{\bigotimes}_i \mathrm{Cl}(L_i \oplus L_i^*, \mathrm{ev})$$

and check that ρ is the tensor product of $\rho_i : \mathrm{Cl}(L_i \oplus L_i^*) \to \mathrm{sEnd}(S_i)$. To show ρ is an isomorphism, it suffices to show that each ρ_i is. This is an easy check.

□

Example 4.26. $(\mathbb{R}^{2d}, |\cdot|^2)$ has a spinor module over $\mathbb{C}$.

Corollary 4.27. *When* (V, q) *is polarized over a field* k*, then:*

1. *Any finite dimensional indecomposable* $\mathrm{Cl}(V, q)$ *module (ungraded) is isomorhpic to* S*.*
2. *Any finite dimensional indecomposable* $\mathrm{Cl}(V, q)$ *supermodule is isomorphic to* S *or* ΠS*, the parity reversed spinors.*

4.1.2 Projective Actions

There is a projective action of $O(V)$ on S, where S is a spinor module, denoted $\Theta : O(V) \to \mathrm{PGL}_K(S) = \mathrm{Aut}_K(S)/K^\times$. To define this, notice that there is an induced action $O(V) \to \mathrm{Aut}\,\mathrm{Cl}(V, q) \otimes K$ given by $g \mapsto \mathrm{Cl}(g)$. A fact about matrix algebras is that all of their automorphisms are inner. This means there exits $F(g) \in \mathrm{Cl}(V, q)_k^\times$ such that $\mathrm{Cl}(g)(a) = F(g) a F(g)^{-1}$. This is well-defined mod scalars, so we get $F : \mathrm{Cl}(V) \to \mathrm{Cl}(V)^\times / K^\times$ such that $\mathrm{Cl}(g) = \mathrm{ad} F(g)$. Then we set $\Theta = \rho \circ F$, which is a homomorphism.

There is also a projective action on the orthogonal Lie alebra given by taking the derivative of $\mathrm{Cl} : O(V) \to \mathrm{Aut}\,\mathrm{Cl}(V, q)$:

$$\delta = D(\mathrm{Cl}) : \mathfrak{o}(V) \to \mathrm{Der}\,\mathrm{Cl}(V, q)$$

[5]These are the Heisenberg relations from Quantum Mechanics.

Just as $\mathrm{Cl}(g) \in \mathrm{Aut}\,\mathrm{Cl}(V)$ is inner for $g \in O(V)$, so $\delta(\xi)$ is also inner. This means that $\delta(\xi) = [f(\xi), \cdot]$ where $f(\xi) \in \mathrm{Cl}(V, q)$. This $f(\xi)$ is defined mod K and it can be normalized such that $\xi(v) = [f(\xi), v]$. There is a formula for $f(\xi)$ given a basis $(e_1, ..., e_{2n})$ for V. It is:

$$f(\xi) = \frac{1}{4} \sum_{i,j} \xi_{ij} e_i \cdot e_j \in \mathrm{Cl}^0(V)$$

Please consult Tim's online notes for a proof.

4.1.3 Spin Groups

Let (V, q) be nondegenerate over a field k. The multiplicative group $\mathrm{Cl}(V, q)^\times$ acts of $\mathrm{Cl}(V, q)$ by $u \cdot v = uvu^{-1}$. The *Clifford group* G is the normalizer of V in $\mathrm{Cl}(V, q)$. In other words:

$$G = \{g \in \mathrm{Cl}(V, q)^\times \mid gvg^{-1} \in V \; \forall v \in V\}$$

Certainly G acts on $\mathrm{Aut}\, V$, and actually this action is by isometries, defining a homomorphism $\alpha : G \to O(V)$. If $u \in V$ and $q(u) \neq 0$, then $\alpha(u)$ is $-$(reflection in $u^\perp\}$. Moreover, since $O(V)$ is generated by reflections, the homomorphism $\alpha : G \to O(V)$ is surjective. Now we claim there is an exact sequence:

$$1 \longrightarrow K^\times \longrightarrow G \xrightarrow{\alpha} O(V) \longrightarrow 1$$

Let $G^+ = G \cap \mathrm{Cl}^0(V, q)$. Then one can check that we have another exact sequence:

$$1 \longrightarrow K^\times \longrightarrow G^+ \xrightarrow{\alpha} SO(V) \longrightarrow 1$$

This is nearly the Spin group. Given a monomial $g = v_1 \cdots v_r \in G$, note that $\beta(g)g \in K^\times$, where β is the principal anti-automorphism. Then define the spinor norm $\nu : G \to K^\times$ by $g \mapsto \beta(g)g$. Now we can define the Spin group:

Definition 4.28. $\mathrm{Spin}(V, q) = G^+ \cap \ker \nu$.

We also have a central extension:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V) \xrightarrow{\alpha} SO(V) \longrightarrow 1$$

which presents $\mathrm{Spin}(V)$ as a double ocver of $SO(V)$. Recall the projective representation Θ of $SO(V)$ on spinors. Then this lifts to a linear action by the spin group, i.e. we have a diagram:

$$\begin{tikzcd} \mathrm{Spin}(V) \arrow[r, "\rho"] \arrow[d, "\alpha"] & \mathrm{Aut}_K(S) \arrow[d] \\ SO(V) \arrow[r, "\Theta"] & \mathrm{Aut}_K(S)/K^\times \end{tikzcd}$$

where ρ is the restriction of $\rho : \mathrm{Cl}(V, q) \to \mathrm{Aut}_K(S)$, the spinor action we constructed earlier.

4.1.4 Spin Structures

Let $(V, |\cdot|^2)$ be a positive definite inner product space over $\mathbb{R}$ of dimension n. Then the group $\mathrm{Spin}(V) \subset \mathrm{Cl}^0(V)^\times$ consists of elements of the form $e_1 e_2 \cdots e_{2r}$, where the e_i are unit vectors. We can view $\mathrm{Cl}^0(V)^\times$ as a Lie group and $\mathrm{Spin}(V)$ as a closed Lie subgroup. The Lie algebra $\mathfrak{spin}(V) \subset (\mathrm{Cl}^0(V), [\cdot, \cdot])$, where we are viewing Cl^0 as a Lie algebra. Specifically, $\mathfrak{spin}(V) = \{[x, y] \mid x, y \in V\}$. Recall the map $f : O(V) \to \mathfrak{spin}(V) \subset \mathrm{Cl}^0(V)$ characterized by $[f(\xi), \cdot]$ acting on $\mathrm{Cl}(V)$ describes the infinitecimal action of $\xi \in \mathfrak{o}(V)$ by derivations of $\mathrm{Cl}(V)$. $\mathrm{Spin}(V)$ acts on V by inner automorphisms of $\mathrm{Cl}(V)$ and defines a short exact sequence:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V) \xrightarrow{\alpha} SO(V) \longrightarrow 1$$

Therefore $\mathrm{Spin}(V)$ is a compact Lie group because $SO(V)$ is compact. Note also that $D\alpha = f^{-1}$.

Notice that $SO(V) = \exp(\mathfrak{o}(V))$. Then there is a subgroup $\exp(\mathfrak{spin}(V)) \subset \mathrm{Spin}(V)$, where the exponential is taken in the Clifford algebra. In fact, this is an equality. Since $\exp(-)$ is surjective, then $\mathrm{Spin}(V)$ is connected and $\alpha : \mathrm{Spin}(V) \to SO(V)$ admits no continuous splitting. This shows that $\mathrm{Spin}(\mathbb{R}^n, |\cdot|^2) \to SO(n)$ is the unqe $2:1$ Lie group homomorphism $G \to SO(n)$ with G connected. We denote $\mathrm{Spin}(n) := \mathrm{Spin}(\mathbb{R}^n, |\cdot|^2)$.

Recall for $n = 2m = \dim(V)$, we had the Clifford spinor representation $\rho : \mathrm{Cl}(V) \otimes \mathbb{C} \to \mathrm{sEnd}(S)$. This restricts to $\rho^{\pm} : \mathrm{Spin}(2m) \to \mathrm{Aut}_{\mathbb{C}}(S^{\pm})$. If n is odd, there is still a homomorphism $\rho : \mathrm{Cl}(V) \otimes \mathbb{C} \to \mathrm{End}(S)$, but with S not graded as before. To see what this is, we can write $V_{\mathbb{C}} = \mathbb{C} \oplus (L \oplus L^*)$ so that $\mathrm{Cl}(V) \cong \mathrm{Cl}(\mathbb{C}) \widetilde{\otimes}\, \mathrm{sEnd}(S')$, with $S' = \Lambda^{\bullet} L^*$ and $D = \mathrm{Cl}(\mathbb{C}) = \mathbb{C}[\epsilon]/(\epsilon^2 + 1)$. Then this means that $\mathrm{Cl}(V) \cong \mathrm{sEnd}_D(D \widetilde{\otimes} S')$.

Definition 4.29. If (S, ρ) is a super representation of $\mathrm{Cl}(V)$, a hermitian inner product $(\cdot,\cdot)$ on S is called *spin* if $\rho(v) \in U(S)$, where $U(S)$ is the Lie algebra of skew adjoint endomorphisms. In other words:

$$(\rho(v)s_1, s_2) + (s_1, \rho(v)s_2) = 0$$

Notice that for $g = v_1 \cdots v_{2r} \in \mathrm{Cl}^0(V)$, we have:

$$(\rho(g)s_1, s_2) = -(\rho(v_2 \cdots v_{2r}, \rho(v_1)s_2)$$

continuing for all elements of g, we find $(\rho(g)s_1, s_2) = (s_1, \rho(\beta g)s_2)$. Therefore:

$$(\rho(g)s_1, \rho(g)s_2) = (s, \rho(g\beta(g))s_2)$$

This if $g \in \mathrm{Spin}(V)$, we have $g\beta(g) = 1$ and hence $\rho(g) \in U(S)$. Thus spin hermitian products exist.

Dimension 2

Consider the case $n = 2$. Then $\mathrm{Spin}(2) = SO(2)$ and the map $\alpha : SO(2) \to SO(2)$ is angle doubling. S^+, S^- are the representations of $SO(2) = U(1)$ on $\mathbb{C}$ where $e^{i\theta}$ acts as $e^{\pm i\theta}$. Notice that $S^- = (S^+)^*$. The Clifford map $\rho : \mathbb{R}^2 \to u(S)$. Since it must reverse the parity on $S = S^+ \oplus S^-$, it should take the form:

$$\rho(v) = \begin{pmatrix} 0 & \rho^-(v) \\ \rho^+(v) & 0 \end{pmatrix}$$

where $\rho^-(v) = -\rho^+(v)^\dagger$. Notice that $\rho^+(v)^\dagger \rho^+(v) = |v|^2\, \mathrm{id}$. Then one can show that $\rho^+ : \mathbb{C} \to \hom_{\mathbb{C}}(S^+, S^-) = (S^-)^{\otimes 2}$ is a $\mathbb{C}$-linear isometry.

Definition 4.30. We define a spin structure on a 2 dimensional oriented inner product space $(V, |\cdot|^2)$ (i.e. a hermitian line) to be a hermitian line L and a $\mathbb{C}$ linear isometry $\rho : V \to L^{\otimes 2}$. In a sense, a spin structure on V is a choice of square root on that line.

Given a spin structure, set $S^- = L, S^+ = L^*$. Construct a Clifford action of V on $S^+ \oplus S^-$ via the above. One can then check that $S = S^+ \oplus S^-$ is a spinor representation.

Remark 4.31. The definition of spin structure applies equally to a rank 2 oriented Euclidean vector bundle $V \to M$.

Dimension 3

Now consider $n = 3$. The symplectic group $\mathrm{Sp}(1)$ is the group of unit length quaternions. This acts on the quaternions $\mathbb{H}$ by multiplication. One can also check that $Sp(1) = SU(2)$. The spin covering of $\beta : Sp(1) \to SO(\mathrm{im}(\mathbb{H}))$ sends $q \mapsto (x \mapsto qxq^{-1})$. Here we are regarding $\mathrm{im}(\mathbb{H})$ as the three dimensional vector space $\mathrm{span}(i, j, k)$. Since $\ker \beta = \pm 1$, a dimension count argument shows that this is a 2-1 covering. Therefore $\mathrm{Spin}(3) = Sp(1) = SU(2)$. The spinors in this case end up being $S = \mathbb{C}^2$ and $\mathrm{Spin}(3)$ acts by the defining representation of $SU(2)$. The Clifford map $\rho : \mathbb{R}^3 \to U(S)$ is actually an isometry $\rho : \mathbb{R}^3 \to \mathfrak{su}(S)$.

Definition 4.32. Suppose V is a three dimensional inner product space over $\mathbb{R}$. A spin structure on V is a 2-dimensional vector space S, a unit length volume form $\Omega \in \Lambda^2 S^*$, and an isometry $\rho : V \to \mathfrak{su}(S)$ satisfying the condition that if (e_1, e_2, e_3) is an oriented orthonormal basis of V then $\rho(e_1)\rho(e_2)\rho(e_3) = I_S$.

Then $\mathrm{Spin}(V)$ can be seen as the symmetries of the spin structures.

Dimension 4

Now the case $n = 4$. We claim that $\mathrm{Spin}(4) = Sp(1) \times Sp(1)$. To see this, define $\gamma : Sp(1) \times Sp(1) \to SO(\mathbb{H})$ by $(q_1, q_2) \mapsto (x \mapsto q_1 x q_2^{-1})$, where $Sp(1)$ is seen as the set of unit length quaternions. If $(q_1, q_2) \in \ker\gamma$, then $q_1 x = x q_2$ for all $x \in \mathbb{H}$. Taking $x = 1$, get $q_1 = q_2$, so $\ker\gamma = \ker\beta = \pm(1, 1)$. Since $\dim(Sp(1) \times Sp(1)) = 6 = \dim SO(4)$, we get that γ is a $2 : 1$ covering map and hence $\mathrm{Spin}(4) = Sp(1) \times Sp(1)$.

The spinors will be $S^+ = \mathbb{H}$ and $S^- = \mathbb{H}$. The spin group $Sp(1) \times Sp(1)$ acts on S^+ via the first projection and acts on S^- via the second projection. Those actions preserve the $\mathbb{H}$ structure and hermitian metrics.

Definition 4.33. A spin structure on a four dimensional Euclidean vector space over $\mathbb{R}$ is a pair of rank 2 vector spaces S^+, S^- with $\mathbb{H}$ structures and an oriented isometry $\rho^+ : V \to \hom_{\mathbb{H}}(S^+, S^-)$.

Once again, $\mathrm{Spin}(V)$ can be seen as the group of symmetries of spin structures. It is the set of pairs $(g, \tilde{g})$ with $g \in SO(V)$ and $\tilde{g} \in \begin{bmatrix} SU(S^+) & 0 \\ 0 & SU(S^-) \end{bmatrix} \subset SU(S^+ \oplus S^-)$. such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\rho} & \mathrm{End}(S) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle Ad(\tilde{g})} \\ V & \longrightarrow & \mathrm{End}(S) \end{array}$$

where g is thought of as an element of $SO(V)$.

Remark 4.34. An $\mathbb{H}$ structure on a hermitian vector space E amounts to a $\mathbb{C}$-antilinear isometry $J : E \to E$ (this defines the action of $j \in \mathbb{H}$. Then $\Omega(u, v) = (u, Jv)$ defines an element $\Omega \in \Lambda^2 E^*$, and vice versa. Then with $(E, (\cdot, \cdot))$ given, the map J is equivalently given by a complex symplectic form Ω. For $n = 4$, spinor bundles $S^\pm$ come with unit volume forms $\Omega_\pm \in \Lambda^2(S^\pm)^*$.

4.1.5 Topology of spin and Spin^c structures

The more general definition of a spin structure is:

Definition 4.35. If $V \to M$ is a rank n real vector bundle, a *spin structure* on V is $\mathfrak{s} = (\mathrm{Spin}(V), \tau)$ is a principal $\mathrm{Spin}(n)$-bundle $\mathrm{Spin}(V) \to M$ and an isomorphism $\tau : \mathrm{Spin}(V) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \to V$.

A spin structure on a manifold is a spin structure on T^*M. We often think of the orientation and metric on the bundle as given in advance. In that case, the spin structure amounts to the principal bundle $\mathrm{Spin}(V)$ and an isomorphism $\mathrm{Spin}(V) \times_{\mathrm{Spin}(n)} SO(n) \cong SO(V)$.

Spinor bundles are $\mathbb{S} = \mathrm{Spin}(V) \times_{\mathrm{Spin}(n)} S$, where S is the standard spinor representation for $\mathrm{Spin}(n)$. If the dimension n is even, we have a decomposition $\mathbb{S} = \mathbb{S}^- \oplus \mathbb{S}^+$.

Proposition 4.36. *If a spin structure in V exists, then the isomorphism classes of spin structures on V form a torsor (a freely transitive action) for the group $H^1(M, \mathbb{Z}/2)$. Moreover, a spin structure exists only if the second Stiefel-Whitney class vanishes: $w_2(V) = 0$.*

Let $(V, |\cdot|^2)$ be a positive definite, real inner product space. Then define Spin$^c(V)$ to be the subgroup of $\mathrm{Cl}^0(V \otimes \mathbb{C})^\times$ generated by the scalars $U(1)$ and $\mathrm{Spin}(V)$. Note that $\mathrm{Spin}(V) \cap U(1) = \{\pm 1\}$. Then one can write:

$$\mathrm{Spin}^c(V) = \mathrm{Spin}(V) \times U(1)/\pm(1, 1)$$

For example, $\mathrm{Spin}^c(4) = SU(2) \times SU(2) \times U(1)/\pm(1, 1, 1)$. There is a short exact sequence:

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Spin}^c(V) \longrightarrow SO(V) \longrightarrow 1$$

Where the right map is projection onto $\mathrm{Spin}(V)/\pm 1 = SO(V)$. More generally, the definition is:

Definition 4.37. If $V \to M$ is a rank r vector bundle, a Spin^c *structure* is a pair $\mathfrak{s} = (\mathrm{Spin}^c V, \tau)$, where $\mathrm{Spin}^c V \to M$ is a principal $\mathrm{Spin}^c(r)$ bundle and τ is an isomorphism $\mathrm{Spin}^c V \times_{\mathrm{Spin}^c(r)} \mathbb{R}^r \to V$.

Just as before, we have the spinor bundle $\mathbb{S} = \mathrm{Spin}^c V \times_{\mathrm{Spin}^c(r)} S$ which decomposes into two components when r is even. There is also a Clifford map $\rho : V \to u(S)$ of odd parity when r is even.

For dimension 2, a Spin^c structure on V is a pair of hermitian line bundles L_+, L_- and a $\mathbb{C}$ linear isometry $V \to \hom_{\mathbb{C}}(L_+, L_-)$. In the spin case, we would have $L_+ = (L_-)^*$.

Proposition 4.38. *If $V \to M$ admits a Spin^c structure, then, the Spin^c structures form a torsor for $H^2(M, \mathbb{Z})$.*

4.2 Dirac Operators and the Lichnérowicz formula ❖

Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold. Recall that the Levi-Civita connection ∇ on TM is characterized by the properties:

$$d\langle u, v\rangle = \langle \nabla u, v\rangle + \langle u, \nabla v\rangle$$
$$\nabla_u v - \nabla_v u - [u, v] = 0$$

where u, v are vector fields. The associated curvature is $R = \nabla \cdot \nabla \in \Omega^2_M$, where:

$$R_{u,v} = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}$$

Given coordinates x_i, we have:

$$R_{ijkl} = \langle R_{\partial_i, \partial_j}(\partial_k), \partial_l\rangle$$

Under the S_4 of permutations on the indices, R_{ijkl} transforms acording to the sign of the permutation. Moreover there is the Bianchi identity:

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

4.2.1 Clifford Connections

Definition 4.39. Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold and let S be a complex vector bundle. Suppoze ρ is a Clifford map: $\rho : T^*M \to u(S)$, $\rho(e)^2 = -|e|^2 \,\mathrm{id}_S$. Then a *Clifford connection* is a unitary connection $\tilde{\nabla}$ on S such that "ρ is parallel:"

$$[\tilde{\nabla}_v, \rho(e)] = \rho(\nabla_v e)$$

where e is a local section of T^*M.

Proposition 4.40. *When S is the spinor bundle of a spin structure on T^*M, there is a distinguished Clifford connection ∇^{Spin}. The formula for this connection in local coordinates at x is:*

$$\nabla^{\mathrm{Spin}}_{\partial_i} = \partial_i + f(A_i) = \partial_i + \frac{1}{4}\sum_{\alpha,\beta} A_i^{\alpha\beta} \rho_\alpha \rho_\beta$$

where we write the Levi-Civita connection ∇_{∂_i} as $\partial_i + A_i$, with $A_i(x) \in \mathfrak{so}(n)$, and $\rho_\alpha = \rho(\partial/\partial x_\alpha)$.

Proposition 4.41. *When S is the spinor bundle of a Spin^c structure $\mathfrak{S}$ (e.g. of a spin structure), the Clifford connections form an affine space modeled on $\Omega^1_M(i\mathbb{R})$.*

Recall that any Spin^c structure $\mathfrak{S}$ has an associated line bundle $L_{\mathfrak{S}}$. It is a fact that a Clifford connection $\tilde{\nabla}$ determines (and is determined by) a unitary connection $\tilde{\nabla}^0$ on $L_{\mathfrak{S}}$. Moreover:

$$(\tilde{\nabla} + a)^0 = \tilde{\nabla}^0 + 2a \qquad (a \in \Omega^1_M(i\mathbb{R}))$$

Defining $F^0(\tilde{\nabla}) := F_{\tilde{\nabla}^0}$, we also have:

$$F^0(\tilde{\nabla} + a) = F^0(\tilde{\nabla}) + 2da$$

The curvature for a Clifford connection is then:

$$F_{\tilde{\nabla}} = f(R) + \frac{1}{2}F^0(\tilde{\nabla}) \otimes \mathrm{id}_S$$

4.2.2 Dirac Operators

Let $(M, \langle\cdot,\cdot\rangle)$ be an oriented Riemannian manifold, $\mathfrak{S}$ be a Spin^c structure with associated spinor bundle S, ∇ be the L-C connection, and $\tilde{\nabla}$ be a Clifford connection.

Definition 4.42. The *Dirac operator* for $\tilde{\nabla}$ is $D : \Gamma(S) \to \Gamma(S)$ given by the composition:

$$\Gamma(S) \xrightarrow{\tilde{\nabla}} \Gamma(T^*M \otimes S) \xrightarrow{\rho} \Gamma(S)$$

where $\rho(e \otimes \phi) = \rho(e)\phi$.

In the even dimensional case, where $S = S^+ \oplus S^-$, we have $D : \Gamma(S^\pm) \to \Gamma(S^\mp)$ (because $\rho(e)$ does this). Note also that D is formally self adjoint:

$$\int_M \langle D\phi, \chi\rangle \mathrm{vol} = \int_M \langle \phi, D\chi\rangle \mathrm{vol}$$

One can verify that the symbol of D is is given by $\sigma_D(e) = \rho(e)$, so indeed D is a generalized Dirac operator (in the sense that its symbol satisfies the Clifford relation).

4.2.3 Lichnérowicz Formula

In a Spin^c manifold M, then the Lichnérowicz formula is:

$$D^2 = \tilde{\nabla}^*\tilde{\nabla} + \frac{1}{4}\kappa \cdot \mathrm{id}_S + \frac{1}{2}\rho(F^0(\tilde{\nabla}))$$

where $\tilde{\nabla}^* : \Gamma(S) \to \Gamma(S)$ is the formal adjoint to $\tilde{\nabla}$, κ is the scalar curvature. We are thinking of ρ acting on Λ^2_M by:

$$\rho(e \wedge f) = \frac{1}{2}[\rho(e), \rho(f)]$$

See Tim's notes for a proof of this formula. In the case where $\tilde{\nabla} = \nabla^{\mathrm{Spin}}$, we get:

$$D^2 = (\nabla^{\mathrm{Spin}})^*\nabla^{\mathrm{Spin}} + \frac{1}{4}\kappa$$

because $F^0(\nabla^{\mathrm{Spin}}) = 0$.

5. The Seiberg-Witten Equations

This is the final section of the course, which uses the theory developed thus far to state the Seiberg-Witten equations. We will introduce the configuration space on which the SW equations operate and write several variants of the SW equations, which include a perturbed version for later use. What follows after are then important results that characterize solutions to the SW equations, like compactness, existence, and transversality.

To write the SW equations, we need to do a bit more work on Spin^c structures in 4 dimensions. The setup is a Riemannian manifold (X^4, g), with a Spin^c structure $\mathfrak{s} = (\mathbb{S}, \rho)$, where $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ (where $\mathbb{S}^\pm$ is a rank 2 hermitian bundle). The clifford map $\rho : T^*X \to u(\mathbb{S})$ exchanges $\mathbb{S}^+, \mathbb{S}^-$ and satisfies $\rho(e)^2 = -|e|^2 \operatorname{id}$. Furthermore, there is an orientation condition given as follows. Let $(e_1, ..., e_4)$ be an oriented basis for T_x^*X and write $-e_1e_2e_3e_4 \in \mathrm{Cl}^0(T_x^*X)$. This has the property that $\omega.v = -v.\omega$ for all $v \in T_x^*X$ so that ω is central in Cl^0 and $\omega^2 = 1$. The extension of ρ to Cl gives ω an action on $\mathbb{S}$, with ± 1 eigenspaces preserved by Cl^0 exchange by T_x^*X. These are necessarily $\mathbb{S}^\pm$. The condition is that $\omega = 1$ on $\mathbb{S}^+$ and $\omega = -1$ on $\mathbb{S}^-$. These are all of the conditions necessary for the Spin^c structure.

Lemma 5.1. $\Lambda^2\mathbb{S}^+ \cong \Lambda^2\mathbb{S}^-$ *canonically.*

Proof:

Recall $\mathrm{Spin}^c(4) = SU(2) \times SU(2) \times U(1)/\pm(1,1,1)$. This can be identified with the group $G = \{(A, B) \in U(2) \times U(2) \mid \det(A) = \det(B)\}$. The isomorphism is $(A, B, z) \mapsto (zA, zB)$. We have projections $p^\pm : G \to U(2)$, the first and second projections. Then $\mathbb{S}^\pm = \mathrm{Spin}^c(T^*X) \times_{G,p_\pm} \mathbb{C}^2$, and hence $\Lambda^\pm = \mathrm{Spin}^c(T^*X) \times_{G,\det \circ p_\pm} \mathbb{C}$. But $\det \circ p_\pm$ is the map $\lambda : [A, B, z] \mapsto z^2$, so $\Lambda^2\mathbb{S}^\pm$ are both identified with $L_{\mathfrak{s}} = \mathrm{Spin}^c(T^*X) \times_{\mathrm{Spin}^c(4),\lambda} \mathbb{C}$.

□

In light of this, define $\det \mathfrak{s} := \Lambda^2\mathbb{S}^+$. Extend ρ to act on 2 forms in the following way:

$$\rho : \Lambda^2 T^*X \to \mathrm{End}^0(S)$$

$$\rho(e \wedge f) = \frac{1}{2}[\rho(e), \rho(f)]$$

where $\mathrm{End}^0(S)$ is the space of endomorphisms that preserve the $\mathbb{Z}/2$ grading. This is the composition:

$$\Lambda^2 T^*X \to \mathfrak{so}(T^*X) \to \mathfrak{spin}(T^*X) \subset \mathrm{Cl}^0(T^*X) \to \mathrm{End}\,\mathbb{S}$$

$$e \wedge f \mapsto \bar{x} = (x \mapsto \langle x, f\rangle e - \langle x, e\rangle f) \mapsto f(\bar{x}) \mapsto \rho(f(\bar{x}))$$

where f was constructed before. The Hodge star $*$ acting on $\Lambda^2 T^*X$ is equivalent to the action of ω on $\mathbb{S}$. For example:

$$\omega \cdot e_1e_2 = e_3e_4$$

$$*(e_1 \wedge e_2) = e_3 \wedge e_4$$

The upshot is that:

$$\rho(\Lambda^+) = \begin{bmatrix} \mathfrak{s}u(\mathbb{S}^+) & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho(\Lambda^-) = \begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{s}u(\mathbb{S}^-) \end{bmatrix}$$

where $\Lambda^\pm$ are the self/anti-self dual forms. Therefore a self-dual 2-form corresponds exactly to trace-free endomorphism of $\mathbb{S}^+$.

5.1 The Configuration Space and SW Equations ❖

Let $\mathcal{A}_{\mathrm{Cl}}(\mathbb{S}^+)$ be the space of clifford connections ∇ on $\mathbb{S}^+$. Recall that there is an associated connection on $L_{\mathfrak{s}}$ that we called $L_{\mathfrak{s}}$. The Gauge group $G = G(\mathfrak{s})$ acts on fibers $\mathbb{S}_x$ commuting with ρ and is canonically identified with the Gaugue group of $L_{\mathfrak{s}}$, which is $\mathbb{C}^\infty(X, U(1))$. Thsi isomorphism is given by $u \mapsto \det(u|_{\mathbb{S}^+})$. We define the configuration space to be:

$$\mathcal{A}_{\mathrm{Cl}}(\mathbb{S}^+) \times \Gamma(\mathbb{S}^+)$$

which is acted upon by G via $u.(A, \phi) = (u^*A, u.\phi)$. The SW equations happen on this configuration space. The first one is:

1. **The Dirac Equation:** $\boxed{D_A^+\phi = 0,}$ where D_A is the Dirac operator $\rho\circ\nabla_A$ and D_A^+ is the part sending $\Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$. For fixed A, this is a linear elliptic operator on spinors (with symbol ρ). This equation leaves A unconstrained. There is also Gauge invariance:

$$D_{u^*A}^+ = \rho\nabla_{u^*A} = \rho(u\nabla_A u^{-1}) = u \circ \rho \circ \nabla_A \circ u^{-1}$$
$$\Rightarrow D_{u^*A}^+(u.\phi) = u\rho\nabla_A\phi u D_A^+\phi$$

   i.e. $D_A^+\phi = 0 \iff D_{u^*A}^+ = 0$.

2. **The Curvature Equation:** $\boxed{\frac{1}{2}\rho(F(A^0)^+) - (\phi\phi^*)_0 = 0.}$ This equation constrains the gaugue orbit of A. Here, A^0 is a connection in $\det\mathfrak{s}$ and so its curvature $F(A^0)^+$ is an imaginary valued self-dual 2 form. Then $\rho(F(A^0)^+) \in i\mathfrak{su}(\mathbb{S}^+)$, which is the space of trace-free self-adjoint endomorphisms of $\mathbb{S}^+$. Additionally, $\phi \in \Gamma(\mathbb{S}^+)$ and $\phi\phi^* \in \operatorname{End}\mathbb{S}^+$ acts as $(\phi\phi^*)(\chi) = (\chi, \phi)\phi$. The zero subscript on $\phi\phi^*$ is the trace-free part of the endomorphsm, which is:

$$(\phi\phi^*)_0 = \phi\phi^* - \frac{1}{2}|\phi|^2\,\mathrm{id}_{\mathbb{S}^+} \in i\mathfrak{su}(\mathbb{S}^+)$$

   The solutions to this equation are also Gauge invariant. This follows from $\psi = u\phi \Rightarrow \psi\psi^* = |u|^2\phi\phi^* = \phi\phi^*$ and the curvature is unchanged by Gaugue transformations.

The left hand side of these equations define a map $\mathcal{F} : \mathcal{A}_{\mathrm{Cl}}(\mathbb{S}^+) \times \Gamma(\mathbb{S}^+) \to \Gamma(i\mathfrak{su}(\mathbb{S}^+)) \times \Gamma(\mathbb{S}^-)$. The SW equations are equivalently $\mathcal{F} = 0$. We define the perturbed curvature equation by:

$$\frac{1}{2}\rho(F(A^0)^+ - 4i\eta) - (\phi\phi^*)_0 = 0$$

where $\eta \in \Omega_X^+$. The Dirac and perturbed equations are $\mathcal{F}_\eta = 0$ in a similar way.

5.2 Linearization ❖

We would like to linearize $\mathcal{F}_\eta$. One can check that:

$$\mathcal{F}_\eta(A + a\,\mathrm{id}_{\mathbb{S}^+}, \phi + \chi) - \mathcal{F}(A, \phi) = \begin{bmatrix} \rho(d^+a) - (\phi\chi^* + \chi\phi^* + \chi\chi^*)_0 \\ D_A^+\chi + \frac{1}{2}\rho(a)(\phi + \chi) \end{bmatrix}$$

So the derivative $D_{(A,\phi)}\mathcal{F}_\eta$ is:

$$D_{(A,\phi)}\mathcal{F}_\eta \begin{bmatrix} a \\ \chi \end{bmatrix} = \begin{bmatrix} \rho\circ d^+ & 0 \\ 0 & D^+A \end{bmatrix} \begin{bmatrix} a \\ \chi \end{bmatrix} + \begin{bmatrix} -(\phi\chi^* + \chi\phi^*)_0 \\ \frac{1}{2}\rho(a)\phi \end{bmatrix}$$

This is not an elliptic operator. The Gauge "fix" is comes by imposing the Coulomb gauge equation: fix $A_0 \in \mathcal{A}_{\mathrm{Cl}}$ reference connection, and write $A = A_0 + a\,\mathrm{id}_{\mathbb{S}^+}$. Then impose $d^*a = 0$. Then define:

$$\mathcal{F}'_\eta(A, \phi) := \begin{bmatrix} \frac{1}{2}(F(A^0)^+ - 4i\eta) - \rho^{-1}(\phi\phi^*)_0 \\ d^*a \\ D_A^+\phi \end{bmatrix}$$

Then the derivative of this is:

$$D_{(A,\phi)}\mathcal{F}'_\eta \begin{bmatrix} a \\ \chi \end{bmatrix} = \begin{bmatrix} d^+ & 0 \\ d^* & 0 \\ 0 & D_A \end{bmatrix} \begin{bmatrix} a \\ \chi \end{bmatrix} + \text{0th order terms}$$

At the level of symbols, $D\mathcal{F}'_\eta$ is the direct sum of $d^+ \oplus d^*$ and D_A^+, which are both elliptic. Therefore $D\mathcal{F}'_\eta$ is elliptic.

5.3 Warmup on Indices of Operators ❖

If $E, F \to M$ are vector bundles with Euclidean metrics and M^n is a closed manifold, and if $\delta : \gamma(F) \to \Gamma(E)$ is elliptic, then the formal adjoint $\delta^* : \Gamma(F) \to \Gamma(E)$ is also elliptic. We will work with the C^k topology on $\Gamma(E), \Gamma(F)$ for some $k \gg 0$. A few observations:

- $\ker(\delta^*) \cong \operatorname{coker}(\delta)$.
- δ is Fredholm, i.e. its image is closed and $\ker\delta, \operatorname{coker}\delta$ are finite dimensional.
- Its index $\operatorname{ind}\delta := \dim\ker\delta - \dim\operatorname{coker}\delta$ depends only on the symbol σ_δ.
- The kernels $\ker\delta, \ker\delta^*$ comprise C^∞ sections.

Atiyah-Singer give us a formula for $\operatorname{ind}\delta$. In the case of generalized Dirac operators, assume $E = E^+ \oplus E^-$ a $\mathbb{Z}/2$ graded Clifford module over (T^*M, g) (and is a complex vector bundle). The Dirac operator $D^\pm : \Gamma(E^\pm) \to \Gamma(E^\mp)$ is a first order differential operator, that is formally self-adjoint, whose symbol gives a Clifford map. Define the bundle $W = \operatorname{End}_{\mathrm{Cl}(T^*M)} E \to M$. The formula is:

$$\operatorname{ind}_{\mathbb{C}}(D^+) = \int_M \hat{A}(T^*M)\cdot \operatorname{ch}(W) \qquad \text{(Atiyah-Singer Index Theorem)}$$

where $\hat{A}$ is a series in the Pontryagin classes:

$$\hat{A} = 1 - \frac{1}{24}p_1 + \ldots \in H^{4*}(M, \mathbb{Q})$$

and ch is the Chern character:

$$\operatorname{ch} = 1\cdot \text{rank} + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \ldots \in H^{2*}(M, \mathbb{Q})$$

In the case of a spin Dirac operator in a spin 4 manifold X, we have W a trivial $\mathbb{C}$ line bundle. Then:

$$\operatorname{ind}_{\mathbb{C}} D^{spin} = \int_M \hat{A}(T^*M) = -\frac{1}{24}p_1(T^*M)[X] = -\frac{1}{8}\tau(M)$$

If we twist the spin structure $\mathfrak{s}$ to a Spin^c structure $L\otimes\mathfrak{s}$, where L is a $\mathbb{C}$ line bundle, we get $W = L$. In this case, we get:

$$\operatorname{ind}_{\mathbb{C}} D_A^+ = \int_X (1 + c_1 L + \frac{1}{2}c_1(L)^2)(1 - p_1/24) = \frac{1}{8}(c_1(\mathfrak{s})[X] - \tau(X))$$

This remains valid for any closed Spin^c 4 manifold not necessarily spin.

5.3.1 Index of the linearized SW operator

The index of the linear operator $D\mathcal{F}'_\eta$, by the decomposition we described, is evidently $\operatorname{ind}_\mathbb{R} D_A^+ + \operatorname{ind}_\mathbb{R}(d^* \oplus d^+)$. For the first of these, we can use the Atiyah Singer formula given above. For the second, we note that $\ker(d^* \oplus d^+), \operatorname{coker}(d^* \oplus d^+)$ are isomorphic to $H^{odd}(\mathcal{E}^\bullet), H^{even}(\mathcal{E}^\bullet)$, where $\mathcal{E}^\bullet$ is the signature complex we from Definition 3.11. Therefore by the Hodge theorem and Theorem 3.12, we have $\operatorname{ind}(d^* \oplus d^*) = b_1 - (1 + b_2^+)$. Therefore:

$$\begin{aligned}
\operatorname{ind}_\mathbb{R} D(\mathcal{F}'_\eta) &= 2 \operatorname{ind}_\mathbb{C} D_A^+ + \operatorname{ind}(d^* \oplus d^+) \\
&= \frac{1}{4}(c_1(\mathfrak{s})^2[x] - \tau(x)) + b_1 - 1 - b_2^+ \\
&= \frac{1}{4}(c_1(\mathfrak{s})^2[X] - 2\chi(X) - 3\tau(X)) \\
&= d(\mathfrak{s})
\end{aligned}$$

Remark 5.2. It can be shown that $d(\mathfrak{s})$ is the Euler number $e(\mathbb{S}^+)[X]$.

5.3.2 Bounds for solutions to the SW equations

We are working towards compactness of solutions to the SW equations. This is the property that if (A_i, ϕ_i) is a sequence of solutions, then there exists a sequence of gauge transformations $u_i \in \mathcal{G}$ such that $(u_i^* A_i, u_i.\phi_i)$ converges smoothly to a solution (A_∞, ϕ_∞).

To move toward proving this, we state an inequality for Laplacians. Let (M, g) be a Riemannian manifold, $E \to M$ be a euclidean vector bundle, and ∇ be an orthogonal covariant derivative on E. The connection has a formal adjoint $\nabla^* : \Gamma(T^*M \otimes E) \to \Gamma(E)$. A fact we haven't proved is that, for $s \in \Gamma(E)$, we have:

$$\frac{1}{2} d^* d(|s|^2) = \langle \nabla^* \nabla s, s \rangle - |\nabla s|^2$$

For a proof, see Tim's notes. As a consequence, $\frac{1}{2} d^* d|s|^2 \leq \langle \nabla^* \nabla s, s \rangle$.

Lemma 5.3. *If p is a local maximum for $f \in \mathbb{C}^\infty(M)$, then $(d^* df)(p) \geq 0$.*

Proof idea:

> With the standard metric on $\mathbb{R}^n$ with $p = 0$, we have $d^*(a_i dx_i) = -\frac{\partial a_i}{\partial x_i}$. For this metric, $d^* df = -\sum \partial_i^2 f$. So the Lemma holds by the second derivative test. In general, you can choose coordinates such that $g = g_{std} + \sum h_{ij} x_i x_j$. Then $d^* = \pm * d*$ defined in this metric still satisfies $(d^* df)(0) = -\left(\sum_i \partial_i^2 f\right)(0)$.

□

Lemma 5.4. *For $\phi, \chi \in \Gamma(\mathbb{S}^+)$, we have:*

$$((\phi\phi^*)_0 \chi, \chi) = |\chi|^2 (\chi, \phi) = -\frac{1}{2} |\chi|^2 |\phi|^2$$

In particular, $((\phi\phi^*)_0 \phi, \phi) = \frac{1}{2}|\phi|^4$. Moreover, for $\eta \in \Omega_X^2$, $\phi \in \Gamma(\mathbb{S})$, we have:

$$(\rho(\eta)\phi, \phi) \leq |\eta||\phi|^2$$

We will not prove this. Returning to the (perturbed) SW equations $\mathcal{F}_\eta = 0$, for $\eta \in \Omega_X^+$. Suppose $\mathcal{F}_\eta(A, \phi) = 0$ is a solution. Then our first basic estimate is:

$$d^* d(|\phi|^2) + \frac{1}{2}(\mathrm{scal}_g - 8|\eta|)|\phi|^2 + |\phi|^4 \leq 0 \tag{5.3.1}$$

To see this, we start with:

$$\begin{aligned}\frac{1}{2}d^*d(|\phi|^2) &\le (\nabla_A^*\nabla_A\phi, \phi)\\ &= (D_A^-D_A^+\phi - \frac{1}{4}\mathrm{scal}_g\phi - \frac{1}{2}\rho(F(A^0))\phi, \phi)\end{aligned}$$

The second line comes from the Liechnérowicz Forumla. Now we impose the SW equations: $D_A^+\phi = 0$ and $\frac{1}{2}\rho(F(A^0))\phi = 2i\rho(\eta)\phi - (\phi\phi^*)_0\phi$. Therefore:

$$\frac{1}{2}d^*d(|\phi|^2) \le -\frac{1}{4}\mathrm{scal}_g|\phi|^2 + 2|\eta||\phi|^2 - \frac{1}{2}|\phi|^4$$

which is what we wished to show.

Theorem 5.5 (Pointwise bound on $|\phi|$)**.** *Define the continuous function $s = \max(8|\eta| - scal_g, 0) \ge 0$. If $\mathcal{F}_\eta(A,\phi) = 0$, then $\max|\phi|^2 \le \frac{1}{2}\max s$, where these are maxima taken over points in the compact manifold X.*

Proof:

The basic pointwise estimate 5.3.1 gives us $d^*d(|\phi|^2) + |\phi|^4 \le \frac{s}{2}|\phi|^2$. Take a point x where $|\phi|^2$ achieves its maximum. Then $d^*d(|\phi|^2)(x) \ge 0$ by Lemma 5.3, so $|\phi|^4 \le \frac{s}{2}|\phi|^2$ at x. If $\phi \equiv 0$, result is trivial. Otherwise, we get $|\phi|^2(x) \le \frac{1}{2}s(x) \le \frac{1}{2}\max s$.

□

Remark 5.6. The sign of $(\phi\phi^*)_0$ in the SW equations is critical to this result. Additionally, if $\eta = 0$ and $\mathrm{scal}_g \ge 0$, then the only solutions to $\mathcal{F}_\eta = 0$ are those with $\phi \equiv 0$, which means $F(A^0)^+ = 0$.

Proposition 5.7. *If $\mathcal{F}_\eta(A,\phi) = 0$, then $|F(A^0)^+ - 4i\eta| \le \frac{1}{4}\max s$*

Proof:

By the curvature equation, we have $\rho(F(A^0)^+ - 4i\eta) = (\phi\phi^*)_0$. Therefore:

$$\begin{aligned}|F(A^0) - 4i\eta| &\le |\rho(F(A^0)^+ - 4i\eta)|_{op}\\ &= |(\phi\phi^*)_0| = \frac{1}{2}|\phi|^2 \le \frac{1}{4}\max s\end{aligned}$$

□

Proposition 5.8. *For any d_0, among those Spin^c structures whose index $d(\mathfrak{s}) \ge d_0$, only finitely many isomorphism classes admit solutions to $\mathcal{F} = 0$.*

Proof:

Set $F = iF(A^0) \in \Omega_X^2$. Then by Chern-Weil theory, we have $\frac{1}{2\pi}[F] = c_1(\mathfrak{s})$. Then:

$$\begin{aligned}c_1(\mathfrak{s})[X] &= \frac{1}{4\pi^2}\int_X F \wedge F\\ &= \frac{1}{4\pi^2}\int_X (F^+ + F^-) \wedge (F^+ + F^-)\\ &= \frac{1}{4\pi^2}\int_X (F^+)^2 + (F^-)^2\\ &= \frac{1}{4\pi^2}\int_X (|F^+|^2 - |F^-|^2) vol_X\end{aligned}$$

Therefore:

$$\frac{1}{4\pi^2}\int_X |F|^2 vol_X = \frac{1}{4\pi^2}\int_X (|F^+|^2 + |F^-|^2) vol_X = -c_1(\mathfrak{s})^2 + \frac{1}{2\pi^2}\int_X |F^+|^2 vol_X$$

Set $S = \max s$. Then $|F^+| \leq \frac{S}{4}$ when $\mathcal{F}(A, \phi) = 0$. Thus:

$$\int_X |F^+|^2 vol_X \leq \frac{S^2}{16} vol(X)$$

Putting this back into the previous equation:

$$\begin{aligned}\frac{1}{4\pi^2}\int_X |F|^2 vol_X &\leq -c_1(\mathfrak{s})^2[X] + \frac{1}{32\pi^2} S^2 vol(X) \\ &\leq -4d(\mathfrak{s}) - 2\chi - 3\tau + \frac{1}{32\pi^2} S^2 vol(X) \\ &\leq -4d_0 \underbrace{-2\chi - 3\tau + \frac{1}{32\pi^2} S^2 vol(X)}_{=:C(x,g)}\end{aligned}$$

The finite dimensional vector space $H^2_{dR}(X)$ has a norm $||\cdot||$ defind by $||c|| = \min_{[\omega]=c} |\omega|_{L^2} = ||c_{harm}||_{L^2}$. Applying this to $c_1(\mathfrak{s})$ gives:

$$||c_1(\mathfrak{s})||^2 = \frac{1}{2\pi}||[F]||^2 \leq \frac{1}{4\pi^2}\int_X |F|^2 vol_X \leq -4d_0 + C(X, g)$$

In particular, it is uniformly bounded. So $c_1(\mathfrak{s}) \in H^2(X;\mathbb{Z}) \cap$ (ball), which is a finite set. The map $\mathfrak{s} \mapsto c_1(\mathfrak{s})$ has fibers $H^1(X, \mathbb{Z}/2)$, which is finite.[a] Therefore we have only finitely many such $\mathfrak{s}$.

[a] Actually Spin^c structures with $c_1(\mathfrak{s})$ fixed correspond to the 2 torsion of $H^2(X;\mathbb{Z})$, which is the image of the Bockstein homomorphism $\beta : H^1(X;\mathbb{Z}/2) \to H^2(X;\mathbb{Z})$, so the fibers are actually a quotient of $H^1(X;\mathbb{Z}/2)$.

□

Remark 5.9. The set of Spin^c structures on X has an action of $H^2(X;\mathbb{Z})$ (which corresponds to line bundles via $c_1(-)$) via tensoring with the line bundle. Moreover $\Lambda^2(L \otimes \mathbb{S}^+) = L^2 \otimes \Lambda^2\mathbb{S}^+$.

5.4 Elliptic Theory and Applications to SW Compactness ❖

The suggested reference for this section is [9].

5.4.1 Sobolev Spaces

Let $U \subset \mathbb{R}^n$ be open and consider $C^\infty(U)$. For all $p > 1$, we have the L^p norm on this space and the Sobolev (p, k) norm (for $k = 0, 1, 2, ...$):

$$||f||_p = \left(\int_U |f|^p\right)^{1/p}$$
$$||f||_{p,k} = \sum_{|\alpha|\leq k} ||D^\alpha f||_p$$

In the second line, α denotes a multi-index $\alpha = (i_1, ..., i_\ell)$, with $1 \leq i_1 \leq ... \leq i_\ell \leq n$ and $D^\alpha = \partial_{x_{i_1}} \cdots \partial_{x_{i_\ell}}$. There is also the C^k norm:

$$||f||_{C^k} = \sum_{0\leq|\alpha|\leq k} \sup_U |D^\alpha f|$$

This is, in a sense, the (p, ∞) norm as used above. To extend this to a manifold, let M be a compact oriented n manifold with a real vector bundle $E \to M$ of rank r. We define Sobolev norms on $\Gamma(M, E)$ via taking an open cover $M = \bigcup U_i$ such that $E|_{U_i}$ is trivial (with trivialization τ_i). Pick a partition of unity $\{\rho_i\}$ subordinate to this cover. For $s \in \Gamma(M, E)$, define:

$$||s||_{p,k} = \sum_i ||\tau_i \circ (\rho_i s)||_{p,k}$$

Let $L^p_k(E)$ be the completion of $\Gamma(M, E)$ with respect to $||\cdot||_{p,k}$. This is a Banach space (and Hilbert when $p = 2$). Notice that C^k sections of E form a Banach space with norm $||\cdot||_{C^k}$ and the inclusion $C^{k+1} \hookrightarrow C^k$ is compact. There will a similar result for Sobolev spaces.

A few important facts that we won't prove are:

- Define the scaling weight $w(k, p) = k-\frac{n}{p}$. This is the weight with which $||D^\alpha f||_p$ ($|\alpha| = k$) on $\mathbb{R}^n$ transitions under dilation $x \mapsto rx$. Then the Sobolev inequality is:

$$k > \ell \text{ and } w(k, p) \leq w(\ell, q) \Longrightarrow ||\cdot||_{p,k} \leq C||\cdot||_{\ell,q}$$

  and therefore there is a bounded inclusion $L^q_\ell(E) \hookrightarrow L^p_k(E)$.

- The Rellich lemma: If $k > \ell$ and $w(k, p) < w(\ell, q)$, then the above inclusion is compact.
- The Morrey inequality: Given $\ell \geq 0$ satisfying $\ell < w(k, p)$, then there exists a bounded inclusion $L^p_k \hookrightarrow C^\ell$. This allows us to view Sobolev sections as continuous sections.
- As a consequence of above, $\bigcap_{k\geq k_0} L^p_k = C^\infty$ for any k_0.

5.4.2 Elliptic Estimates

Let $D : \Gamma(E) \to \Gamma(F)$ be a differential operator of order m over a compact oriented manifold M. This naturally extends to a bounded map $D : L^p_{k+m}(E) \to L^p_k(E)$.

Theorem 5.10 (Elliptic estimate)**.** *If D is elliptic of order m, then:*

$$||s||_{2,k+m} \leq C_k(||Ds||_{2,k} + ||s||_{2,k})$$

for some C_k. Moreover, if we take $s \in (\ker D)^\perp$ (the L^2 complement of the kernel), then:

$$||s||_{2,k+m} \leq C_k||Ds|_{2,k}$$

Corollary 5.11. *Let D be elliptic of order m with formal L^2 adjoint D^* with respect to some euclidean metric. Then:*

1. *(Elliptic regularity): If $S \in L^2_m(E)$ with $Ds = 0$, then $s \in C^\infty(E)$.*
2. *The unit ball in $\ker D \subset L^2_m$ is compact, and hence $\dim \ker D < \infty$.*
3. *$\operatorname{im} D$ is a closed in $L^2(E)$.*
4. *$\operatorname{im} D = (\ker D^*)^\perp$, the complement of $\ker D^*$ with respect to L^2.*
5. *$\operatorname{coker} D \cong \ker D^*$ (and hence is also finite dimensional since D^* is elliptic).*
6. *$D : L^2_{k+m} \to L^2_k$ is Fredholm. Its index $\operatorname{ind} D = \dim \ker D - \dim \operatorname{coker} D$ depends only on the symbol of D.*

See Tim's notes for a proof. A more general version of Theorem 5.10 is:

Theorem 5.12. *If D is elliptic of order m, then $||s||_{p,k+m} \leq C_{p,k}(||Ds||_{p,k} + ||s||_{p,k})$. If one takes s from a complement to $\ker D$ then $||s||_{p,k+} \leq C_{p,k}||Ds||_{p,k}$.*

5.5 The Compactness Theorem ❖

Theorem 5.13. *Let (X^4, g) be a closed, oriented Riemannian manifold, and $\mathfrak{s}$ a Spin^c structure, $\eta \in \Omega^+_X$. Let (A_j, ϕ_j) be a sequence of solutions to the SW equations $F_\eta = 0$. Write $A_j = A_0 + a_j \cdot i\,\mathrm{id}_{\mathbb{S}^+}$ and assume $d^* a_j = 0$ (Coulomb gauge) and that $(a_j)_{harm}$ is a bounded sequence in $\mathcal{H}^1(X)$ with its L^2 norm.*[6] *Then a subsequence converges in C^∞ to a smooth limiting solution (A, ϕ).*

[6]Note these two assumptions are always achievable by gauge transformaitions.

To prove this we will need to know about Sobolev multiplication.

Lemma 5.14. *Over the closed oriented 4-manifold X, multiplication of C^∞ functions extends to bounded linear maps $L^2_k(X) \otimes L^2_\ell(X) \to L^2_\ell(X)$, where $k \geq 3, k \geq \ell$. In particular, $L^2_k(X)$ is an algebra (with continuous multiplication) for $k \geq 3$.*

For a proof of this, see Tim's notes. As a consequence, for vector bundles E, F, one has bounded multiplication $L^2_k(E) \otimes L^2_\ell(F) \to L^2_\ell(E \otimes F)$. Additionally, we have multiplications:

$$\begin{aligned} L^1_2(E) \otimes L^3_1(F) &\to L^3(E \otimes F) \\ L^2_2(E) \otimes L^3_1(F) &\to L^2_1(E \otimes F) \\ L^2_3(E) \otimes L^2_2(F) &\to L^2_2(E \otimes F) \end{aligned} \tag{5.5.1}$$

It is an exercise to verify these multiplications. For $k \geq 3$, we have a (topological) gauge group $\mathcal{G}_k$ of L^2_k gauge transformations: $\mathcal{G}_k \cong L^2_k$ maps $X \to S^1 \subset \mathbb{C}$. The L^2_k configurations are elements of the form $(A = A_0 + a, \phi)$, where $A_0 \in \mathbb{C}^\infty$ and $a \in L^2_k$. Then $\mathcal{G}_k$ acts on L^2_k configurations. Also, the SW equations make sense in L^2_k, $k \geq 3$: the differential terms (e.g. $(D^+_A\phi)$) map L^2_k to L^2_{k-1}, and the quadratic terms (e.g. $(\phi\phi^*)_0$) map L^2_k to L^2_k by the Lemma.

We build in Coulomb gauge transformations to the SW equations:

$$\mathcal{F}'_\eta : L^2_k(iT^*X) \times L^2_k(\mathbb{S}^+) \to L^2_{k-1}(X)_0 \times L^2_{k-1}(i\mathfrak{su}(\mathbb{S}^+)) \times L^2_{k-1}(\mathbb{S})$$

$$(a, \phi) \mapsto (d^*a, \mathcal{F}_\eta(A_0 + a \cdot \mathrm{id}, \phi))$$

This map takes the form:

$$\mathcal{F}'_\eta = \mathcal{D} + q + c$$

where $\mathcal{D}$ is a linear first order differential operator, q is a quadratic term, and c is a constant. These explicitly are:

$$\mathcal{D}\begin{bmatrix} a \\ \phi \end{bmatrix} = \begin{bmatrix} d^*a \\ \rho(d^+a) \\ D^-_{A_0}\phi \end{bmatrix}$$

$$q\begin{bmatrix} a \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ (\phi\phi^*)_0 \\ \frac{1}{2}\rho(a)\phi \end{bmatrix}$$

$$c = \begin{bmatrix} 0 \\ \frac{1}{2}\rho(F(A^0_0)^+ - 4i\eta) \\ 0 \end{bmatrix}$$

Positive Feedback Loop (elliptic bootstrapping)

Proposition 5.15. *Fix $k \geq 3$. Consider a set of solutions $\gamma = (a, \phi)$ to $F'_\eta(a, \phi) = 0$. Assume a bound $||\gamma||_{L^2_k} \leq c_k$. Then there is a bound $||\gamma||_{L^2_{k+1}} \leq c_{k+1}$. (Here the norm of γ means the sum of the norms of a and ϕ).*

Proof:

The operator $\mathcal{D}$ as above has elliptic estimates:

$$||\gamma||_{L^2_{k+1}} \leq C(||D\gamma||_{L^2_k} + ||\gamma||_{L^2_k})$$

Use the SW equations and our assumption on the bound of γ:

$$\begin{aligned} ||\gamma||_{L^2_{k+1}} &\leq C\left(||q(\gamma)+c||L^2_k + ||\gamma||_{L^2_k}\right) \\ &\leq C(||q(\gamma)||_{L^2_k} + C' + c_k) \\ &= C''(1+||q(\gamma)||_{L^2_k}) \\ &= C''(1+D||\gamma||^2_{L^2_k}) \end{aligned}$$

In the final line, we used that q is quadratic and the Sobolev multiplication lemma. Then the fact that $||\gamma||^2_{L^2_k} \leq c_k^2$ concludes the proof.

□

Corollary 5.16. *if $\gamma_j = (a_j, \phi_j)$ is a sequence of solutions to $\mathcal{F}'_\eta = 0$ converging (a,ϕ) in L^2_3. Then (a,ϕ) is C^∞, $\mathcal{F}'_\eta(a,\phi) = 0$, and convergence is in C^∞ (i.e. it is in C^ℓ for any ℓ).*

A priori, having $\mathcal{F}_\eta(A,\phi) = 0$ implies that there exists $\kappa = \kappa(X,g,\eta)$ such that $||\phi||_{C^0}, ||F(A^0)^+||_{C^0} \leq \kappa$. Let $A = A_0 + a \cdot \mathrm{id}$. If $\mathcal{F}'_\eta(a,\phi) = 0$, then $d^*a = 0$ an we can write $a = a_{harm} + a'$, where $a' \in \operatorname{im} d^*$ by the Hodge theorem.

Lemma 5.17. *For all $p > 1$, there exists a uniform bound:*

$$||a'||_{L^p_1} \leq k_p$$

on solutions to $\mathcal{F}'_\eta = 0$, where $k_p = k_p(X,g,\eta,A_0)$.

Proof:

Note that $F(A^0)^+ = F((A^0_0)^+ + 2d^+a$, and so $||d^+a||_{C^0} \leq \kappa'$ by the a priori C^0 estimates stated above. The operator $d^* \oplus d^+$ is elliptic and $\ker(d^* \oplus d^+) = \mathcal{H}^1$. Then by Theorem 5.12, for $b \in (\mathcal{H}^1)^{\perp, L^2}$ we have $||b||_{L^p_1} \leq C||(d^* \oplus d^+ b||_{L^p}$. Since $a' \in \operatorname{im} d^* \subset (\mathcal{H}^1)^\perp$, we have:

$$||a'||_{L^p_1} \leq C||d^+a'||_{L^p} = C||d^+a||_{L^p} \leq C\kappa' \mathrm{vol}(X)^{1/p}$$

□

Now we need to get from L^p_1 bounds to L^2_3 convergence, which would allow us to apply Corollary 5.16.

Lemma 5.18. *Suppose that $\gamma_j = (a_j, \phi_j)$ is a sequence of solutions to $\mathcal{F}'_\eta = 0$, converging in L^2_1 to an L^2_1-limit $\gamma = (a,\phi)$. Then $\gamma_j \to \gamma$ in L^2_3.*

Proof:

We will use the bounded multiplication maps 5.5.1 to successively improve the convergence, $L^2_1 \to L^3_1$, $L^2_2 \to L^2_3$. To begin, we use the elliptic estimate for $\mathcal{D}$:

$$\begin{aligned} ||\gamma_i - \gamma_j||_{L^3_1} &\leq C(||D\gamma_i - D\gamma_j||_{L^3} + ||\gamma_i - \gamma_j||_{L^3}) \\ &= C(||q(\gamma_i) + c - q(\gamma_j) - c||_{L^3} + ||\gamma_i - \gamma_j||_{L^3}) \qquad \text{(by SW eq.'s)} \\ &= C(||q(\gamma_i) - q(\gamma_j)||_{L^3} + ||\gamma_i - \gamma_j||_{L^3} \end{aligned}$$

Since q is quadratic, there exists a bininear form b such that $q(\gamma) = b(\gamma, \gamma 0$. Then $q(\gamma_i) - q(\gamma_j) = b(\gamma_i - \gamma_j, \gamma_i + \gamma_j)$. Since γ_i converge in L^2_1, for all $\epsilon > 0$ there exists i_0 such that $||\gamma_i - \gamma_{i_0}||_{L^2_1} \leq \epsilon$ for all $i \geq i_0$. Then:

$$b(\gamma_i - \gamma_j, \gamma_i + \gamma_j) = b(\gamma_i - \gamma_j, \gamma_i + \gamma_j - 2\gamma_{i_0}) + 2b(\gamma_i - \gamma_j, 2\gamma_{i_0})$$

$$\begin{aligned}\implies ||q(\gamma_i) - q(\gamma_j)||_{L^3} &\leq ||b(\gamma_i - \gamma_j, \gamma_i\gamma_j - 2\gamma_{i_0})||_{L^3} + 2C||\gamma_{i_0}||_{C^0}||\gamma_i - \gamma_j||_{L^3} \\ &\leq C'\left(||\gamma_i - \gamma_j||_{L^3_1} \cdot ||\gamma_i + \gamma_j - 2\gamma_{i_0}||_{L^2_1} + ||\gamma_{i_0}||_{C^0} \cot ||\gamma_i - \gamma_j||_{L^3}\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(By Sob. mult.)} \\ &\leq C'(2\epsilon||\gamma_i - \gamma_j||_{L^3_1} + ||\gamma_{i_0}||_{C^0}||\gamma_i - \gamma_j||_{L^3})\end{aligned}$$

Now:

$$\begin{aligned}||\gamma_i - \gamma_j||_{L^3_1} &\leq C(2\epsilon||\gamma_i - \gamma_j||_{L^3_1} + (1 + ||\gamma_{i_0}||_{C^0})||\gamma_i - \gamma_j||_{L^3}) \\ \iff (1 - 2\epsilon C)||\gamma_i - \gamma_j||_{L^3} &\leq C(1 + ||\gamma_{i_0}||_{C^0})||\gamma_i - \gamma_j||_{L^3}\end{aligned}$$

Pick $\epsilon = \frac{1}{4C}$, so that $1 - 2\epsilon C = \frac{1}{2}$. Pick a constant i_0, so that:

$$||\gamma_i - \gamma_j||_{L^3_1} \leq D||\gamma_i - \gamma_j||_{L^3} \leq D'||\gamma_i - \gamma_j||_{L^2_1}$$

So that γ_i is Cauchy in L^3_1. A similar argument shows that it is L^2_2, L^2_3.

□

The upshot of this lemma is that if a sequence of solutions converges in L^2_1, it converges in C^∞ and $||a'||_{L^p_1}$ is bounded. The last step, then, is to show L^2_1 convergence (done in the proof below). Note that if we assume a_{harm} bounded, we have $||a||_{L^p_1}$ bounded.

Proof of Theorem 5.13:

We have to produce a subsequence of (a_j, ϕ_j) converging in L^2_1. By the note above, we are assuming that $(a_j)_{harm}$ is bounded. Since we have an L^2_1 bound on a_j and we can easily show that there is an L^2_1 bound on ϕ_j, so $||\gamma_j||_{L^2_1}$ is bounded. It is a fact that a bounded sequence in a separable Hilbert space has a weakly convergent subsequence, i.e. there exists $\gamma \in L^2_1$ such that $\langle \gamma - \gamma_j, \beta_{L^2_1}\rangle \to 0$ for all β. Moreover, if $||\gamma||_{L^2_1} = \limsup ||\gamma_j||_{L^2_1}$, the convergence is strong. This is in fact the case for our situation (see Tim's notes for details).

□

5.6 Transversality ❖

The configurations $(A, \phi) \in \mathcal{A}_{\mathrm{Cl}}(\mathbb{S}^+) \times \Gamma(\mathbb{S}^+)$ form a space $\mathcal{C}(\mathfrak{s})$; those of class L^2_k form the class $\mathcal{C}(\mathfrak{s})_k$. Fix $k \geq 4$, and consider the gauge group $\mathcal{G}_{k-1}$, which acts continuously on $\mathcal{C}(\mathfrak{s})_k$.[7] We distinguish two types of configurations for which we will discuss transversality:

- The irreducible configurations $\mathcal{C}^{irr}(\mathfrak{s}) = \{(A, \phi) \mid \phi \not\equiv 0\}$. The gaugue group $\mathcal{G}$ acts freely on these, since if $u \in \mathcal{G}$ and $u^*A = A$ then u =constant; but then if $u \neq 1$, $u \cdot \phi \neq \phi$.
- The reducible configurations $\mathcal{C}^{red}(\mathfrak{s}) = \{(A, 0\}\}$. In this case, the stablilizer of the $\mathcal{G}$ actions is $U(1) = \{\text{constant gauge transformations}\}$.

Fixing some $x \in X$, let $\mathcal{G}^x = \{u \in \mathcal{G} \mid u(x) = 1\}$. Then $\mathcal{G}^x$ acts freely on $\mathcal{C}(\mathfrak{s})$, and $\mathcal{G} = \mathcal{G}^x \times U(1)$. Note that $U(1)$ acts *semi-freely* on $\mathcal{C}(\mathfrak{s})/\mathcal{G}^x$, meaning it acts freely on $\mathcal{C}^{irr}(\mathfrak{s})/\mathcal{G}^x$ and trivially on $\mathcal{C}^{red}(\mathfrak{s})/\mathcal{G}^x$.

5.6.1 Reducible Solutions

For a reduible configuration $(A, 0 \in \mathcal{C}^{red}(\mathfrak{s})$, the equation $\mathcal{F}'_\eta(a, 0) = 0$ (where $A = A_0 + a$) simplifies to $d^*a = 0, F(A^0)^+ - 2i\eta = 0$. For $\eta = 0$, the second equation says we're studying $U(1)$ instantons. Even for $\eta \neq 0$, the theory is similar. Write $\eta = \eta_{harm} + \eta'$, for $\eta' \in \operatorname{im} d^+$ (which we can do by the signature complex from Definition 3.11).

[7] From here out, we might omit the k's

Proposition 5.19. *There exists a reducible solution to $\mathcal{F}'_\eta = 0$ (for $\mathfrak{s}$) if and only if $c_1(\mathfrak{s}) + \frac{1}{\pi}[\eta_{harm}] \in \mathcal{H}^-_{[g]} \subset H^2_{dR}(X)$. Moreover, when nonempty, the reducible solutions modulo $\mathcal{G}^x$ form an affine space (torsor) for $H^1(X,\mathbb{R})/H^2(X,\mathbb{Z})$.*

From our generic non-existence theorem for $U(1)$ instantons, we are led to:

Theorem 5.20. *Say $b^+(X) > 0$. Fix a conformal structure $[g]$ and fix a self-dual 2 form $\eta \in \Omega^+_{[g]}$. Then $[g]$ can be approximated in C^r by C^r conformal structures $[g_i]$ (here $r \geq 2$) such that for $[g_i]$ there are no reducible solutions to $\mathcal{F}'_{\eta,g_i} = 0$ for any Spin^c structure.*

Define $\mathcal{W}(\mathfrak{s}) = \{([g],\eta) \mid c_1(\mathfrak{s}) + \frac{1}{\pi}\eta_{harm} \in \mathcal{H}^-_{[g]}\}$. These are the pairs for which reducible solutions exist. The fiber of the projection $\mathcal{W}(\mathfrak{s}) \to \mathrm{Conf}_X$ onto the conformal structure part, denoted $\mathcal{W}(\mathfrak{s})_{[g]}$, is a codimension $b^+(X)$ affine linear subspace of $\Omega^+_{[g]}$. When $b^+(X) = 1$, this means that $\mathcal{W}(\mathfrak{s})$ is a codimension 1 submanifold, which is called the *wall*. It divides the space of all pairs into two components, called chambers.

Proposition 5.21. *Say $b^+(X) > 0$, and fix $([g_0],\eta_0),([g_1],\eta_1)$ not on $\mathcal{W}(\mathfrak{s})$. Then:*

1. *If $b^+(X) > 1$, any interpolating path $([g_t],\eta_t)$ can be approximated by one which avoids $\mathcal{W}(\mathfrak{s})$.*
2. *If $b^+(X) = 1$, then any interpolating path can be approximated by one transverse to the wall.*

5.6.2 Irreducible Solutions

First, we quote a unique continuation theorem: Let L be a linear elliptic operator over a connected manifold X. Suppose $Lu = 0$ and $u \equiv 0$ on an open set $U \subset X$. Then $u \equiv 0$. To get transversality, we will allow η to vary. More precisely, write $\eta = \omega + \eta'$, with $\omega = \eta_{harm}$ and $\eta' \in \operatorname{im} d^+$, and let η' vary with ω fixed. The parametric SW map is now:

$$\mathcal{F}^{par}_\omega : \operatorname{im}(d^+ : L^2_{k+1} \to L^2_k) \times L^2_k(iT^*X) \times L^2_k(\mathbb{S}^+) \to L^2_{k-1}(X)_0 \times L^2_{k-1}(i\mathfrak{su}(\mathbb{S}^+)) \times L^2_{k-1}(\mathbb{S}^+)$$
$$(\eta', a, \phi) \mapsto (d^*a, \mathcal{F}'_{\omega+\eta'}(a,\phi))$$

Theorem 5.22. *If $\mathcal{F}^{par}_\omega(\eta', a, \phi) = 0$, then $D_{(\eta',a,\phi)}\mathcal{F}^{par}_\omega$ is surjective. In other words, the parametric space $\{\mathcal{F}^{par}_\omega = 0\}$ is cut out transversely.*

Proof:

Let D abbreviate $D_{(\eta',a,\phi)}F^{par}_\omega$. Then:

$$D\begin{bmatrix}\delta\\ a\\ \chi\end{bmatrix} = \begin{bmatrix} d^*a \\ \rho(d^+a - 2i\delta) - (\chi\phi^* + \phi\chi^*)_0 \\ D^+_{A_0+a}\chi + \frac{1}{2}\rho(a)\phi \end{bmatrix}$$

This is a Fredholm linear map between Hilbert spaces. Then we claim that the L^2 orthogonal complement to $\operatorname{im} D$ is zero. From this, $\operatorname{im} D$ must be dense in L^2, and therefore also dense in L^2_{k-1}. The Fredholm property tells us $\operatorname{im} D$ is closed in L^2_{k-1}. Together, these mean that D is surjective. Thus, we only need to prove the claim.

Take (f, α, ψ) in the L^2 complement to $\operatorname{im} D$. To begin, notice that:

$$D\begin{bmatrix}\delta\\ 0\\ 0\end{bmatrix} = \begin{bmatrix} 0 \\ -2i\rho(\delta) \\ 0 \end{bmatrix}$$

So for all δ, $\langle \rho(\delta), \alpha\rangle_{L^2} = 0$. If δ were allowed to be an arbitrary element, then $\rho(\delta) \in \Gamma(i\mathfrak{su}(\mathbb{S}^+))$ would be also arbitrary and so we would necessarily have $\alpha = 0$. However, we are assuming that $\delta \in \operatorname{im} d^+$, which amounts to a finite codimension linear constraint on $\rho(\delta)$, and so we can still conclude $\alpha = 0$.

Now observe:

$$D\begin{bmatrix}0\\ a\\ 0\end{bmatrix} = \begin{bmatrix} d^*a \\ \rho(d^+a) \\ \frac{1}{2}\rho(a)\phi \end{bmatrix}$$

Take $a = d^*b$, so that the first component of the RHS above vanishes. By assumption, the RHS is L^2 orthogonal to $[f\ 0\ \psi]$. This is trivial in the first two components, but the third component says that $\rho(d^*b)\phi$ is L^2 orthogonal to ψ for all b. The irreduciblity assumption allows us to take $x \in X$ such that $\phi(x) \neq 0$. Then one can manufacture b such that $\rho(d^*b)\phi(x) \neq 0$. Better yet, one can manufacture b_1, b_2 such that $(\rho(d^*b_1)\phi(x), \rho(d^*b_2)\phi(x))$ is an orthonormal basis for $\mathbb{S}_x$. By taking b_i supported near x, the orthogonality condition means that $\psi = 0$ in a neighborhood of x.

Finally, observe:

$$D \begin{bmatrix} 0 \\ 0 \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ -(\phi\chi^* + \chi\phi^*) \\ D_A^+\chi \end{bmatrix}$$

Which says $\langle D_A^+\chi, \psi\rangle_{L^2} \equiv 0$ for all χ, which is equivalent to saying that $D_A^-\psi = 0$. Therefore ψ is a solution to a Dirac equation, and we may cite the unique continuation theorem stated on the outset. Namely, $\psi = 0$ everywhere.

Now, we must show that $f = 0$. At this point, our orthogonality condition is $\langle f, d^*a\rangle_{L^2} \equiv 0$ for all a, so $\langle df, a\rangle_{L^2} = 0 \Rightarrow df = 0$. Thus f is constant. Since f is assumed to be mean zero, we must have $f = 0$. Thus $(f, \alpha, \psi) = (0, 0, 0)$.

□

From the inverse function theorem in Banach spaces, we get that the solutions to $\mathcal{F}_\omega^{par} = 0$ form a smooth submanifold of the domain. To make this more precise, let U, V, P be Banach spaces (we're thinking of P as the "parameter space" of values of η above) and let $F = (f_1, f_2) : U \to V \times P$ be a smooth map. For all $p \in P$, set $M_p = F^{-1}(0, p)$. In the above setup, the corresponding objects are $F = \mathcal{F}'_\omega, P = \operatorname{im} d^+, p = \eta'$. We want to show that M_p is cut out transversely (i.e. $(0, p)$ is a regular value for F) for generic $p \in P$. Define the parametric function:

$$F^{par} : U \times P \to V \times P$$

$$(x, p) \mapsto (f_1(x), f_2(x) - p)$$

Set $M^{par} = (F^{par})^{-1}(0, 0)$. We suppose that $(0, 0)$ is a regular value of F^{par}. Then the inverse function theorem for Banach spaces shows that M^{par} is a submanifold of $U \times P$. Define $\Pi : M^{par} \to P$, so that $M_p = \Pi^{-1}(p)$. Thus we'd like generic fibers of Π to be cut out transversely.

Lemma 5.23. *If $F(x) = (0, p)$ (i.e. $F^{par}(x, p) = (0, 0)$), then:*

1. $\ker(D_x F) = \ker(D_{(x,p)}\Pi)$.
2. $\operatorname{coker}(D_x F) \cong \operatorname{coker}(D_{(x,p)}\Pi)$.

Hence, $(0, p)$ is a regular value for F if and only if p is a regular value for Π.

Proof:

For 1), just unravel the definitions and it will be evident. For 2), the fact that M^{par} is cut out transversely means, given $(x, p) \in M^{par}$ and a tangent vector (v, q) at $F(x, p)$, we have a solution $(\dot{x}, \dot{p})$ to $(D_x f_1)(\dot{x}) = v$ and $(D_x f_2)(\dot{x}) - \dot{p} = q$. The substance here is that $D_x f_1$ surjects.

In general, if $L_1 : U \to V, L_2 : U \to P$ are linear maps, with L_1 surjective, then the inclusion $P \to V \times P$ induces an isomorphism $P/L_2(\ker(L_1)) \cong \operatorname{coker}(L_1, L_2)$. Taking $L_i = D_x f_i$, we get $\operatorname{coker} D\Pi = P/Df_1(\ker(Df_1) \cong \operatorname{coker} DF$, the second isomorphism by what we just claimed.

□

5.6.3 Generalizing Sard's Theorem

So, finding trasversely cut out M_p is equivalent to finding regular values of Π. The finite dimensional Sard's theorem says that the regular values of a map have measure zero, but in infinite dimensions "measure zero" has no meaning. Smale's solution and generalization of Sard's theorem to infinite dimensions is as follows.

Definition 5.24. A subspace S of a topological space T is a *Baire subspace* (also known as residual) if it is the intersection of countably many open dense subsets.

The Baire category theorem states that, if the topology on T comes from a complete metric, then Baire subsets are dense. Notice that the countable intersection of Baire subspaces is again Baire, by definition. In this case, Sard's theorem can be reformulated as: the regular values of a smooth map of finite dimensional manifolds are a Baire subspace of M, and hence are dense. It is important to note that, at this point, this is version of Sard's theorem is still false for infinite dimensional manifolds.

Definition 5.25. Let Y, Z be Banach manifolds (and second countable). A C^∞ map $\Phi : Y \to Z$ is called a *Fredholm map* if $D_y\Phi : T_yY \to T_{\Phi(y)}Z$ is a Fredholm map of Banach spaces for all $y \in Y$.

Theorem 5.26 (Sard-Smale)**.** *If $\Phi : Y \to Z$ is a Fredholm map, its regular values are a Baire subspace of Z.*

More discussion about this theorem can be found in Tim's notes. Returning to $F : U \to V \times P$, assume that F is Fredholm. Then lemma 5.23 implies that Π is also Fredholm.[8] Then Theorem 5.26 gives a Baire set of parameters $p \in P$ such that M_p is cut out transversely, and hence is a manifold. Thus, we deduce:

Theorem 5.27. *For a Baire subspace of $\eta' \in P = \operatorname{im} d^+$, all irreducible solutions to $\mathcal{F}'_{\omega+\eta'} = 0$ are cut out transversely.*

For such η', set $\eta = \omega + \eta'$. Set $\tilde{M}^{irr}_\eta = \{(a,\phi) \in \mathbb{C}(\mathfrak{s})^{irr} \mid \mathcal{F}'_\eta(a,\phi) = 0\}$. This is then a manifold whose tangent spaces are $\ker D$, where D is the linearized SW map. Since $\operatorname{coker} D = 0$ by transversality, the dimension of $\tilde{M}_\eta$ is $\dim\ker D = \operatorname{ind} D$. This manifold has a free action of $U(1)$ by constant gauge transformations. Thus we have a quotient $M^{irr}_\eta = \tilde{M}^{irr}_\eta / U(1)$, which is the orbit space. The dimension of this orbit space is:

$$\dim M^{irr}_\eta = d(\mathfrak{s}) = \frac{1}{4}(c_1(\mathfrak{s})^2[X] - 2\chi + 3\tau)$$

hence $\dim \tilde{M}^{irr}_\eta = d(\mathfrak{s}) + 1$.

Corollary 5.28. *Suppose $b^+(X) > 0$. Then for generic (in a Baire sense) metrics g and generic g-self dual 2 forms $\eta' \in \operatorname{im} d^+$, and with ω_g the g-self dual harmonic representative of a fixed class $w \in H^2_{dR}(X)$, the SW moduli space $\tilde{M}_{\omega_g+\eta'} = \{\mathcal{F}'_{\omega_g+\eta'} = 0\}$ consists only of irreducibles and is a compact manifold of dimension $d(\mathfrak{s}) + 1$, admitting a free $U(1)$ action by constant gauge transformations.*

5.7 The Diagonalization Theorem ❖

The goal of this subsection is to prove:

Theorem 5.29. *If X is a closed oriented 4 manifold with Q_X positive definite, then Q_X is diagonalizable over $\mathbb{Z}$, i.e. $Q_X \cong \langle 1\rangle^{b_2(X)}$.*

Remark 5.30. With an assumption on $\pi_1(X)$, this is due to Donaldson in 1982. The proof we give here is due to Kronheimer-Mrowka (unpublished).

Some preliminary facts:

- An equivalent statement is that Q_X negative definitie implies Q_X diagonalizable.
- It is enough to prove this in the case where $b_1(X) = 0$. Namely, if $b_1(X) > 0$, then there exists a 4- manifold Y with $b_1Y < b_1X$ and $Q_Y \cong Q_X$. Such a Y can be found using surgery theory: suppose $h \in H_1(X, \mathbb{Z})$ is non-torsion and primitive. Then let $Y = X_\gamma =$ surgery on an embedded loop γ with $[\gamma] = h$. That is, let Y be the gluing of X minus a neighborhood of γ along the boundary with $D^2 \times S^2$. Then the claim is that $H_1(Y) = H_1(X)/[\gamma]$, hence $b_1(Y) = b_1(X) - 1$, and $Q_Y \cong Q_X$.

[8] One should also check that $\operatorname{im} D\Pi$ is closed

5.7.1 SW Moduli Spaces

Let X be a 4-manifold with $b_1 X = 0$ and $b^+ = 0$ (Q_X negative definite). The operator $d^* \oplus d^* : \Omega^1 \to \Omega^0 \oplus \Omega^+$ has kernel $\mathcal{H}^1 = 0$ and cokernel $\mathbb{R} \oplus \mathcal{H}^+$ (imposing mean 0 on the first target of the codomain would make the cokernel 0). Let $\mathfrak{s}$ be a Spin^c structure and denote the dirac operators $D_A^+ : \gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$. These have index:

$$\mathrm{ind}_{\mathbb{R}}\, D_A^+ = \frac{1}{4}(c^2[X] - \tau_X) = \frac{1}{4}(c^2[X] + b_2)$$

where $c \equiv c_1(\mathfrak{s})$. Therefore $d(\mathfrak{s}) = \frac{1}{4}(c^2[X] + b_2) - 1$. Since c is characteristic, $c^2[X] \equiv \tau_X \bmod 8$, which means $d(\mathfrak{s})$ is odd. Recall that the irreducible solutions to SW are $M_\eta^{irr}(\mathfrak{s}) = \tilde{M}_\eta(\mathfrak{s})/U(1)$. For generic η, $\tilde{M}_\eta^{irr}$ is cut out transversely and has dimension $d(\mathfrak{s}) + 1$. Therefore M_η^{irr} is a manifold of dimension $d(\mathfrak{s})$, since the $U(1)$ action is free.

If $d(\mathfrak{s}) < 0$, we must have M_η^{irr} empty, so only reducible solutions might exist. We thus make the **hypothesis** that our choice of $\mathfrak{s}$ is such that $c_1(\mathfrak{s})^2[X] + b_2(X) > 0$. Write $d(\mathfrak{s}) = 2k - 1$ since it is odd. Since $b^+ = b_1 = 0$, there is a unique gauge orbit of reducible solutions $[A_0, 0]$, i.e. $F(A_0)^+ = 2\pi\eta$, because the Picard torus is trivial in this case. So, if we denote by $\mathbb{R}$ the gauge orbit of the reducible solution, the full SW moduli space is $\tilde{M}_\eta = \{R\} \cup \tilde{M}_\eta^{irr}$. This is a compact space with a distinguished point R whose complement is a $2k$ manifold and the action of $U(1)$, free on M_η^{irr}, is trivial on R. They key thing is to understand the structure of $\tilde{M}_\eta$ near R.

Let $D := D_{(A_0,0)}\mathcal{F}'_\eta$. Then:

$$D \begin{bmatrix} b \\ \chi \end{bmatrix} = \begin{bmatrix} d^* b \\ \rho(d^+ b) \\ D_{A_0}^+ \chi \end{bmatrix}$$

Since D is a sum of $d^* \oplus d^+$ and $D_{A_0}^+$, we have that $\ker D = \ker D_{A_0}^+$ and $\mathrm{coker}\, D = \mathrm{coker}\, D_{A_0}^+ = \ker D_{A_0}^-$, since the kernel and cokernel of $d^* \oplus d^+$ are both trivial. We assume that $R = [A, 0]$ is regular, meaning $\mathrm{coker}\, D = 0 \iff \mathrm{coker}\, D_{A_0}^+ = 0$ (this will be true for generic A, but this we will arrange later). Under this **hypothesis**, $\tilde{M}_\eta$ is a manifold of dimension $2k$, even at the point R. Since $2k > 0$, this manifold is nonempty and hence there are irreducible solutions. The group $U(1)$ fixes R and so acts $T_R(\tilde{M}_\eta) = \ker D_{A_0}^+$. This the action of $U(1) \subset \mathbb{C}$ by scalar multiplication on a complex vector space.

Lemma 5.31. *Suppose Q^{2k} is a manifold with a $U(1)$ action with $q \in Q$ is fixed by $U(1)$ and suppose also that the $U(1)$ action on T_qQ has a single weight $N \in \mathbb{Z}$, i.e. $T_qQ \cong (\mathbb{C}^{\otimes N}) \otimes \mathbb{R}^k$ as a representation of $U(1)$ (the action is given by t^N, $t \in U(1)$). Then q has a neighborhood equivariantly modeled on a neigborhood of 0 in $(\mathbb{C}^{\otimes N}) \otimes \mathbb{R}^k$.*

Proof:

The idea is to average over $U(1)$ to get a $U(1)$ invariant metric g near q. Then use the associated exponential map to give a chart.

□

By this lemma, our fixed point R has a neighborhood modeld on $\mathbb{C}^k$ with a $U(1)$ action by scalar multiplication. If we remove a small $U(1)$ invariant ball around R, the result is a compact manifold $\tilde{N}$ of dimension $2k$ with $\partial\tilde{N} = S^{2k-1}$ with a free circle action restricting to the standard action on S^{2k-1}. The quotient by $U(1)$ is N, a $2k-1$ dimensional manifold with $\partial N = \mathbb{CP}^{k-1}$. When $k-1$ is even, we find that $\mathbb{CP}^{k-1}$ bounds a manifold, contradicting oddness of its Euler characteristic.

In the case where $k-1$ is odd, we have to do a bit more work. We have the principal $U(1)$ bundle $\tilde{N} \to N$ whose associated $\mathbb{C}$ bundle has a chern class c_1 restricting to ∂N as a generator of $H^2(\mathbb{CP}^{k-1}, \mathbb{Z})$. Reducing mod 2, this means that $w_2 = c_1 \bmod 2$ restricts on ∂N to the nonzero class of $H^2(\mathbb{CP}^{k-1}, \mathbb{Z}/2) = \mathbb{Z}/2$. Therefore $w_2^{k-1}|_{\partial N} \neq 0$, i.e. $\langle w_2^{k-1}, [\partial N]\rangle \neq 0$. But $[\partial W] = 0$ in $H_*(N, \mathbb{Z}/2)$, which is a contradiction.

We are thus at a contradiction, which means one of our two **hypotheses** must be wrong. We take it as a fact that the second of these (regularity of R) can be attained. This means that our first hypothesis is wrong: $\nexists$ such that $c_1(\mathfrak{s})^2[X] + b_2(X) > 0$. In other words, the positive definite unimodular lattice $-Q_x$ of rank b_2 admits no characteristic vector c such that $c^2 < b_2$. Enter Noam Elkies:

Theorem 5.32 (Elkies). *Let Λ be a positive definite rank N unimodular lattice. If all its characteristic vectors c have $|c|^2 \geq N$, then Λ is diagonalizable.*

Therefore, the diagonalization Theorem now follows (we just need to verify the regularity hypothesis).

Proof of the Regularity Assumption

Lemma 5.33. *Fix $k \geq 3$ and a reference connection A_0, and introduce the parametric Dirac map:*

$$D^{par} : L^2_k(iT^*X) \times L^2_k(\mathbb{S}^+) \to L^2_{k-1}(\mathbb{S}^-)$$

$$(a, \phi) \mapsto D^+_{A_0+a}\phi = D^+_{A_0}\phi + \rho(a)\phi$$

Restricting to the domain where $\phi \neq 0$, then D^{par} has 0 as a regular value.

Proof:

The derivative can be computed to act as:

$$(D_{(a,\phi)}D^{par})(b, \chi) = D^+_{A_0}\chi + \rho(a)\chi + \rho(b)\phi = D^+_{A_0+a}\chi + \rho(b)\chi$$

We wish to show this is surjective. It is enough to show that any ψ which is L^2 orthogonal to the image must vanish. Taking $b = 0$, we see that ψ is L^2 orthogonal to $D^+_{A_0+a}\chi$ for all χ. Therefore $\psi \in \ker D^-_{A_0+a}$. To prove $\psi \equiv 0$, it suffices to show that $\psi = 0$ on an open set (by the unique continuation theorem we quoted at the beginning of §5.6.2). Taking $\chi = 0$, we have $\psi \perp \rho(b)\phi$ for all b, hence there exists $x \in X$ such that $\phi(x) \neq 0$. The othogonality condition is:

$$\int_X (\rho(b)\phi, \psi)\text{vol} = 0$$

for all b. Near x, ϕ is nonvanishing so there exists b such that $\rho(b)\phi = \psi$ on U. Taking a cutoff function σ for U, we take $b' = \sigma b$. Therefore $\int_X \sigma|\psi|^2 = 0$, hence $\psi = 0$ near x.

□

We now have a Banach manifold $(D^{par})^{-1}(0)$ with a projection map Π to $L^2(iT^*X)$, sending $(a, \phi) \to a$. As before, Π is a Fredholm map and:

$$\Pi^{-1}(a) = \{\phi \neq 0 \mid D^+_{A_0+a}\phi = 0 = \ker(D^+_{A_0+a}) \setminus 0$$

The regular values of Π are those such that $\operatorname{coker} D^+_{A_0+a} = 0$. The Sard-Smale theorem says that tehse are a Baire set. Thus, for generic a, $D^+_{A_0+a}$ has trivial cokernel.

Now set $A = A_0 + a$ and define $\eta(a) = \frac{1}{2i}F(A^0)^+ \in \operatorname{im} d^+$ since $\mathcal{H}^+ = 0$. We simultaneously apply two conditions on a (both Baire conditions):

1. $\operatorname{coker} D^+_{A_0+a} = 0$.
2. 0 is a regular value of $\mathcal{F}_\eta = 0$ on irreducible configurations.

So then R is regular and $\tilde{M}^{irr}_\eta$ is transverse. Note that the Coulomb gauge is inconvenient, since d^*a may be nonzero. But this can be fixed by considering solutions $\mathcal{F}'_\eta =$ constant (all of the analysis we have done on these solutions still hold in this case).

5.8 Seiberg Witten Invariants

To define the SW invariants, some preliminaries. Let X^4 be closed and oriented.

Definition 5.34. A *homology orientation* for X the equivalence class of a triple (H_+, H_-, o) where H^+, H^- are positive/negative definite subspaces of $H^2_{dR}(X)$ such that $H^2_{dR}(X) = H^+ \oplus H^-$ and where o is an orientation for the vector space $H^1_{dR}(X) \oplus H^0_{dR}(X)^* \oplus (H^+)^*$. Equivalently, o is a choice of positive ray in the line $\det H^1 \otimes (\det H^0)^* \otimes (\det H^+)^*$. Two triples (H^+, H^-, o) and (K^+, K^-, o') are equivalent if the canonical isomorphism $H^+ \cong K^+$ sends o to o'.

There is a two element set o_X of homology orientations and moreover if $\phi : X \to X'$ is an orientation preserving map, then there is an induced map $o_\phi : o_{X'} \to o_X$.

Suppose $\mathfrak{s}(\mathbb{S}, \rho)$ is a Spin^c structure on (X^4, g). Then it has a conjugate $\bar{\mathfrak{s}} = (\overline{\mathbb{S}}, \rho)$, where $\overline{\mathbb{S}}$ is the conjugate vector bundle to $\mathbb{S}$. It is the same as $\mathbb{S}$ as a real vector bundle, but the action of i changes as $i \cdot \phi = -i \cdot \phi$, where the first ϕ is thought of as an element of $\overline{\mathbb{S}}$ and the second is thought of as an element of $\mathbb{S}$. It turns out that $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$.

Now we can state the SW invariants and their properties, which are formulated depending on the value of $b_+(X)$.

When $b_+(X) > 1$

Suppose that $b_+(X) > 1$. Then the invariants are a map:

$$SW_{X,\sigma} : \text{Spin}^c(X) \to \mathbb{Z}$$

where Spin$^c(S)$ is the set of isomorphism classes of Spin^c structures[9] and σ is a homology orientation. The properties of this map will be:

1. $SW_{X,-\sigma} = -SW_{X,\sigma}$.
2. If $\Phi : X' \to X$ preserves orientation and a diffeomoerphism, then $SW_{X',o_{\Phi(\sigma)}}(\Phi^*\mathfrak{s}) = SW_{X,\sigma}(\mathfrak{s})$.
3. $SW_{X,\sigma}$ has finite support.
4. $SW_{X,\sigma}(\bar{\mathfrak{s}}) = (-1)^{1-b_1+b^+} SW_{X,\sigma}(\mathfrak{s})$.
5. If $SW_{X,\sigma}(\mathfrak{s}) \neq 0$, then $d(\mathfrak{s})$ is nonnegative and even.

Remark 5.35. It should be noted that, in general, there is no relation between the SW invariants of X and $\overline{X}$.

Remark 5.36. In the case where $d(\mathfrak{s}) = 0$ (in fact, this is the main case), the invariant $SW_{X,\sigma}(\mathfrak{s})$ is a signed count of the finite set of SW solutions modulo gague. The signs involved are to be explained.

When $b_+(X) > 1$

Now we consider the case when $b_+(X) = 1$, in which we have to deal with the wall mentioned earlier. Let $\mathcal{V} = \{([g] \in \text{Conf}_X, \eta) \mid \eta \in \Omega^+_{[g]}\}$, which has a projection to Conf_X. Let $\mathcal{W}(\mathfrak{s}) \subset \mathcal{V}$ be defined by the condition $c_1(\mathfrak{s}) \in -2i\eta_{harm} + \mathcal{H}^-_{[g]} \subset H^2_{dR}(X)$. This is an affine is an affine subbundle of the vector bundle $\mathcal{V} \to \text{Conf}_X$ of codimension 1. Recall the set $\mathcal{V} \setminus \mathcal{W}(\mathfrak{s})$ has two components, called chambers. Let $\text{Spin}^c(X)_{ch} = \{(\mathfrak{s}, c) \mid c \text{ is a chamber for } \mathfrak{s}\}$. Then the SW invariant is a map of the form:

$$SW_{X,\sigma} : \text{Spin}^c(X)_{ch} \to \mathbb{Z}$$

Some of the properties mentioned above still hold, but not all of them. In particular, the only one that fails is the third one about finite support.

[9] This is a torsor for $H^2(X, \mathbb{Z})$

The wall-crossing formula

Suppose that $b_+ = 1$ and $b_1 = 0$ for simplicity. If $d(\mathfrak{s}) \geq 0$ and even, and if $c_\pm$ denote the chambers for $\mathfrak{s}$, then:

$$|SW_{X,\sigma}(\mathfrak{s}, c_+) - SW_{X,\sigma}(\mathfrak{s}, c_-)| = 1$$

The proof of this will be postponed.

5.8.1 Configuration Spaces (revisited)

Recall the configuration space $\mathcal{C} = \mathcal{C}(\mathfrak{s}) = \{(A, \phi)\}$. There are various quotients of this space. For a fixed base point $x \in X$, we have $B_x = \mathcal{C}/\mathcal{G}^x$ (where $\mathcal{G}^x$ denotes the gaugue transformations that are trivial at the point x. There is also $B = B_x/U(1) = \mathcal{C}/\mathcal{G}$ and $\hat{B}_x = \mathcal{C}/\mathcal{G}_0^x$, the former of which representing the Coulomb gauge slice. Finally there is $\hat{B} = \mathcal{C}/\mathcal{G}_0$. We also distinguished the irreducible solutions $\mathcal{C}^{irr}$, which satisfy $\phi \neq 0$, and so the irr superscript carries through all of the above quotients.

There is a projection $B_x^{irr} \to B^{irr}$, which is a principal $U(1)$ bundle with first Chern class $c \in H^2(B^{irr}, \mathbb{Z})$. Recall that the orbits of $\mathcal{G}_0^x$ are $H^1_{dR}(X) \times \operatorname{im} d^*$ and the $\mathcal{G}^x$ orbits are $H^1(X,\mathbb{R})/H^1(X,\mathbb{Z}) \times \operatorname{im} d^*$. Therefore:

$$\hat{B}_x = H^1(X, \mathbb{R}) \times \operatorname{im} d^* \times \Gamma(\mathbb{S}^+)$$

and moreover:

$$B^{irr} = \mathcal{C}^{irr}/\mathcal{G} \cong \frac{H^1(X,\mathbb{R})}{H^1(X,\mathbb{Z})} \times \operatorname{im} d^* \times \mathbb{P}\Gamma(\mathbb{S}^+) \times (0, \infty)$$

Note that B^{irr} deformation retracts to a copy of $H^1(X,\mathbb{R})/H^1(X,\mathbb{Z}) \times \mathbb{P}H$, where H is the Hilbert space $L^2_k(\mathbb{S}^+)$.

Lemma 5.37. *Let H be a separable complex Hilbert space with an orthonormal basis $\{e_i\}$. Write $\mathbb{CP}^\infty$ as $\bigcup_{n\geq 1} \mathbb{PC}\{e_1, ..., e_n\}$ and let $i : \mathbb{CP}^\infty \to \mathbb{P}H$ be the inclusion. Then i is a weak homotopy equivalence.*

Proof idea:

> Look at spheres $S^\infty = \bigcup_n S\mathbb{C}\{e_1, ..., e_n\}$ and $S(H)$, which are both contractible. There is an induced map $\tilde{i} : S^\infty \to S(H)$. Then there are fibrations showns horizontally below:
>
> $$\begin{tikzcd} U(1) \arrow[r] \arrow[d, equal] & S^\infty \arrow[r] \arrow[d, "\tilde{i}"] & \mathbb{CP}^\infty \arrow[d, "i"] \\ U(1) \arrow[r] & S(H) \arrow[r] & \mathbb{P}H \end{tikzcd}$$
>
> Now compute homotopy exact sequences for the fibrations to see that i_* is an isomorphism on homotopy groups.

□

The upshot is that $H^*(B^{irr}) \cong H^*(H^1(X,\mathbb{R})/H^1(X,\mathbb{Z}) \times \mathbb{CP}^\infty) \cong \Lambda H^1(X,\mathbb{Z})^* \otimes \mathbb{Z}[c]$, the second congruence by Künneth.

5.8.2 Constructing the Invariants

Suppose $b^+ > 0$. For generic (g, η), we get a compact $d(\mathfrak{s})$ dimensional manifold $M_\eta \subset B^{irr}$ of solutions to $\mathcal{F}_\eta = 0$ modulo $\mathcal{G}$ (here we pick (g, η) such that there do not exist reducibles).

Proposition 5.38. *There's a real line bundle $\underline{\det \operatorname{ind}} \to B$ and a canonical isomorphism $\underline{\det \operatorname{ind}}|_{M_\eta} \cong \det TM_\eta$ for all (g, η) such that M_η is regular. Moreover, a choice of σ, a homology orientation, determines an orientation for $\underline{\det \operatorname{ind}}$. Thus, σ orients TM_η.*

Thus, the element $[M_\eta] \in H_{d(\mathfrak{s})}(B^{irr})$ has a sign depending on σ. With this in mind, we define:

$$SW_{X,\sigma} = \begin{cases} \langle c^{d(\mathfrak{s})/2}, [M_\eta]\rangle & d(\mathfrak{s}) \geq 0, \text{ even} \\ 0 & \text{else} \end{cases}$$

One can define a more elaborate SW invariant value din $\Lambda H^1(X)$ capturing $[M_\eta]$ in full. This can be shown to be independent of the choice of (g, η) by considering a path of such pairs (see Tim's notes for details).

5.8.3 A few words on $\underline{\det \operatorname{ind}}$

If H_0, H_1 are Hilbert spaces (both over $\mathbb{R}$ or both over $\mathbb{C}$), then $\mathrm{Fred}(H_0, H_1)$ is an open subset of the bounded linear maps $H_0 \to H_1$. The index $\operatorname{ind} : \mathrm{Fred}(H_0, H_1) \to \mathbb{Z}$ is locally constant. It is a fact that there is a well-defined virtual vector bundle (in the sense of K theory) $\underline{\operatorname{ind}} \to \mathrm{Fred}(H_0, H_1)$ with fibers over some L are stably isomorphic to $\ker(L) - \operatorname{coker}(L)$, thought of as a virtual vector space. Then $\underline{\operatorname{ind}}$ has a well-defined determinant line $\underline{\det \operatorname{ind}} \to \mathrm{Fred}(H_0, H_1)$ with $\underline{\det \operatorname{ind}}_L \cong \det(\ker L) \otimes \det(\operatorname{coker} L)^*$. So if L is surjective, then $\underline{\det \operatorname{ind}}_L = \det(\ker L)$.

5.9 Symplectic 4-manifolds ❖

5.9.1 The canonical Spin^c structure

Let $(V, \langle , \rangle)$ be a $2n$ dimensional real inner product space and let $J \in SO(V)$ be a complex structure on V (i.e. $J^2 = -\mathrm{id}$). Then (V, J) is a complex vector space with $i \cdot v = Jv$. Then we have a decomposition $V^* \otimes \mathbb{C} = \hom_{\mathbb{R}}(V, \mathbb{C}) = V^{1,0} \oplus V^{0,1}$, where $V^{1,0} = \hom_{\mathbb{C}}(V, \mathbb{C})$ (i.e. the $+i$ eigenspace of J^*) and $V^{0,1}$ is the $\mathbb{C}$ antilinear maps (the $-i$ eigenspace of J^*). Both are isotropic with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}}$, so that $V^* \otimes \mathbb{C}$ is polarized. This defines a spinor representation $S = \Lambda^\bullet V^{0,1}$, which are spinors for $\mathrm{Cl}(V^* \otimes \mathbb{C})$. The Clifford map $\rho : V^* \to \mathrm{End}_{\mathbb{C}} S$ is $\rho(e) = \sqrt{2}(e^{0,1} \wedge \cdot - \iota(\overline{e^{1,0}}))$, where ι denotes metric contraction. Then $S = S^+ \oplus S^-$, where $S^+ = \Lambda^{even} V^{0,1}$ and $S^- = \Lambda^{odd} V^{0,1}$.

Globally, if (M^{2n}, g) is Riemannian, an almost complex structure $J \in \Gamma(SO(TM))$ determines an orientation for TM and a Clifford module $\rho : T^*M \to \mathrm{End}_{\mathbb{C}}(\Lambda^\bullet_{\mathbb{C}}(T^*M)^{0,1}) = \mathrm{End}(\Lambda^{0,\bullet}(T^*M))$. This is actually a Spin^c structure $\mathfrak{s}_J$ on M, and moreover homotopic J's induce isomorphic $\mathfrak{s}_J$.

Lemma 5.39. *If J is an orientation compatible almost complex structure on X, then $d(\mathfrak{s}_J) = 0$.*

Proof:

We have $S^+ = \Lambda^{0,0} \oplus \Lambda^2_{\mathbb{C}}(T^*X)^{0,1}$. Denoting the second summand by $\Lambda^{0,2}_J$, we have $\Lambda^2 S^+ = \Lambda^{0,2}_J = (\Lambda^{2,0}_J)^* = \Lambda^2_{\mathbb{C}}(TX, J)$. Therefore $c_1(\Lambda^2 S) = c_1(\Lambda^2_{\mathbb{C}}(TX, J)) = c_1(TX, J)$. Moreover, $p_1(TX) = c_1^2(TX, J) - 2c_2(TX, J) = c_1(\mathfrak{s}_J)^2 - 2e(TX)$. Evaluating on $[X]$:

$$p_1(TX)[X] = c_1(\mathfrak{s}_J)^2[X] - 2\chi(X)$$

But by Hirzebruch $p_1(TX)[X] = 3\tau(X)$. Therefore $d(\mathfrak{s}_J) = 0$.

□

Remark 5.40. The converse actually holds too.

Definition 5.41. If (M^{2n}, ω) is symplectic, then a *compatible* almost complex structure J is one for which $g(u, v) := \omega(u, Jv)$ defines a Riemannian metric.

The set of such J's form a contractible space. Therefore (M^{2n}, ω) has a canonical Spin^c structure $\mathfrak{s}_{can} = \mathfrak{s}_J$ for any J compatible. Any of these are isomorphic because all such J are homotopic. We also denote $K_M := \det_{\mathbb{C}}(T^*M, J)$ (the canonical bundle) and $k = c_1(K_M) \in H^2(M, \mathbb{Z})$ (its chern class).

5.9.2 Taubes's Constraints

Let $(X^4, \omega 0$ be a symplectic oriented by $\omega \wedge \omega$ and J a compatible almost complex structure for $\mathfrak{s}_{can}$. Then $S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$ and $S^- = \Lambda^{0,1}$.

Theorem 5.42. *WIth things defined as above, there exists a canonical solution (A_{can}, ϕ_{can}) to the Dirac equation $D_A^+\phi = 0$ for $\mathfrak{s}_{can}$. Moreover, for $\tau > 0$, $(A_{can}, \sqrt{\tau}\phi_{can})$ is a solution to the SW equations $\mathcal{F}_{\eta(\tau)} = 0$, where $\eta(\tau) = iF(A^0_{can})^+ + \frac{1}{2}\tau\omega$.*

The parameter τ is called the Taubes parameter and $(A_{can}, \sqrt{\tau}\phi_{can})$ is called the Taubes monopole. Trivialize the $H^2(X)$-torsor $\text{Spin}^c(X)$ by taking $\mathfrak{s}_{can}$ as an origin. Recall that the conjugation invariance of $SW_{X,\sigma} : H^2(X) \to \mathbb{Z}$ amounts to $SW_{X,\sigma}(k - e) = \pm SW_{X,\sigma}(e)$.

Taubes's Constraints: Assume $b^+(X) > 1$. Then there is a canonical homology orientation σ for which:

1. The Taubes monopole is the unique solution for $\mathcal{F}_{\eta(\tau)} = 0$ (mod gauge) provided $\tau \gg 0$. It is regular and one has:
$$SW_{X,\sigma}(0) = 1, \qquad SW_{X,\sigma}(K) = (-1)^{1-b_1+b^+}$$
2. If $SW_{X,\sigma}(e) \neq 0$ then:
$$0 \leq e \cdot [\omega] \leq K \cdot [\omega]$$
with equality in one of these if and only if $e = 0$ or K.

Why are these constraints? Consider the following question. Suppose M^{2n} is a closed oriented manifold, $w \in H^2(M, \mathbb{R})$. Does there exist a symplectic form ω such that $[\omega] = w$? There are two obvious constraints for this; namely $w^n[M] > 0$ and there needs to be an almost complex structure J inducing the orientation of M. In dimensions ≥ 6, these are the only known constraints. In dimension 4, we also have the above Taubes constraints. When X is simply connected, these are the only additional constraints that are known.

Corollary 5.43. *If $b^+(X) > 1$, then $K \cdot [\omega] \geq 0$.*

Corollary 5.44 (Taubes)**.** *There exists a symplectic surface*[10] *$S \subset X$ with $[S] = PD(k)$.*

Corollary 5.45. *If X is a $K3$ surface or a 4 torus (so that k is trivial), then $SW(0) = 1$ and $SW(e) = 0$ for $e \neq 0$.*

5.9.3 Geometry of almost complex manifolds

Let (M, J) be an almost complex manifold. Recall we had the decomposition $T^*M \otimes \mathbb{C} = \hom_{\mathbb{R}}(TM, \mathbb{C}) = T^{1,0}M \oplus T^{0,1}M$. We have projections $\pi^{1,0} = \frac{1}{2}(1 - iJ) : T^*M \to T^{0,1}M$ and $\pi^{0,1} = \frac{1}{2}(1 + iJ) : T^*M \to T^{0,1}M$ which are $\mathbb{C}$ linear. We also have the following decomposition:

$$\begin{aligned} \Lambda^k_{\mathbb{R}}(T^*M) \otimes \mathbb{C} &= \Lambda^k_{\mathbb{C}}(T^*M \otimes \mathbb{C}) \\ &= \Lambda^k_{\mathbb{C}}(T^{1,0} \oplus T^{0,1}) \\ &= \bigoplus_{p+q=k} \Lambda^{p,q} \end{aligned}$$

Where $\Lambda^{p,q} = \Lambda^p_{\mathbb{C}} T^{1,0} \oplus \Lambda^q_{\mathbb{C}} T^{0,1}$. Denote $\Omega^{p,q} = \Gamma(\Lambda^{p,q})$. We also have the following maps coming from the exterior derivative d:

$$\frac{1}{2}(1 - iJ)d = \partial_J : \Omega^{p,q} \to \Omega^{p+1,q}$$
$$\frac{1}{2}(1 + iJ)d = \bar{\partial}_J : \Omega^{p,q} \to \Omega^{p,q+1}$$

[10] A symplectic surface is one for which $\omega|_S$ is a positive area form.

The *Nijenhuis tensor* $N_J : \Lambda^2 TX \to TX$ is defined on vector fields by:

$$N_J(u,v) = [Ju, Jv] - [u,v] - J[Ju, v] - J[u, Jv]$$

This is C^∞ linear. This arrives as an obstruction to the integrability of J (i.e. the existence of holomorphic coordinates). If we complexify, we get $N_J : \Lambda^2_{\mathbb{C}}(TX_{\mathbb{C}}) \to TX_{\mathbb{C}}$. One can check that N_J sends $\Lambda^2 T_{0,1} \to T_{1,0}$. Dualizing this gives a map $N_J^* : \Lambda^{1,0} \to \Lambda^{0,2}$.

Lemma 5.46. *For any function f on M, we have $\bar{\partial}_J f = -\frac{1}{4} N_J^* \circ \partial_J f$.*

5.9.4 More on symplectic 4 manifolds

Let X be a four manifold and take (g, J, ω) a compatible triple (i.e. a triple satisfying $g(u,v) = \omega(u, Jv)$.) At $x \in X$, there exists an orthonormal basis $\{e_1, ..., e_4\}$ for $T_x^* X$ such that:

$$\omega(x) = e_1 \wedge e_2 + e_3 \wedge e_4, \quad Je_1 = e_2, Je_3 = e_4$$

Notice that $|\omega|_g^2 = 2$ and $*_g \omega = \omega$. Therefore ω is self-dual harmonic. One can check that:

$$\Lambda_g^+ \otimes \mathbb{C} = \Lambda^{2,0} \oplus \mathbb{C} \cdot \omega \oplus \Lambda^{0,2}$$

$$\Lambda_g^- \otimes \mathbb{C} = \Lambda_0^{1,1} \ (= \omega^\perp \text{ in } \Lambda^{1,1})$$

If we define $L = \omega \wedge (-) : \Lambda^k \to \Lambda^{k+2}$, then it has an adjoint L^* and $L^* \omega = 2$ and $L^*(\Lambda_0^{1,1}) = 0$. If $\eta \in \Omega_g^+$, then $\eta = \eta^{2,0} + \frac{1}{2} L^* \eta \cdot \omega + \overline{\eta^{2,0}}$. Therefore the relevant data for η is $(\eta^{2,0}, L^* \eta)$.

In the spinor setting, one can check that $\rho(\omega) \in \mathfrak{su}(S^+)$ has the form $\rho(\omega) = \frac{1}{2}[\rho(e_1), \rho(e_2)] + \frac{1}{2}[\rho(e_3), \rho(e_4)]$ and has matrix representation:

$$\rho(\omega) = \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix}$$

where we are decomposing $S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$. Therefore, we can use the action of $\rho(\omega)$ on a spinor $\phi \in \Gamma(S^+)$ to distinguish the summands $\Lambda^{0,0} = \underline{\mathbb{C}}$ and $\Lambda^{0,2}$. Also, for $\beta \in \Lambda^{0,2}$, we have:

$$\rho(\beta) = 2 \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}, \qquad \rho(\bar{\beta}) = 2 \begin{bmatrix} 0 & \bar{\beta} \\ 0 & 0 \end{bmatrix}$$

5.9.5 Almost Kähler Manifolds

Let M^{2n} be a smooth manifold and pick a compatible triple (g, J, ω) as above. We say that such a triple is an *almost Kähler structure* if $d\omega = 0$, i.e ω is symplectic. Morevoer, $\omega \in \Omega_J^{1,1}$ since $\omega(Ju, Jv) = \omega(u,v)$. It is called *Kähler* if J is integrable. At $x \in M$, there exists an orthonormal basis $(e_1, ..., e_n, f_1, ..., f_n)$ for $T_x^* M$ such that:

$$\omega(x) = \sum_j e_j \wedge f_j$$

$$Je_j = f_j$$

In what follows, $(E, (\cdot,\cdot)_E)$ is a hermitian vector bundle over M, which is almost Kähler.

Lemma 5.47. *For a unitary connection A in E, letting $\bar{\partial}_A : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ be the projection of d_A to $\Omega^{p,q+1}(E)$, we have:*

$$\bar{\partial}_A^2 = F_A^{0,2} - \frac{1}{4} N_J^* \circ \partial_A$$

as acting on $\Omega^{0,0}(E) = \Gamma(E)$.

(Almost) Kähler identities: Like above, consider the operator $L_\omega = \omega \wedge (-) : \Omega^{p,q} \to \Omega^{p+1,q+1}$ and let L_ω^* be the adjoint with respect to g. Then:

1. $\bar{\partial}_A = iL_\omega^* \circ \partial A$ on $\Omega_M^{0,1}(E)$.
2. $\partial_A^* = -iL_\omega^* \circ \bar{\partial}_A$ on $\Omega_M^{1,0}(E)$.

Lemma 5.48. *Let ∇_A denote the covariant derivative associated to the connection A. The Weitzenbock formula in this setting is:*

$$\frac{1}{2}\nabla_A^*\nabla_A = \bar{\partial}_A^*\bar{\partial}_A + iL_\omega^*(F_A)$$

Proof:

The covariant derivative splits as $\nabla_A = \partial_A + \bar{\partial}_A$, hence $\nabla_A^* = \partial_A^* + \bar{\partial}_A^*$. Using the Almost Kähler identities:

$$\nabla_A^*\nabla_A = iŁ_\omega^*(-\bar{\partial}_A + \partial_A)(\partial_A + \bar{\partial}_A) = iL_\omega^* \circ [\partial_A, \bar{\partial}_A]$$

Also by the Kähler identities, we have $2\bar{\partial}_A^*\bar{\partial}_A = 2iL_\omega^* \circ \partial_A \circ \bar{\partial}_A$. Moreover, since:

$$L_\omega^* F_A = L_\omega^*(\bar{\partial}_A\partial_A + \partial_A\bar{\partial}_A)$$

we get the result by comparing terms.

□

5.9.6 A canonical solution to the Dirac equation

Let X^4 be a four manifold with an almost Kähler triple (ω, J, g) and consider the canonical Spin^c structure $\mathfrak{s}_{can}$. Then there is a distinguished spinor $\phi_{can} = 1 \in \Gamma(\Lambda^{0,0})$. There is also a distinguished clifford connection $A_{can} \in \mathcal{A}_{\mathrm{Cl}}(S^+)$. It is characterized by $\nabla_{A_{can}}\phi_{can} \in \Omega_X^1(\Lambda^{0,2})$ (whereas in general, $\nabla_A\phi_{can} \in \Omega_X^1(\Lambda^{0,0} \oplus \Lambda^{0,2})$). Let D^+ denote $D_{A_{can}}^+$.

Theorem 5.49. *In the setting above, we have $D^+\phi_{can} = 0$ and D^+ takes the form:*

$$D^+ = \sqrt{2}(\bar{\partial}_J \oplus \bar{\partial}_J^*)$$

Lemma 5.50. *For $\gamma \in \Omega_X^k$ and $\phi \in \Gamma(S)$, we have:*

$$\tilde{\rho}(\nabla\rho(\gamma)\phi) - \rho(\delta\gamma)\phi = \tilde{\rho}(\rho(\gamma(\nabla\phi))$$

where $\delta = d + d^*$ and $\tilde{\rho}$ is:

$$\begin{tikzcd} T^*X \otimes S \arrow[r, "\rho\otimes\mathrm{id}"] \arrow[rr, bend right, "\tilde{\rho}"'] & \mathrm{End}(S) \otimes S \arrow[r, "ev"] & S \end{tikzcd}$$

Proof of Theorem 5.49:

Letting $\Omega = \frac{1}{2i}\rho(\omega) : S^+ \to S^+$, this takes the matrix form $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ by our work in subsection 5.9.4. Thus $\Omega\phi_{can} = -\phi_{can}$ and $\Omega(\nabla_{A_{can}}\phi_{can}) = \nabla_{A_{can}}\phi_{can}$. Since $\delta\omega = d\omega + d^*\omega = 0$, we have:

$$\tilde{\rho}(\nabla\Omega\phi_{can}) = \tilde{\rho}(\Omega\nabla\phi_{can})$$

The left hand side is $D^+(\Omega\phi_{can}) = -D^+\phi_{can}$, whereas the right hand side is $\tilde{\rho}(\nabla\phi_{can}) = D^+\phi_{can}$. Therefore $D^+\phi_{can} = 0$.

The second claim is proved in Tim's notes. The idea is to notice that D^+ and $\sqrt{2}(\bar{\partial} \oplus \bar{\partial}^*)$ have the same symbol and use that both kill ϕ_{can}, so that it suffices to check that they agree on $\beta \in \Gamma(\Lambda^{0,2})$.

□

We return to the SW equations under some Spin^c structure $\mathfrak{s} = L \otimes \mathfrak{s}_{can}$, where L is some line bundle. In this case, we have $S^+ = L \otimes (\Lambda^{0,0} \oplus \Lambda^{0,2}) = L \oplus (L \otimes \Lambda^{0,2})$ and $S^- = L \otimes \Lambda^{0,1}$. Given $A \in \mathcal{A}_{\mathrm{Cl}}(S^+)$, write it as $\nabla = \mathrm{id}_L \otimes \nabla_{can} + \nabla_B \otimes \mathrm{id}$, where B is a unitary connection in L. The associated Dirac operator is $D_B^+ = \sqrt{2}(\bar{\partial}_B \oplus \bar{\partial}_B^*)$ by the previous theorem, and the curvature of A^0 is $F(A^0) = F(\nabla^0_{can}) + 2F(B)$. The second SW equation invokes $F^+ := F(A^0)^+ = F(\nabla^0_{can})^+ + 2F_B^+$. Because we have a decomposition:

$$(iF)^+ = iF^{2,0} + \frac{i}{2}(L_\omega^* F)\omega + \overline{iF^{2,0}}$$

giving F^+ is equivalent to giving $L_\omega^*(iF)$ and $F^{2,0}$. Any spinor ϕ is written as $\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$, where $\alpha \in \Lambda^{0,0}$ and $\beta \in \Lambda^{0,2}$. Then:

$$\phi\phi^* = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}\begin{bmatrix}\bar{\alpha} & \bar{\beta}\end{bmatrix} = \begin{bmatrix}|\alpha|^2 & \alpha\bar{\beta} \\ \bar{\alpha}\beta & |\beta|^2\end{bmatrix}$$

Therefore:

$$(\phi\phi^*)_0 = \begin{bmatrix}\frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2)\end{bmatrix}$$

The curvature SW equation says $\frac{1}{2}\rho(F^+ - 4i\eta) = (\phi\phi^*)_0$ for some $\eta = \Omega^+$. We write $\eta = \frac{1}{4i}F(\nabla^0_{can})^+ + \eta_0$. Then the curvature equation is $\rho(F_B^+ - 2i\eta_0)(\phi\phi^*)_0$. Taubes's choice of η_0 is $\eta_0 = -\frac{1}{4}\tau\omega$ for $\tau \gg 0$. Thus the SW equations with the Taubes parameter τ are:

$$\boxed{\begin{gathered}\bar{\partial}_B\alpha = -\bar{\partial}_B^*\beta \\ F_B^{0,2} = \tfrac{1}{2} = \tfrac{1}{2}\bar{\alpha}\beta \\ L_\omega^*(iF_B) = \tfrac{1}{4}(|\beta|^2 - |\alpha|^2 - \tau)\end{gathered}}$$

The canonical solution (the Taubes monopole) takes $L = \underline{\mathbb{C}}$, B the trivial connection. The solution is $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \sqrt{\tau}\phi_{can}$. The Taubes constraints largely follow from:

Proposition 5.51. *There exists a constant $C = C(X, g, J)$ such that if $e \in c_1(L), e \cdot [\omega] \leq 0$ and (B, α, β) is a SW solution with taubes parameter $\tau > 0$, then L is trivial ($e = 0$) and in suitable gaugue B is trivial, $\beta = 0$ and $\alpha = \sqrt{\tau}\phi$.*

5.10 Applications ❖

5.10.1 Proof of Rokhlin's Theorem

Rokhlin's Theorem is almost an immediate corollary of the Atiyah-Singer index theorem, which says $\mathrm{ind}_{\mathbb{C}} D^+ = -\frac{1}{8}\tau_X$, where τ_X is the signature of a spin 4-manifold X. We note that D^+ is not only $\mathbb{C}$ linear, but it is actually $\mathbb{H}$-linear. Therefore $\mathrm{ind}_{\mathbb{H}} D^+$ is well-defined and is half of $-\frac{1}{8}\tau_X$. Therefore $\frac{\tau_X}{16}$ is an integer.

5.10.2 Symplectic Thom Conjecture

Let X^4 be a closed, oriented manifold, and pick $\sigma \in H_2(X)$. We can represent σ by oriented embedded surfaces $\Sigma \subset X$. The minimal genus problem asks what the minimal genus of all connected representatives of σ is. It is easy to raise the genus of a representative by 1 through gluing a torus contained in a chart of X to Σ. As of yet, there isn't a complete answer to this problem. One can also ask the same question but allowing disconnected surfaces. In such a case, let $\Sigma = \coprod_i \Sigma_i$ and define:

$$\chi_-(\sigma) = \sum_{g(\Sigma_i) > 0} (2g(\Sigma_i) - 2)$$

Then the relevant question is: which Σ minimize χ_- within σ?

Definition 5.52. If $b^+(X) > 1$, a characteristic vector $c \in H^2(X)$ is called a *basic class* if there exists a Spin^c structure $\mathfrak{s}$ with $c_1(\mathfrak{s}) = c$ such that $SW_X(\mathfrak{s}) \neq 0$.

Theorem 5.53 (Adjunction Inequality)**.** *Suppose $b^+(X) > 1$ and suppose $\Sigma \subset X$ is an oriented embedded surface with nonnegative normal bundle (i.e. for each component Σ_i of Σ, the self intersection $\Sigma_i \cdot \Sigma_i$ is nonnegative). Let $\sigma = [\Sigma]$. Then for all basic classes c, we have:*

$$\chi_-(\Sigma) \geq \langle c, \sigma \rangle + \sigma \cdot \sigma$$

Therefore each basic class gives a nontrivial lower bound on $\chi_-(\Sigma)$.

Remark 5.54. There is a generalization of this theorem to surfaces with negative self-intersection due to Ozváth-Szabó.

We will use this to deduce the

Symplectic Thom Conjecture: Let X, Σ be as before and assume there are no spherical components Σ_i. If there exists a symplectic form ω with $\omega|_{T\Sigma} > 0$, then Σ minimizes χ_-.

The heart of the conjecture is the following special case of the adjunction inequality:

Proposition 5.55. *Assume $b^+(X) > 1$. If $\sigma = [\Sigma]$ where Σ has trivial normal bundle, then for all basic classes c, one has that $\chi_-(\Sigma) \geq \langle c, \sigma \rangle$.*

Proof:

Gauss-Bonnet says that for a metric h on Σ, we have $\int_\Sigma \mathrm{scal}_h \mathrm{vol}_h = 4\pi\chi(\Sigma)$ because scalar curvature is twice Gauss curvature. SW theory is related to scalar curvature, so we want to use the fact that the scalar curvature gives you bounds on the spinor and self dual curvature of a solution to the SW equations and play that off of the Gauss-Bonnet formula. The fact that c is a basic class gives us $\mathfrak{s} \in \mathrm{Spin}^c(X)$ such that $SW_(\mathfrak{s}) \neq 0$ and $c_1(\mathfrak{s}) = c$, which implies that there exists a solution to the SW equations for $\mathfrak{s}$ for *any* pair (g, η) (i.e. transversality is a moot condition). So we take $\eta = 0$ and a convenient choice of g in the following sense. There is a tubular neighborhood of Σ, which we'll identify with $\Sigma \times D^2(2)$, where $D^2(r)$ is the closed disk of radius r. Then inside this neighborhood is the annulus $\Sigma \times (D^2(2) \setminus D^2(1)) \cong \Sigma \times S^1 \times [0,1]$. We take a metric g_1 on $\Sigma \times S^1 \times [0,1]$ given by $g_1 = h \oplus dy^2 \oplus dz^2$, where h on Σ has constant scalar curvature and $\mathrm{vol}_h(\Sigma) = 1$, and $y \in S^1$ and $z \in [0,1]$. Now for $t \geq 1$, we replace $\Sigma \times S^1 \times [0,1]$ by $\Sigma \times S^1 \times [0,t]$. This results in a manifold X_t with metric:

$$g_t = \begin{cases} h \oplus dy^2 \oplus dz^2 & \text{on } S^1 \times [0,t] \\ g_1 & \text{elsewhere} \end{cases}$$

Note that $\mathrm{scal}(g_t) = 4\pi(2g(\Sigma_i) - 2)$ on $\Sigma_i \times S^1 \times [0,t]$, where $\Sigma = \coprod_i \Sigma_i$. Then set $s_-(t) = \max(0, -\mathrm{scal}(g_t))$, which is positive. Thus:

$$||s_-(t)||_{L^2} = 4\pi\chi_-(\Sigma)t^{1/2}$$

If (A_t, ϕ_t) solves the SW equations for $(X_t, g_t, \mathfrak{s})$, then we have pointwise bounds:

$$|F_t^+|^2_{g_t} \leq \frac{1}{8} s_-(t)^2$$

where F_t is $F(A_t^0)$. Then:

$$||F_t^+||^2_{\Sigma \times S^1 \times [0,t]} \leq \frac{1}{8}||s_-(t)||^2 = 2\pi^2\chi_-^2 \cdot t$$

This means:

$$||F_t^+||^2_{X_t} \leq 2\pi^2\chi_-^2 \cdot t + S$$

where S is independent of t. We also have:

$$||F_t||^2_{X_t} = ||F_t^2 + ||^2 + ||F_t^-||_{X_t} = 2||F_t||^2_{X_t} + 4\pi^2 c^2[X]$$

by Chern-Weil theory. Therefore we have the bound:

$$||F_t||^2 X_t \leq 4\pi^2\chi_-^2 t + C$$

$$\Rightarrow ||F_t||_{X_t} \leq 2\pi\chi_-\sqrt{t} + \sqrt{C}$$

where C is independent of t. But, we claim that if ω is any closed 2 form on X_t, then we have::

$$||\omega||_{X_t} \geq t^{1/2}\int_\Sigma \omega$$

Taking $\omega = F_t$, we get:

$$||F_t||_{X_t} \geq 2\pi t^{1/2}\langle c, \sigma\rangle$$

because $\frac{i}{2\pi}[F_t] = c$. Therefore:

$$2\pi t^{1/2}\langle c, \sigma\rangle \leq ||F_t||_{X_t} \leq 2\pi\chi_- \cdot t^{1/2} + \sqrt{c}$$

Dividing both sides by $t^{1/2}$ gives:

$$\langle c, \sigma\rangle \leq \chi_- + \frac{\sqrt{c}}{2\pi t^{1/2}}$$

Sending $t \to \infty$ gives the result.

□

With this special case, we can prove the general adjunction inequality via blowups. Suppose $x \in \Sigma \subset X$; then blowing X at x results in a manifold $\tilde{X} \cong X\#\overline{\mathbb{CP}^2}$. We can take Σ to be complex in holomorphic coordinates near x. Let $\widetilde{\Sigma} \subset \widetilde{X}$ be the strict transform of Σ. If e is the exceptional curve, then $H_2(\widetilde{X}) = H_2(X) \oplus \mathbb{Z}e$. If $\sigma = [\Sigma]$ and $\tilde{\sigma} = [\widetilde{\Sigma}] = \sigma - e$, then $(\sigma - e)^2 = \sigma \cdot \sigma - 1$. Then blowing up reduces by 1 the self intersection number by 1 but doesn't change $g(\Sigma)$. Repeatedly doing so will give us a surface with $\Sigma \cdot \Sigma = 0$. One can then deduce the adjunction formula for $\Sigma \cdot \Sigma$ nonnegative from the case of trivial normal bundles once you know the fact $SW_{\tilde{X}}(\tilde{c}) = SW_X(c)$, where $\tilde{c} = c + e$.

As far as the symplectic Thom conjecture, it is a fact that blowing up a symplectic manifold X results in a symplectic manifold $\widetilde{X}$ and $K_{\tilde{X}} = K_X + e$. After doing so, we have $\chi_-(\sigma) \geq \widetilde{K} \cdot \tilde{\sigma} + \tilde{\sigma} \cdot \tilde{\sigma} = K \cdot \sigma + \sigma \cdot \sigma$. Finally, we have the easy adjunction formula in the case where Σ is symplectic:

$$\chi(\Sigma) = k \cdot \sigma + \sigma \cdot \sigma$$

by some topology. But if there are no spheres in Σ, we have $\chi(\Sigma) = \chi_-(\Sigma)$, hence the Symplectic Thom conjecture.

Appendix

A.0 Solutions to Selected Exercises ❖

Excercise 2.3: Recall the following notions: a *Riemannian structure* on a manifold M at each point; it is smoothly varying in the following sense: if X and Y are two smooth vector fields on M, then $\langle X, Y\rangle$ is a smooth function on M. Every manifold can be given a Riemannian structure. We quote a general result in Riemannian geometry which says that every Riemannian manifold has a geodesically convex neighborhood. The intersection of any two such neighbourhood is again geodesically convex. Since a geodesically convex neighbourhood in a Riemannian manifold of dimension n is diffeomorphic to $\mathbb{R}^n$, an open cover consisting of geodesically convex neighbourhoods will be a good cover.

Another way to prove this is to embed M into $\mathbb{R}^N$ for some N and take balls around each point of M sufficiently small.

Exercise 2.7: Fix an integral basis $(e_1, \ldots, e_b)$ of $H^n(M)'$ and take the dual basis $(e_1^*, \ldots, e_b^*)$ defined by $e_i^*(e_j) = \delta_i^j \in \mathbb{Z}$). Denote with L the group homomorphism

$$H^n(M)' \to \hom(H^n(M)'; \mathbb{Z})$$

induced by the cup-product form. Now $L(e_i) = \sum_j Q_M(e_j, e_i)e_j^*$, the matrix of L in these bases is $[Q(e_i, e_j)]_{ij}$. A matrix B over $\mathbb{Z}$ is invertible (over $\mathbb{Z}$) if and only if $\det B \pm 1$.

Exercise 2.22:

1. Let us fix some notation. The matrix representing Λ is the matrix whose columns are a basis for Λ. Call it A. Similarly we will denote with B the matrix representing Λ'. Now notice that the matrix A (resp. B) represents the change of basis from the one spanning Λ (resp Λ') to the standard one.

   Lemma A.1. *Let L_1 be a $\mathbb{Z}$-module, and let $L_2 \subset L_1$ be a submodule of it. Assume $L_1 \simeq L_2 \simeq \mathbb{Z}^n$, for some $n \in \mathbb{N}$. Then*

   $$\left|\frac{L_1}{L_2}\right| < \infty$$

   and moreover, if $L_1 = span_{\mathbb{Z}}\{v_1, \ldots, v_n\}$ and $L_2 = span_{\mathbb{Z}}\{w_1, \ldots, w_n\}$ with $w_i = \sum_{i=1}^n c_{ij}v_j$, then

   $$\left|\frac{L_1}{L_2}\right| = \det C$$

   where $C = \{c_{ij}\}_{i,j}$.

   Proof: One can choose the basis $\{v_i\}_{i=1}^n$ of L_1 and $\{w_i\}_{i=1}^n$ of L_2 in such a way that $w_i = \beta_i v_i$, with $\beta_i \in \mathbb{N}$ for every i. In this way C is a diagonal matrix whose diagonal entries $c_{ii} = \beta_i$. Then clearly

   $$\frac{L_1}{L_2} \simeq \frac{\mathbb{Z}}{\beta_1\mathbb{Z}}v_1 \oplus \cdots \oplus \frac{\mathbb{Z}}{\beta_n\mathbb{Z}}v_n$$

   and the cardinality of the quotient is clearly $\prod_{i=1}^n \beta_i = \det C$.
   So how do we choose such basis? Start with the matrix A, and up to reordering columns and rows (which only change the ordering of the basis), we can assume that a_{11} is the integer with minimum absolute value. Start subtracting the first row/column in order to clean the first column/row (with the only exception being a_{11}). After each passage, make sure that in position a_{11} there is always the integer with minimum absolute value. It's easy to see that this procedure, iterated for all a_{mm} once the $m-1$ rows and column are *clean*, lead to a diagonal matrix as claimed before. □

To conclude the exercise notice

$$A = B \cdot C^{-1}$$

Here you might notice that there is a slight discrepancy between the result and the suggested solution, I think it's a matter of convention of what is the matrix representing a lattice, with our convention the identity holds. What's more important, the following exercises turn out to be correct.

2. Notice that the determinant of the matrix representing $\mathbb{Z}^8$ is trivially 1, since we identify $\mathbb{Z}^8 \subset \mathbb{R}^8$ with the integer span of the standard basis. Now we need to find the matrix representing Γ. Notice that it has rank 8 and that the set of vectors

$$\begin{aligned} w_1 &= 2e_1 \\ w_2 &= e_1 + e_2 \\ w_3 &= e_1 + e_3 \\ &\vdots \\ w_8 &= e_1 + e_8 \end{aligned}$$

is a basis for the vectors $x \in \mathbb{R}^8$ such that $x \cdot x$ is even. Hence for dimension reason that must be a basis for our lattice. With our convention, the resulting matrix B will be

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix}$$

which is easy to see that it has determinant 2.

3. We will solve first point 3: Notice that a matrix for E_8 is the following one:

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1/2 \\ & 1 & & & & & & 1/2 \\ & & 1 & & & & & 1/2 \\ & & & 1 & & & & 1/2 \\ & & & & 1 & & & 1/2 \\ & & & & & 1 & & 1/2 \\ & & & & & & 1 & 1/2 \\ & & & & & & & 1/2 \end{pmatrix}$$

which has determinant 1. Now Consider the equation

$$\det \Gamma = [E_8 \colon \Gamma][\mathbb{Z}^8 \colon E_8]$$

which becomes $2 = [E_8 \colon \Gamma]$ proving the result.

4. We prove it in the previous point. $\det E_8 = 1$

Exercise 2.23: Let L be the tautological bundle on $\mathbb{RP}^n$. We will first compute $w(L)$. Since the pullback of L over a linear embedding $i : \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$ is the tautological bundle over $\mathbb{RP}^1$, it follows that $w_1(L)$ is nonzero because $i^* w_1(L) = w_1(L_0) \neq 0$ by the nontriviality axiom. Since L is a line bundle, there can be no higher degree cohomology classes so $w(L) = 1 + H$.

Now, to compute $w(T\mathbb{RP}^n)$, we claim that there is a natural isomorphism:

$$T\mathbb{RP}^n \cong \hom(L, L^\perp)$$

where $L^\perp$ is the orthogonal complement of L in the trivial bundle $\mathcal{E} = \mathbb{RP}^n \times \mathbb{R}^{n+1}$. The tangent space $T\mathbb{RP}^n$ can be identified with the set of pairs $\{(x, v), (-x, -v)\}$, where $x \in S^n$ and $\in T_x S^n$. Equivalently, to specify such a pair it suffices to provide a linear map on the fibers $\ell : L_x \to L_x^\perp$ sending the line x to the span of v. The claim then follows. Summing both sides with the trivial line bundle $\hom(L, L)$:

$$\begin{aligned} T\mathbb{RP}^n \oplus \hom(L, L) &= \hom(L, L^\perp) \oplus \hom(L, L) \\ &= \hom(L, L^\perp \oplus L) \\ &= \hom(L, \mathcal{E}) \end{aligned}$$

where we used that $L^\perp$ is the orthogonal complement of L in $\mathcal{E}$. Splitting up $\mathcal{E}$ into trivial line bundles gives:

$$T\mathbb{RP}^n \oplus \underline{\mathbb{R}} = \bigoplus^{n+1} \hom(L, \underline{\mathbb{R}}) = (L^*)^{\oplus n+1}$$

Then applying $w(-)$ and using the product axiom gives:

$$w(T\mathbb{RP}^n) \cdot 1 = w(L^*)^{n+1} \implies w(T\mathbb{RP}^n) = (1 + H)^{n+1}$$

where we used that $w(L^*) = -w(L) = 1 + H$, since $L \otimes L^*$ is a trivial bundle.

Exercise 3.19: Let $\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$ be a connection. It can be equivalently seen as a $\mathbb{C}$-linear map $\nabla : \Gamma(M, E) \otimes \Gamma(M, TM) \to \Gamma(M, E)$. Any $s \in \Gamma(M, E)$ determines a section $f^*s \in \Gamma(M', f^*E)$ given by $f^*s(x) = s(f(x))$. Then we claim there is a unique connection $f^*\nabla$ on f^*E satisfying:

$$(f^*\nabla)_v(f^*s) = f^*(\nabla_{df(v)} s) \tag{A.0.1}$$

for all $v \in T_x M'$ and $s \in \Gamma(M, E)$. Uniqueness follows from the fact that, over a trivializing neighborhood U of $y \in M$ there is a basis of nonvanishing sections which pull back to a basis of nonvanishing sections on the trivializing neighborhood $f^{-1}(U)$ of f^*E. Therefore the space of sections over $f^{-1}(U)$ are linear combinations of pullbacks of sections over U. Then defining $f^*\nabla$ only on pullback sections (as in equation A.0.1) determines the connection entirely, given the Leibniz rule.

Note that equation A.0.1 is the statment that the following diagram commutes:

$$\begin{tikzcd} \Gamma(M,E) \otimes T_y(M) \arrow[d, "\nabla"] & \Gamma(M,E) \otimes T_x M' \arrow[l, "\mathrm{id} \otimes df"'] \arrow[r, "f^* \otimes \mathrm{id}"] & \Gamma(M', f^*E) \otimes T_x M' \arrow[d, "f^*\nabla"] \\ \Gamma(M,E) \arrow[rr, "f^*"] & & \Gamma(M', f^*E) \end{tikzcd}$$

where $y = f(x)$. Given any smooth map $g : M'' \to M'$, this diagram can be used to show that $g^*(f^*\nabla) = (f \circ g)^*\nabla$ since $g^* \circ f^* = (f \circ g)^* : \Gamma(M, E) \to \Gamma(M'', (f \circ g)^*E)$.

Exercise 4.3: To show that this is well defined, let s, s' be sections of E with $s(x) = s'(x) = e$. Then we wish to show that $(Ls)(x) - (Ls')(x) = L(s - s')(x) = 0$. Let $\{s_i\}$ be a local basis of sections on a trivializing neighborhood of x. Then we can expand s and s' in this basis:

$$s = \sum_i f_i s_i$$

$$s' = \sum_i g_i s_i$$

where f_i, g_i are scalar functions. Then since $(s - s')(x) = 0$, we must have $f_i(x) - g_i(x) = 0$ due to linear independence of s_i. Now using the C^∞-linearity of L:

$$L(s - s')(x) = \left(\sum_i (f_i - g_i) L s_i \right)(x) = \sum_i (f_i(x) - g_i(x)) L s_i(x) = 0$$

To verify that this is an isomorphism, we first check injectivity. If the map $\sigma^0(L)(x)$ is the zero map for every x, then $(Ls)(x) = 0$ for all x. Therefore Ls is the zero section. Since this is true for any s with $s(x) = e$, we must have that $L = 0$. For surjectivity, let $H \in \Gamma(M, \hom(E, F)$. For every $x \in M$, $H(x) : E_x \to F_x$ is a linear map, and so given a section $s \in \Gamma(M, E)$, there is a natural induced section $H(-)(s(-)) : M \to F$. This defines a map $L : \Gamma(M, E) \to \Gamma(M, F)$ which is C^∞ linear, and hence in $D_0(E, F)$. One can check that $\sigma^0(L) = H$, which shows surjectivity.

Exercise 4.5: We must identify the kernel of $\text{ev} : J^1E \to E$. Over each point $x \in M$, its fiber is the subspace of 1-jets which vanish at x. In other words:

$$\ker(\text{ev})_x = \{(x, [s]) \in (J^1E)_x \mid s(x) = 0\}$$

Thus any element $(x, [x])$ in this subspace can be identified with the action of $d_x s : T_x M \to T_{(x,0)} E$. Recall also that the tangent bundle to E at the zero section splits canoncally as $TE = TM \oplus E$, so that $T_{(x,0)} E = T_x M \oplus E_x$. Projecting $d_x s$ onto the second component then gives a map $\pi_2 d_x s : T_x M \to E_x \in \hom(T_x M, E_x)$. We use this to define an isomorphism $\varphi : \ker(ev) \to \hom(TM, E) \cong (T^*M) \otimes E$. This is clearly an injective homomorphism because on each fiber $\pi_2 d_x s = 0$ if and only if $s \sim 0$. Surjectivity is also clear. Therefore the sequence:

$$0 \longrightarrow (T^*M) \otimes E \longrightarrow J^1E \xrightarrow{ev} E \longrightarrow 0$$

is exact.

As an application, we will compute $j^1(fs) - f \cdot j^1(s)$ for any smooth function $f : M \to \mathbb{R}$ and section $s : M \to E$. Since they both evaluate to the same point, this difference lies in the kernel of the evaluation map, so it should be of the form $\xi \otimes \theta$. By definition, we have:

$$j_x(fs)^1 - f(x) \cdot j^1(s)_x = (x, [fs]) - f(x)(x, [s])$$

Under the isomorphism we just wrote down, as an element of $\hom(T_x M, E_x)$ this is:

$$\begin{aligned} \pi_2 d_x(fs) - f(x)\pi_2 d_x s &= \pi_2(d_x(fs) - f(x) d_x s) \\ &= \pi_2(f(x) d_x s + s(x) d_x f - f(x) d_x s) && \text{(product rule)} \\ &= \pi_2(s(x) d_x f) \\ &= s(x) d_x f \end{aligned}$$

As an element of $(T^*M) \otimes E$, this is $d_x f \otimes s(x)$.

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