BOUNDS ON CERTAIN HIGHER-DIMENSIONAL EXPONENTIAL SUMS VIA THE SELF-REDUCIBILITY OF THE WEIL REPRESENTATION

SHAMGAR GUREVICH AND RONNY HADANI

Abstract. We describe a new method to bound certain higher-dimensional exponential sums which are associated with tori in symplectic groups over finite fields. Our method is based on the self-reducibility property of the Weil representation. As a result, we obtain a sharp form of the Hecke quantum unique ergodicity theorem for generic linear symplectomorphisms of the $2N$-dimensional torus.

0. Introduction

0.1. **Eigenvectors of tori in the Weil representation.** To an $2N$-dimensional symplectic vector space (V, ω) over an odd characteristic finite field $k = \mathbb{F}_q$ and a non-trivial additive character $\psi : \mathbb{F}_q \to \mathbb{C}^*$ one can associate in a functorial manner a Hilbert space $\mathcal{H} = \mathcal{H}(V)$ of dimension q^N equipped with a unitary action of two symmetry groups—the Heisenberg group $H = H(V)$ and the symplectic group $Sp = Sp(V)$. The first action is called the Heisenberg representation and we denote it by $\pi : H \to U(\mathcal{H})$; the latter action is called the Weil representation [39] and we denote it by

$$\rho : Sp \to U(\mathcal{H}).$$

These representations play an important role in discrete harmonic analysis with applications to various disciplines of pure and applied mathematics such as representation theory, number theory, mathematical physics, coding theory and signal processing.

Let $T \subset Sp$ be (the set of rational points of) a maximal torus which, for simplicity, we will assume acts irreducibly on V. The commutative group T acts, via the Weil representation, on the Hilbert space $\mathcal{H}$ and decomposes it into a direct sum of character spaces $\mathcal{H} = \oplus \mathcal{H}_\chi$, where χ runs in the group of characters of T. As it turns out $\dim(\mathcal{H}_\chi) = 1$ for each character χ that appears in the decomposition. In this paper we would like to study the Wigner distribution

$$\langle \varphi | \pi(v) \varphi \rangle \,, \tag{0.1}$$

associated with a unit character vector $\varphi \in \mathcal{H}_\chi$ and a non-zero vector $v \in V \subset H$.

0.2. **Wigner distributions as high-dimensional exponential sums.** We would like to bound the Wigner distribution. A possible method to achieve this goal was developed in [14, 15]. The main idea is to write $\langle \varphi | \pi(v) \varphi \rangle$ as an explicit exponential sum for which we can use the powerful techniques of ℓ-adic cohomology to obtain good bounds.

If we denote by $P_\chi = \frac{1}{|T|}\sum_{g\in T}\chi^{-1}(g)\rho(g)$ the orthogonal projector on the space $\mathcal{H}_\chi$ then $|T|\cdot\langle\varphi|\pi(v)\varphi\rangle = |T|\cdot Tr(\pi(v)P_\chi)$ is equal to

$$c_\chi = \sum_{g\in T}\chi^{-1}(g)\cdot Tr(\pi(v)\rho(g)),$$

which by the formula [16] for the trace of the Heisenberg–Weil representation has the form of an explicit N-dimensional exponential sum over $\mathbb{F}_q$

$$c_\chi = \sum_{g\in T\smallsetminus I}\chi^{-1}(g)\cdot\sigma(\det(g-I))\cdot\psi(\tfrac{1}{2}\omega(\frac{v}{g-I},v)),$$

where $\sigma:\mathbb{F}_q^*\to\mathbb{C}^*$ denote the Legendre character, $\psi:\mathbb{F}_q\to\mathbb{C}^*$ is a non-trivial additive character, and for the rest of this paper $\frac{v}{g-I}=(g-I)^{-1}v$.

It is usually expected that these sums will have the "square root cancellation" phenomenon, i.e., that $|c_\chi|\leq d\sqrt{q}^N$ for some constant d which is independent of q. Using the formalism of ℓ-adic cohomology, we replace the constant c_χ by the alternating sum

$$c_\chi = \sum_{i=0}^{2N}(-1)^i Tr(Fr_{|H_c^i(\mathbf{X},\mathcal{F}_\chi)}), \tag{0.2}$$

of the traces of the Frobenius operator acting on the cohomology groups with compact support $H_c^i(\mathbf{X},\mathcal{F}_\chi)$ associated with a suitable ℓ-adic Weil sheaf $\mathcal{F}_\chi$ which lives on the variety $\mathbf{X}=\mathbf{T}\smallsetminus\mathbf{I}$ where $\mathbf{T}\subset\mathbf{Sp}$ is the algebraic torus (we use boldface letters to denote algebraic varieties) such that that $T=\mathbf{T}(\mathbb{F}_q)$.

Deligne's Theory of weights [7] and a purity argument imply that the eigenvalues of Fr acting on $H_c^i(\mathbf{X},\mathcal{F}_\chi)$ are of absolute value $\sqrt{q}^i$. Moreover, it can be shown that $H_c^i(\mathbf{X},\mathcal{F}_\chi)=0$ for $0\leq i\leq N-1$. This means that in order to obtain the expected bound on c_χ we need

- To show that all but the middle cohomology group vanish.
- To calculate the dimension $d=\dim H_c^N(\mathbf{X},\mathcal{F}_\chi)$.

In [14] we study the case $N=1$ and show that only the first cohomology does not vanishes and $\dim H_c^1(\mathbf{X},\mathcal{F}_\chi)=2$; therefore $|c_\chi|\leq 2\sqrt{q}$. The computations for general N were carried in [15]—indeed all but the middle cohomology group vanish and $\dim H_c^N(\mathbf{X},\mathcal{F}_\chi)=2^N$. Hence, we find that in general

$$|c_\chi|\leq 2^N\sqrt{q}^N. \tag{0.3}$$

However, as we will show in this paper (see the survey [18] and the announcement [19]) the constant 2^N in the above bound is not optimal—in fact we have

$$|c_\chi|\leq 2\sqrt{q}^N. \tag{0.4}$$

Thinking on the Frobenius operator acting on the space $H_c^N(\mathbf{X},\mathcal{F}_\chi)$ as a large matrix

$$Fr = \begin{pmatrix}\lambda_1 & * & * \\ & \ddots & * \\ & & \lambda_{2^N}\end{pmatrix}$$

a possible scenario which we might confront is cancellations between different eigenvalues, more precisely angles, of the Frobenius operator acting on a high-dimensional vector space, i.e., cancellations in the sum $\sum_{j=1}^{2^N}e^{i\theta_j}$, where the angles

$0 \leq \theta_j < 2\pi$ are defined via $\lambda_j = e^{i\theta_j} \cdot \sqrt{q}^N$. This problem is of a completely different nature, which is not accounted for by standard cohomological techniques[1].

0.3. **Sharp bound via self-reducibility.** Our approach to obtain the sharp bound (0.4) is to realize the constant c_χ as a one-dimensional exponential sum over $\mathbb{F}_{q^N}$. This we do using a the self-reducibility property of the Weil representation.

0.3.1. ***Representation theoretic interpretation of the Wigner distribution.*** The character vector φ is a vector in a representation space $\mathcal{H}$ of the Weil representation of the symplectic group $Sp(2N, \mathbb{F}_q)$. The vector φ is completely characterized in representation theoretic terms, as being a character vector of the torus T. As a consequence, all quantities associated to φ, and in particular the Wigner distribution $\langle \varphi | \pi(v) \varphi \rangle$ is characterized in terms of the Weil representation. The main observation to be made is that the character vector φ can be characterized in terms of another Weil representation, this time of a group of a much smaller dimension—the torus T induces an $\mathbb{F}_{q^N}$-structure on (V, ω) and now φ is characterize in terms of the Weil representation of $SL(2, \mathbb{F}_{q^N})$.

0.3.2. ***Self-reducibility property.*** A fundamental notion in our study is that of a *symplectic module structure*. A symplectic module structure is a triple $(K, V, \overline{\omega})$, where K is a finite dimensional commutative algebra over $k = \mathbb{F}_q$, equipped with an action on the vector space V, and $\overline{\omega}$ is a K-linear symplectic form satisfying the property $Tr_{K/k}(\overline{\omega}) = \omega$. Let $\overline{Sp} = Sp(V, \overline{\omega})$ be the group of K-linear symplectomorphisms with respect to the form $\overline{\omega}$. There exists a canonical embedding

$$\iota : \overline{Sp} \hookrightarrow Sp. \tag{0.5}$$

It will be shown that associated to T there exists a canonical symplectic module structure $(K, V, \overline{\omega})$ so that $T \subset \overline{Sp}$. In our case the torus T acts irreducibly on the vector space V, hence, the algebra K is in fact a field with $\dim_K V = 2$ which implies that $K = \mathbb{F}_{q^N}$ and $\overline{Sp} \simeq SL(2, \mathbb{F}_{q^N})$, i.e., using (0.5) we get $T \subset SL(2, \mathbb{F}_{q^N}) \subset Sp$. Consider the Weil representation $(\rho, Sp, \mathcal{H})$ associated with the non-trivial additive character $\psi : \mathbb{F}_q \to \mathbb{C}^*$. Denote by $\overline{\psi} : K \to \mathbb{C}^*$ the additive character $\overline{\psi} = \psi \circ Tr_{K/k}$.

Theorem 0.1 (Self-reducibility property). *The restricted representation* $(\overline{\rho} = \iota^* \rho, \overline{Sp}, \mathcal{H})$ *is the Weil representation associated with* $\overline{\psi}$.

Applying the self-reducibility property to the torus T, it follows that the vector φ can be characterized in terms of the Weil representation of $SL(2, \mathbb{F}_{q^N})$. Therefore, we can apply the result obtained in [14] and get the sharp bound $|c_\chi| \leq 2\sqrt{q}^N$. Knowing that $|T|$ is of order of q^N, we obtain the sharp bound on the Wigner distribution

$$|\langle \varphi | \pi(v) \varphi \rangle| \leq \frac{2 + o(1)}{\sqrt{q}^N}, \tag{0.6}$$

for every non zero vector $v \in V$.

[1]We thank R. Heath-Brown for pointing out to us [21] about the phenomenon of cancelations between Frobenius eigenvalues in the presence of high-dimensional cohomologies.

We would like now to explain why the bounds on the Wigner distributions are of interest.

0.4. **Quantum chaos problem.** One of the main motivational problems in quantum chaos is [2, 3, 30, 34] describing eigenstates

$$\widetilde{H}\varphi = \lambda\varphi, \quad \varphi \in \mathcal{H},$$

of a chaotic Hamiltonian $\widetilde{H} = Op(H) : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. We deliberately use the notation $Op(H)$ to emphasize the fact that the quantum Hamiltonian $\widetilde{H}$ is a quantization of a Hamiltonian $H : M \to \mathbb{C}$ where M is a phase space—usually a cotangent bundle of a configuration space $M = T^*X$, in which case $\mathcal{H} = L^2(X)$. In general, describing φ is considered to be an extremely complicated problem. Nevertheless, for a few mathematical models of quantum mechanics rigorous results have been obtained. We shall proceed to describe one of these models.

0.4.1. ***Hannay–Berry model.*** In [20] Hannay and Berry explored a model for quantum mechanics on the two-dimensional symplectic torus $(\mathbb{T}, \omega)$. Hannay and Berry suggested to quantize simultaneously the functions on the torus and the linear symplectic group $\Gamma \simeq SL(2, \mathbb{Z})$. One of their main motivations was to study the phenomenon of quantum chaos in this model [30, 32]. More precisely, they considered an ergodic discrete dynamical system on the torus which is generated by a hyperbolic automorphism $A \in \Gamma$. Quantizing the system we replace—the phase space $(\mathbb{T}, \omega)$ by a finite dimensional Hilbert space $\mathcal{H}$; observables, i.e., functions $f \in C^{\infty}(\mathbb{T})$ by operators $\pi(f) \in End(\mathcal{H})$; and symmetries by a unitary representation $\rho : \Gamma \to U(\mathcal{H})$ which, in particular, enables one to associate to A a unitary operator $\rho(A)$ acting on $\mathcal{H}$.

0.4.2. *The **Shnirelman theorem.*** Analogous with the case of the Schrödinger equation, consider the following eigenstates problem:

$$\rho(A)\varphi = \lambda\varphi.$$

A fundamental result, valid for a wide class of quantum systems which are associated to ergodic dynamics, is Shnirelman's theorem [36], asserting that in the semi-classical limit "almost all" eigenstates become equidistributed in an appropriate sense.

A variant of Shnirelman's theorem also holds in our situation [5]. More precisely, we have that in the semi-classical limit $\hbar \to 0$ for "almost all" eigenstates φ of the operator $\rho(A)$ the corresponding Wigner distribution $\langle\varphi|\pi(\cdot)\varphi\rangle : C^{\infty}(\mathbb{T}) \to \mathbb{C}$ approaches the phase space average $\int_{\mathbb{T}} \cdot|\omega|$. In this respect, it seems natural to ask whether there exist exceptional sequences of eigenstates? Namely, eigenstates that do not obey the Shnirelman's rule ("scarred" eigenstates). It was predicted by Berry [2],[3], that "scarring" phenomenon is not expected to be seen for quantum systems associated with "generic" chaotic dynamics. However, in our situation the operator $\rho(A)$ is not generic, and exceptional eigenstates were constructed. Indeed, it was confirmed mathematically in [9] that certain $\rho(A)$-eigenstates might localize. For example, in that paper a sequence of eigenstates φ was constructed, for which the corresponding Wigner distribution approaches the measure $\frac{1}{2}\delta_0 + \frac{1}{2}|\omega|$ on $\mathbb{T}$.

0.4.3. ***Hecke quantum unique ergodicity.*** A quantum system that obeys the Shnirelman rule is also called quantum ergodic. Can one impose some natural conditions on the eigenstates so that no exceptional eigenstates will appear? Namely, *quantum unique ergodicity* will hold. This question was addressed by Kurlberg and Rudnick in [28], and they formulated a rigorous notion of Hecke quantum unique ergodicity for the cases $\hbar = \frac{1}{p}$, p a prime. Their basic observation is that the degeneracies of the operator $\rho(A)$ are coupled with the existence of symmetries—there exists a commutative group of operators that commute with $\rho(A)$ and which can be computed effectively. In more detail, the representation ρ factors through the Weil representation of the quotient group $Sp \simeq SL(2, \mathbb{F}_p)$. We denote by $T_A \subset Sp$ the centralizer of the element A, now considered as an element of the quotient group. We call the group T_A the *Hecke torus* associated with A. The Hecke torus acts semisimply on $\mathcal{H}$; therefore we have a decomposition into a direct sum of Hecke eigenspaces $\mathcal{H} = \oplus\, \mathcal{H}_\chi$, where χ runs in the group of character of T_A. Consider a unit eigenstate $\varphi \in \mathcal{H}_\chi$ and the corresponding Wigner distribution $C^\infty(\mathbb{T}) \to \mathbb{C}$ defined by $f \mapsto \langle \varphi | \pi(f) \varphi \rangle$. The main statement in [28] proves an explicit bound on the semi-classical asymptotic—for sufficiently large p they obtained $| \langle \varphi | \pi(f) \varphi \rangle - \int_{\mathbb{T}} f |\omega| | \leq C_f / p^{1/4}$, where C_f is a constant that depends only on the function f. In addition, in [31, 32] Kurlberg and Rudnick conjectured the following stronger bound:

$$\left| \langle \varphi | \pi(f) \varphi \rangle - \int_{\mathbb{T}} f |\omega| \right| \leq \frac{C_f}{\sqrt{p}}, \tag{0.7}$$

for sufficiently large prime p.

A particular case, which implies (0.7), of the above inequality is when $f = \xi$ a non-trivial character. In this case the integral $\int_{\mathbb{T}} \xi |\omega|$ vanishes and the bound (0.6) for the case with $N = 1$ and $k = \mathbb{F}_p$ gives $| \langle \varphi | \pi(\xi) \varphi \rangle | \leq (2 + o(1))/\sqrt{p}$, proving the conjecture [14].

0.4.4. ***The higher-dimensional Hannay–Berry model.*** The higher dimensional Hannay–Berry model is obtained as a quantization of the $2N$-dimensional symplectic torus $(\mathbb{T}, \omega)$ acted upon by the group $\Gamma \simeq Sp(2N, \mathbb{Z})$ of linear symplectic autmorphisms. It was first constructed in [13], where, in particular, a quantization of the whole group of symmetries Γ was obtained. Again, in the case $\hbar = \frac{1}{p}$ the quantization of Γ factors through the Weil representation of $Sp \simeq Sp(2N, \mathbb{F}_p)$. Considering a regular ergodic element $A \in \Gamma$, i.e., A generates an ergodic discrete dynamical system and it is regular in the sense that it has distinct eigenvalues over $\mathbb{C}$. It is natural to ask whether quantum unique ergodicity will hold true in this setting as well, as long as one takes into account the whole group of Hecke symmetries? Interestingly, the answer to this question is no. Several new results in this direction have been announced recently. In the case where the automorphism A is *non-generic*, meaning that it has an invariant Lagrangian (and more generally co-isotropic) sub-torus $\mathbb{T}_L \subset \mathbb{T}$, an interesting new phenomenon was revealed. There exists a sequence $\{\varphi_\hbar\}$ of Hecke eigenstates which might be related to the physical phenomenon of "localization" known in the literature (cf. [22],[27]) as "scars". We will call them *Hecke* scars. These states are localized in the sense that the associated Wigner distribution converges to the Haar measure μ on the invariant

Lagrangian sub-torus

$$\langle \varphi_\hbar | \pi(f) \varphi_\hbar \rangle \rightarrow \int_{\mathbb{T}_L} f\mu, \text{ as } \hbar \rightarrow 0, \tag{0.8}$$

for every smooth observable f. These special Hecke eigenstates were first established in [12]. The semi-classical interpretation of the localization phenomena (0.8) was announced in [26].

The above phenomenon motivates the following definition:

Definition 0.2. *We will call an element $A \in \Gamma$ generic if it is regular and admits no non-trivial invariant co-isotropic sub-tori.*

Remark 0.3. *The collection of generic elements constitutes an open subscheme of Γ. In particular, a generic element need not be ergodic automorphism of $\mathbb{T}$. However, in the case where $\Gamma \simeq SL_2(\mathbb{Z})$ every ergodic (i.e., hyperbolic) element is generic. An example of generic elements which is not ergodic is given by the Weyl element $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.*

For the sake of simplicity let us assume now that the automorphism A is *strongly generic*, i.e., it has no non-trivial invariant sub-tori. This case was first considered in [15], where using the bound (0.3) we obtain that for a fixed non-trivial character ξ of $\mathbb{T}$

$$|\langle \varphi | \pi(\xi) \varphi \rangle| \leq \frac{m_\chi \cdot (2 + o(1))^N}{\sqrt{p}^N}, \tag{0.9}$$

for a sufficiently large prime number p, where $m_\chi = \dim \mathcal{H}_\chi$.

In particular, using the bound (0.9) we have:

Theorem 0.4 (Hecke quantum unique ergodicity)**.** *Consider an observable $f \in C^\infty(\mathbb{T})$ and a sufficiently large prime number p. Then*

$$\left| \langle \varphi | \pi(f) \varphi \rangle - \int_{\mathbb{T}} f d\mu \right| \leq \frac{C_f}{\sqrt{p}^N},$$

where $\mu = |\omega|^N$ is the corresponding volume form and C_f is an explicit computable constant which depends only on the function f.

The new method, using the self-reducibility property applied to the torus T_A, leads to a bound similar to (0.6) and to a sharp form of Theorem 0.4.

Theorem 0.5 (Sharp bound)**.** *Let ξ be a non-trivial character of $\mathbb{T}$. For sufficiently large prime number p the following bound holds:*

$$|\langle \varphi | \pi(\xi) \varphi \rangle| \leq \frac{m_\chi \cdot (2 + o(1))^{r_p}}{\sqrt{p}^N}, \tag{0.10}$$

where the number r_p is an integer between 1 and N that we call the symplectic rank of T_A.

Remark 0.6. *If the torus T_A acts irreducibly on $V \simeq \mathbb{F}_p^{2N}$ then $r_p = 1$; and if it splits, i.e., $T_A \simeq \mathbb{F}_p^{*N}$ then $r_p = N$. In general (see Subsection 5.2) the distribution of the symplectic rank r_p in the set $\{1, ..., N\}$ is governed by the Chebotarev density theorem applied to a suitable Galois group G. For example, in the case where*

$A\in Sp(4,\mathbb{Z})$ is strongly generic then G is the symmetric group S_2 and we have the density law

$$\lim_{x\to\infty}\frac{\#\{r_p=r\mid p\leq x\}}{\pi(x)}=\tfrac{1}{2},\qquad r=1,\ 2,$$

where $\pi(x)$ denotes the number of primes up to x.

0.5. **Quantum unique ergodicity for statistical states.** As in harmonic analysis, we would like to use Theorem 0.5 concerning the Hecke eigenstates in order to extract information on the spectral theory of the operator $\rho(A)$ itself. For the sake of simplicity, let us assume again that A is strongly generic, i.e., it acts on the torus $\mathbb{T}$ with no non-trivial invariant sub-tori. The following is a possible reformulation of the quantum unique ergodicity statement—one which is formulated for the automorphism A itself instead of the all Hecke group of symmetries. The element A acts via the Weil representation ρ on the space $\mathcal{H}$ and decomposes it into a direct sum of $\rho(A)$-eigenspaces

$$\mathcal{H}=\bigoplus\mathcal{H}_\lambda. \tag{0.11}$$

Considering an $\rho(A)$-eigenstate φ and the corresponding projector P_φ one usually studies the Wigner distribution $\langle\varphi|\pi(f)\varphi\rangle=Tr(\pi(f)P_\varphi)$ which, due to the fact that $rank(P_\varphi)=1$, is sometimes called a *"pure state"*. In the same way, we might think about a Hecke–Wigner distribution $\langle\varphi|\pi(f)\varphi\rangle=Tr(\pi(f)P_\chi)$, attached to a T_A-eigenstate φ, as a "pure Hecke state". Following von Neumann [38] we suggest the possibility of looking at the more general *"statistical state"* defined by a non-negative self-adjoint operator D—called the von Neumann density operator—normalized to have $Tr(D)=1$. For example, to the automorphism A we can attach the natural family of density operators $D_\lambda=\frac{1}{m_\lambda}P_\lambda$, where P_λ is the orthogonal projector on the eigenspace $\mathcal{H}_\lambda$ (0.11), and $m_\lambda=\dim(\mathcal{H}_\lambda)$. Consequently, we obtain a family of statistical states

$$Tr(\pi(\cdot)D_\lambda).$$

Theorem 0.7. *Let ξ be a non-trivial character of $\mathbb{T}$. For a sufficiently large prime number p we have*

$$|Tr(\pi(\xi)D_\lambda)|\leq\frac{m\cdot(2+o(1))^{r_p}}{\sqrt{p}^N}, \tag{0.12}$$

where $1\leq r_p\leq N$ is an integer which is determined by A, and $m=\max\dim\mathcal{H}_\chi$, where the maximum is taking over the characters of the Hecke torus T_A.

Theorem 0.7 follows from the fact that the Hecke torus T_A acts on the spaces $\mathcal{H}_\lambda$, hence, we can use the Hecke eigenstates and the bound (0.10).

In particular, using the bound (0.12), and the explicit information on m (see Theorem 2.15) we obtain:

Theorem 0.8 (Quantum unique ergodicity for statistical states)**.** *Consider an observable $f\in C^\infty(\mathbb{T})$ and a sufficiently large prime number p. Then*

$$\left|Tr(\pi(f)D_\lambda)-\int_{\mathbb{T}}fd\mu\right|\leq\frac{C_f}{\sqrt{p}^N},$$

where $\mu=|\omega|^N$ is the corresponding volume form and C_f is an explicit computable constant which depends only on the function f.

0.6. **Results.**

(1) *Bounds on higher-dimensional exponential sums.* The main result of this paper is a new method to bound certain higher-dimensional exponential sums associated with tori in $Sp(2N, \mathbb{F}_q)$. Our method is based on the self-reducibility property of the Weil representation. As an application we prove the Hecke quantum unique ergodicity theorem for generic linear symplectomorphisms of the higher-dimensional tori.

(2) *Self-reducibility of the Weil representation.* The main technical result of this paper is the proof of the self-reducibility of the Weil representation. This property was described first by Gèrardin in [10]. However, our proof is slightly different and in particular applies to the Weil representation over any local field of characteristic different from two. In order to keep the paper self contained we decided to present our proof in detail.

(3) *Multiplicities.* We present a simple method, using the self-reducibility property, to compute the dimension of the character spaces for the action of the tori in the Weil representations.

(4) *Two-dimensional Wigner distributions.* We describe a new proof for the bound on the Wigner distributions associated with tori in $SL(2, \mathbb{F}_q)$. It uses direct geometric calculations, using the new character formula (1.3), and avoids the use of the equivariant property of the Deligne sheaf [14].

0.7. **Structure of the paper.** Apart from the introduction, the paper consists of five sections and two appendices. In Section 1 we present preliminaries from the theory of the Heisenberg–Weil representation. Section 2 constitutes the main technical part of this work. Here we formulate and prove the self-reducibility property of the Weil representation. Section 3 deals with the main application of the paper—bounds on the higher-dimensional Wigner distributions. In Section 4 we introduce the Hannay–Berry model of quantum mechanics on the higher-dimensional tori and in Section 5 we apply the bounds on the Wigner distribution to obtain the Hecke quantum unique ergodicity theorem. Finally, in Appendices A and B we supply the proofs for the statements that appear in the body of the paper.

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1. The Heisenberg–Weil representation

In this section, we denote by $k = \mathbb{F}_q$ the finite field of q elements and odd characteristic.

1.1. **The Heisenberg representation.** Let (V, ω) be a $2N$-dimensional symplectic vector space over the finite field k. There exists a two-step nilpotent group $H = H(V, \omega)$ associated to the symplectic vector space (V, ω). The group H is called the *Heisenberg group*. It can be realized as the set $H = V \times k$ equipped with the following multiplication rule:

$$(v,z)\cdot(v',z') = (v+v', z+z'+\tfrac{1}{2}\omega(v,v')).$$

The center of H is $Z(H) = \{(0,z) : z \in k\}$. Fix a non-trivial central character $\psi : Z(H) \longrightarrow \mathbb{C}^*$. We have the following fundamental theorem:

Theorem 1.1 (Stone–von Neumann). *There exists a unique* (*up to isomorphism*) *irreducible representation* $(\pi, H, \mathcal{H})$ *with central character* ψ*, i.e.,* $\pi(z) = \psi(z)\cdot Id_{\mathcal{H}}$ *for every* $z \in Z(H)$.

We call the representation π appearing in Theorem 1.1 the *Heisenberg representation* associated with the central character ψ.

1.2. **The Weil representation.** Let $Sp = Sp(V,\omega)$ denote the group of linear symplectic automorphisms of V. The group Sp acts by group automorphisms on the Heisenberg group through its action on the vector space V, i.e., $g\cdot(v,z) = (gv,z)$. A direct consequence of Theorem 1.1 is the existence of a projective representation $\widetilde{\rho} : Sp \to PGL(\mathcal{H})$. The classical construction [39] works as follows. Considering the Heisenberg representation π and an element $g \in Sp$ we define a new representation π^g acting on the same Hilbert space via $\pi^g(h) = \pi(g(h))$. Because these irreducible representations share the same central character then by Theorem 1.1 they are isomorphic and $\dim\, Hom_H(\pi,\pi^g) = 1$. Choosing for every $g \in Sp$ a non-zero operator $\widetilde{\rho}(g) \in Hom_H(\pi,\pi^g)$ we obtained the required projective representation. In other words the projective representation $\widetilde{\rho}$ is characterized by the formula

$$\widetilde{\rho}(g)\pi(h)\widetilde{\rho}(g)^{-1} = \pi(g(h)), \tag{1.1}$$

for every $g \in Sp$, $h \in H$.

It is a deep fact that over finite fields of odd characteristic this projective representation has a linearization that we will call the Weil representation.

Theorem 1.2 (Weil representation). *There there exists a unique*[2] *representation*

$$\rho : Sp \longrightarrow GL(\mathcal{H}),$$

satisfying the identity (*1.1*).

1.3. **The Heisenberg–Weil representation.** Let J denote the semi-direct product $J = Sp \ltimes H$. The group J is sometimes referred to as the *Jacobi* group. The compatible pair (ρ,π) is equivalent to a single representation

$$\tau : J \longrightarrow GL(\mathcal{H}),$$

of the Jacobi group defined by the formula $\tau(g,h) = \rho(g)\pi(h)$. In this paper, we would like to adopt the name *Heisenberg–Weil* representation for referring to the representation τ.

[2]Unique except in the case when $k = \mathbb{F}_3$ and $\dim(V) = 2$. For the natural choice in this case see [17].

1.4. **The character of the Heisenberg–Weil representation.** The absolute value of the characters $ch_\rho : Sp \to \mathbb{C}$ of the Weil representation and $ch_\tau : J \to \mathbb{C}$ of the Heisenberg–Weil representation was described in [23, 24], but the phases have been made explicit only recently in [16]. Denote by $\sigma : \mathbb{F}_q^* \to \mathbb{C}^*$ the Legendre (quadratic) character. The following formulas are taken from [16]:

$$ch_\rho(g) = \sigma((-1)^N \cdot \det(g - I)), \tag{1.2}$$

$$ch_\tau(g, v, z) = ch_\rho(g) \cdot \psi(\tfrac{1}{2}\omega(\frac{v}{g - I}, v) + z), \tag{1.3}$$

for every $g \in Sp$ such that $g - I$ is invertible and every $(v, z) \in H$.

1.5. **Application to multiplicities.** Let us start with the two-dimensional case. Let $T \subset Sp \simeq SL(2, \mathbb{F}_q)$ be a maximal torus. The torus T acts semisimply on $\mathcal{H}$, decomposing it into a direct sum of character spaces $\mathcal{H} = \oplus \mathcal{H}_\chi$ over the characters of T. As a consequence of having the explicit formula (1.2), we obtain a simple description for the multiplicities $m_\chi = \dim \mathcal{H}_\chi$ (cf. [1, 10, 37]). Denote by $\sigma_T : T \to \mathbb{C}^*$ the unique quadratic character of T.

Theorem 1.3 (Multiplicities formula)**.** *We have $m_\chi = 1$ for any character $\chi \neq \sigma_T$. Moreover, $m_{\sigma_T} = 2$ or 0, depending on whether the torus T is split or inert, respectively.*

For a proof see Appendix A.1.

Using the orthogonality relation for characters we obtain:

Corollary 1.4. *The character ch_ρ when restricted to the punctured torus $T \smallsetminus I \subset Sp$ equals σ_T or $-\sigma_T$ depending on whether T is split or inert, respectively.*

In Subsection 2.4 we use the self-reducibility property and extend Theorem 1.3 to the higher-dimensional Weil representations.

2. Self-reducibility of the Weil representation

In this section, unless stated otherwise, the field k is an arbitrary local or finite field of characteristic different from two.

2.1. **Symplectic module structures.** Let K be a finite-dimensional commutative algebra over the field k. Let $Tr : K \to k$ be the trace map associating to an element $x \in K$ the trace of the k-linear operator $m_x : K \to K$ obtained by left multiplication by the element x. Consider a symplectic vector space (V, ω) over k.

Definition 2.1. *A symplectic K-module structure on (V, ω) is an action $K \otimes_k V \to V$, and a K-linear symplectic form $\overline{\omega} : V \times V \to K$ such that*

$$Tr \circ \overline{\omega} = \omega. \tag{2.1}$$

Given a symplectic module structure $(K, V, \overline{\omega})$ on a symplectic vector space (V, ω), we denote by $\overline{Sp} = Sp(V, \overline{\omega})$ the group of K-linear symplectomorphisms with respect to the form $\overline{\omega}$. The compatibility condition (2.1) gives a natural embedding

$$\iota_S : \overline{Sp} \hookrightarrow Sp. \tag{2.2}$$

.

2.2. **Symplectic module structure associated with a maximal torus.** Let $T \subset Sp$ be a maximal torus.

2.2.1. *A* ***particular case***. For simplicity, let us assume first that T acts irreducibly on the vector space V, i.e., there exists no non-trivial T-invariant subspaces. Let $A = Z(T, End(V))$ be the centralizer of T in the algebra of all linear endomorphisms. Clearly (due to the assumption of irreducibility) A is a division algebra. In addition we have:

Claim 2.2. *The algebra A is commutative.*

For a proof see Appendix A.2.

In particular, Claim 2.2 implies that A is a field. Let us now describe a special quadratic element in the Galois group $Gal(A/k)$ of all the automorphisms of A which leave the field k fixed. Denote by $(\cdot)^t : End(V) \to End(V)$ the symplectic transpose characterized by the property $\omega(Rv, u) = \omega(v, R^t u)$ for all $v, u \in V$, and every $R \in End(V)$. It can be easily verified that $(\cdot)^t$ preserves A, leaving the subfield k fixed, hence, it defines an element $\Theta \in Gal(A/k)$ satisfying $\Theta^2 = Id$. Denote by

$$K = A^{\Theta},$$

the subfield of A consisting of the elements fixed by Θ.

Proposition 2.3 (Hilbert's theorem 90)**.** *We have* $\dim_K V = 2$.

For a proof see Appendix A.2.

Corollary 2.4. *We have* $\dim_K A = 2$.

As a corollary, we have the following description of T. Denote by $N_{A/K} : A \to K$ the standard norm map.

Corollary 2.5. *We have* $T = S(A) = \left\{a \in A : N_{A/K}(a) = 1\right\}$.

For a proof see Appendix A.3.

The symplectic form ω can be lifted to a K-linear symplectic form $\overline{\omega}$ which is invariant under the action of the torus T. This is the content of the following proposition:

Proposition 2.6 (Existence of canonical symplectic module structure)**.** *There exists a canonical T-invariant K-linear symplectic form $\overline{\omega} : V \times V \to K$ satisfying the property $Tr \circ \overline{\omega} = \omega$.*

For a proof see Appendix A.4.

Concluding, we obtained a T-invariant symplectic K-module structure on V.

Let $\overline{Sp} = Sp(V, \overline{\omega})$ denote the group of K-linear symplectomorphisms with respect to the symplectic form $\overline{\omega}$. We denote the embedding (2.2) by $i_S : \overline{Sp} \hookrightarrow Sp$. The elements of T commute with the action of K and preserve the symplectic form $\overline{\omega}$ (Proposition 2.6); hence, we can consider T as a subgroup of $\overline{Sp}$. By Proposition 2.3 we can identify $\overline{Sp} \simeq SL(2, K)$, and using (2.2) we obtain

$$T \subset SL(2, K) \subset Sp. \tag{2.3}$$

Moreover, we see that T consists of the set of K-rational points of a maximal algebraic torus $\mathbf{T} \subset \mathbf{SL}_2$ (we use bold face to denote algebraic varieties).

2.2.2. ***General case***. Here, we drop the assumption that T acts irreducibly on V. By the same argument as before one can show that the algebra $A = Z(T, End(V))$ is commutative, yet, it may no longer be a field. The symplectic transpose $(\cdot)^t$ preserves the algebra A, and induces an involution $\Theta : A \to A$. Let $K = A^\Theta$ be the subalgebra consisting of elements $a \in A$ fixed by Θ. Following the same argument as in the proof of Proposition 2.3, one shows that V is a free K-module of rank 2. Following the same arguments as in the proof of Proposition 2.6, one shows that there exists a canonical symplectic form $\overline{\omega} : V \times V \to K$, which is K-linear and invariant under the action of the torus T. Concluding, associated to a maximal torus T there exists a T-invariant symplectic K-module structure $(K, V, \overline{\omega})$.

Denote by $\overline{Sp} = Sp(V, \overline{\omega})$ the group of K-linear symplectomorphisms with respect to the form $\overline{\omega}$. We have a natural embedding $\iota_S : \overline{Sp} \hookrightarrow Sp$ and we can consider T as a subgroup of $\overline{Sp}$. Finally, we have $\overline{Sp} \simeq SL(2, K)$ and T consists of the K-rational points of a maximal torus $\mathbf{T} \subset \mathbf{SL}_2$. In particular, the relation (2.3) holds also in this case

$$T \subset SL(2, K) \subset Sp.$$

We shall now proceed to give a finer description of all objects discussed so far. The main technical result is summarized in the following lemma:

Lemma 2.7 (Symplectic decomposition)**.** *We have a canonical decomposition*

$$(V, \omega) = \bigoplus_{\alpha \in \Xi} (V_\alpha, \omega_\alpha), \tag{2.4}$$

into (T, A)-invariant symplectic subspaces. In addition, we have the following associated canonical decompositions:

1. $T = \prod T_\alpha$, where T_α consists of elements $g \in T$ such that $g_{|V_\beta} = Id$ for every $\beta \neq \alpha$.
2. $A = \oplus A_\alpha$, where A_α consists of elements $a \in A$ such that $a_{|V_\beta} = Id$ for every $\beta \neq \alpha$. Moreover, each subalgebra A_α is preserved under the involution Θ.
3. $K = \oplus K_\alpha$, where $K_\alpha = A_\alpha^\Theta$. Moreover, K_α is a field and $\dim_{K_\alpha} V_\alpha = 2$.
4. $\overline{\omega} = \oplus \overline{\omega}_\alpha$, where $\overline{\omega}_\alpha : V_\alpha \times V_\alpha \to K_\alpha$ is a K_α-linear T_α-invariant symplectic form satisfying $Tr \circ \overline{\omega}_\alpha = \omega_\alpha$.

For a proof see Appendix A.5.

Definition 2.8. *We will call the set Ξ (2.4) the symplectic type of T and the number $|\Xi|$ the symplectic rank of T.*

Using the results of Lemma 2.7, we have an isomorphism

$$\overline{Sp} \simeq \prod \overline{Sp}_\alpha, \tag{2.5}$$

where $\overline{Sp}_\alpha = Sp(V_\alpha, \overline{\omega}_\alpha)$ denotes the group of K_α-linear symplectomorphisms with respect to the form $\overline{\omega}_\alpha$. Moreover, for every $\alpha \in \Xi$ we have $T_\alpha \subset \overline{Sp}_\alpha$. In particular, under the identifications $\overline{Sp}_\alpha \simeq SL(2, K_\alpha)$, there exist the following sequence of inclusions of groups:

$$T = \prod T_\alpha \subset \prod SL(2, K_\alpha) = SL(2, K) \subset Sp, \tag{2.6}$$

and for every $\alpha \in \Xi$ the torus T_α coincides with the K_α-rational points of a maximal torus $\mathbf{T}_\alpha \subset \mathbf{SL}_2$.

2.3. **Self-reducibility of the Weil representation.** In this subsection we assume that the field k is a finite field[3]. Let $(\tau, J, \mathcal{H})$ be the Heisenberg–Weil representation associated with a central character $\psi : Z(J) = Z(H) \to \mathbb{C}^*$. Recall that $J = Sp \ltimes H$ and τ is obtained as a semi-direct product $\tau = \rho \ltimes \pi$ of the Weil representation ρ and the Heisenberg representation π. Let $T \subset Sp$ be a maximal torus.

2.3.1. *A particular case.* For clarity of presentation, let us assume first that T acts irreducibly on V. Using the results of the previous section, there exists a symplectic module structure $(K, V, \overline{\omega})$ where K/k is a field extension of degree $[K : k] = N$. The group $\overline{Sp} = Sp(V, \overline{\omega})$ is embedded as a subgroup $\iota_S : \overline{Sp} \hookrightarrow Sp$. Our goal is to describe the restriction

$$(\overline{\rho} = \iota_S^* \rho, \overline{Sp}, \mathcal{H}). \tag{2.7}$$

Define an auxiliary Heisenberg group

$$\overline{H} = V \times K, \tag{2.8}$$

and the multiplication is given by $(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\overline{\omega}(v, v'))$.

There exists a homomorphism

$$\iota_H : \overline{H} \to H, \tag{2.9}$$

given by $(v, z) \mapsto (v, Tr(z))$.

Consider the pullback $(\overline{\pi} = \iota_H^* \pi, \overline{H}, \mathcal{H})$.

Proposition 2.9. *The representation $(\overline{\pi} = \iota_H^* \pi, \overline{H}, \mathcal{H})$ is the Heisenberg representation associated with the central character $\overline{\psi} = \psi \circ Tr$.*

For a proof see Appendix A.6.

The group $\overline{Sp}$ acts by automorphisms on the group $\overline{H}$ through its tautological action on the V-coordinate. This action is compatible with the action of Sp on H, i.e., we have $\iota_H(g \cdot h) = \iota_S(g) \cdot \iota_H(h)$ for every $g \in \overline{Sp}$, and $h \in \overline{H}$. The description of the representation $\overline{\rho}$ (2.7) now follows easily (cf. [10])

Theorem 2.10 (Self-reducibility property—particular case)**.** *The representation $(\overline{\rho}, \overline{Sp}, \mathcal{H})$ is the Weil representation associated with the Heisenberg representation $(\overline{\pi}, \overline{H}, \mathcal{H})$.*

For a proof see Appendix A.7.

Remark 2.11. *We can summarize the result in a slightly more elegant manner using the Jacobi groups. Let $J = Sp \ltimes H$ and $\overline{J} = \overline{Sp} \ltimes \overline{H}$ be the Jacobi groups associated with the symplectic spaces (V, ω) and $(V, \overline{\omega})$ respectively. We have a homomorphism $\iota : \overline{J} \to J$ given by $\iota(g, h) = (\iota_S(g), \iota_H(h))$. Let $(\tau, J, \mathcal{H})$ be the Heisenberg–Weil representation of J associated with a character ψ of the center $Z(J)$ (note that $Z(J) = Z(H)$), then the pullback $(\iota^* \tau, \overline{J}, \mathcal{H})$ is the Heisenberg–Weil representation of $\overline{J}$ associated with the character $\overline{\psi} = \psi \circ Tr$ of the center $Z(\overline{J})$.*

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[3]We remark that the results continue to hold true also for local fields of characteristic $\neq 2$, i.e., with the appropriate modification, replacing the group Sp with its double cover $\widetilde{Sp}$ [39].

2.3.2. *The general case.* Here, we drop the assumption that T acts irreducibly on V. Let $(K, V, \overline{\omega})$ be the associated symplectic module structure. Using the results of Subsection 2.2.2, we have decompositions

$$(V,\omega) = \bigoplus_{\alpha\in\Xi}(V_\alpha,\omega_\alpha), \quad (V,\overline{\omega}) = \bigoplus_{\alpha\in\Xi}(V_\alpha,\overline{\omega}_\alpha), \tag{2.10}$$

where $\overline{\omega}_\alpha : V_\alpha \times V_\alpha \to K_\alpha$. Let $\overline{H} = V \times K$ be the Heisenberg group associated with $(V, \overline{\omega})$ (cf. 2.8). There exists (cf. 2.9) an homomorphism $\iota_H : \overline{H} \to H$. Let us describe the pullback $\overline{\pi} = \iota_H^* \pi$ of the Heisenberg representation. First, we note that the decomposition (2.10) induces a corresponding decomposition of the Heisenberg group $\overline{H} = \prod \overline{H}_\alpha$, where $\overline{H}_\alpha$ is the Heisenberg group associated with $(V_\alpha, \overline{\omega}_\alpha)$.

Proposition 2.12. *There exists an isomorphism*

$$(\overline{\pi}, \overline{H}, \mathcal{H}) \simeq (\bigotimes \overline{\pi}_\alpha, \prod \overline{H}_\alpha, \bigotimes \mathcal{H}_\alpha),$$

where $(\overline{\pi}_\alpha, \overline{H}_\alpha, \mathcal{H}_\alpha)$ is the Heisenberg representation of $\overline{H}_\alpha$ associated with the central character $\overline{\psi}_\alpha = \psi \circ Tr_{K_\alpha/k}$.

For a proof see Appendix A.8.

Let $\iota_S : \overline{Sp} \hookrightarrow Sp$ be the embedding (2.2). Our next goal is to describe (cf. [10]) the restriction $\overline{\rho} = \iota_S^* \rho$. Recall the decomposition $\overline{Sp} = \prod \overline{Sp}_\alpha$ (see (2.5)).

Theorem 2.13 (Self-reducibility property—general case)**.** *There exists an isomorphism*

$$(\overline{\rho}, \overline{Sp}, \mathcal{H}) \simeq (\bigotimes \overline{\rho}_\alpha, \prod \overline{Sp}_\alpha, \bigotimes \mathcal{H}_\alpha),$$

where $(\overline{\rho}_\alpha, \overline{Sp}_\alpha, \mathcal{H}_\alpha)$ is the Weil representation associated with the Heisenberg representation $\overline{\pi}_\alpha$.

For a proof see Appendix A.9.

Remark 2.14. *As before, we can state an equivalent result using the Jacobi groups $J = Sp \ltimes H$ and $\overline{J} = \overline{Sp} \ltimes \overline{H}$. We have a decomposition $\overline{J} = \prod \overline{J}_\alpha$, where $\overline{J}_\alpha = \overline{Sp}_\alpha \ltimes \overline{H}_\alpha$. Let τ be the Heisenberg–Weil representation of J associated with a character ψ of the center $Z(J)$ (note that $Z(J) = Z(H)$). Then the pullback $\overline{\tau} = \iota^*\tau$ is isomorphic to $\otimes\overline{\tau}_\alpha$, where $\overline{\tau}_\alpha$ is the Heisenberg–Weil representation of $\overline{J}_\alpha$, associated with the character $\overline{\psi}_\alpha = \psi \circ Tr_{K_\alpha/k}$ of the center $Z(\overline{J}_\alpha)$.*

2.4. **Application to multiplicities.** Let $T \subset Sp$ be a maximal torus. The torus T acts, via the Weil representation ρ, on the space $\mathcal{H}$, decomposing it into a direct sum of character spaces $\mathcal{H} = \oplus \mathcal{H}_\chi$. We would like to compute the multiplicities $m_\chi = \dim(\mathcal{H}_\chi)$. Using Lemma 2.7, we have (see (2.6)) a canonical decomposition of T

$$T = \prod T_\alpha, \tag{2.11}$$

where each of the tori T_α coincides with a maximal torus inside $\overline{Sp} \simeq SL(2, K_\alpha)$, for some field extension $K_\alpha \supset k$. In particular, by (2.11) we have a decomposition

$$\mathcal{H}_\chi = \bigotimes_{\chi_\alpha : T_\alpha \to \mathbb{C}^*} \mathcal{H}_{\chi_\alpha}, \tag{2.12}$$

where $\chi = \prod \chi_\alpha : \prod T_\alpha \to \mathbb{C}^*$. Hence, by Theorem 2.13 and the result about the multiplicities in the two-dimensional case (see Proposition (1.3)), we can compute the integer m_χ. Denote by σ_{T_α} the quadratic character of T_α (note that by Proposition (1.3) the quadratic character σ_{T_α} cannot appear in the decomposition (2.12) if the torus T_α is inert).

Theorem 2.15 (Multiplicities formula—higher dimensional)**.** *We have*

$$m_\chi = 2^l,$$

where $l = |\{\alpha : \chi_\alpha = \sigma_{T_\alpha}\}|$.

3. Bounds on higher-dimensional Wigner distributions

Let $(\rho, Sp, \mathcal{H})$ be the Weil representation associated with a $2N$-dimensional vector space over an odd characteristic finite field $k = \mathbb{F}_q$. Consider a maximal torus $T \subset Sp$ and the associated decomposition of $\mathcal{H}$ into a direct sum of character spaces $\mathcal{H} = \oplus\, \mathcal{H}_\chi$. For a character vector $\varphi \in \mathcal{H}_\chi$ we will bound the Wigner distribution $\langle \varphi | \pi(v) \varphi \rangle$ where $v \in V$ is not contained in any proper T-invariant subspace. We will explain how to use the self-reducibility property for this purpose.

3.1. **The completely inert case.** It will be convenient to assume first that the torus T is completely inert, i.e., acts irreducibly on V.

Theorem 3.1 (Bound on Wigner distributions—inert case)**.** *For every non-zero vector* $v \in V$ *we have*

$$|\langle \varphi | \pi(v) \varphi \rangle| \leq \frac{2}{\sqrt{q}^N}.$$

To get this bound we proceed as follow. The torus T acts irreducibly on the vector space V. Invoking the result of Section 2.2.1, there exists a canonical symplectic module structure $(K, V, \overline{\omega})$ associated to T. Recall that in this particular case the algebra K is in fact a field, $K = \mathbb{F}_{q^N}$, and $\dim_K V = 2$. Let $\overline{J} = \overline{Sp} \ltimes \overline{H}$ be the Jacobi group associated to the two-dimensional symplectic vector space $(V, \overline{\omega})$. There exists a natural homomorphism $\iota : \overline{J} \to J$. Invoking the results of Section 2.3.1, the pullback $\overline{\tau} = \iota^* \tau$ is the Heisenberg–Weil representation of $\overline{J}$, i.e., $\overline{\tau} = \overline{\rho} \ltimes \overline{\pi}$. The orthogonal projector P_χ on the space $\mathcal{H}_\chi$ can be written in terms of the Weil representation $\overline{\rho}$ as $P_\chi = |T|^{-1} \sum_{g \in T} \chi^{-1}(g) \overline{\rho}(g)$. Since $\dim \mathcal{H}_\chi = 1$ (Theorem 2.15) we realize that $\langle \varphi | \pi(v) \varphi \rangle = Tr(P_\chi \pi(v))$ where $\varphi \in \mathcal{H}_\chi$. Overall, we have

$$\langle \varphi | \pi(v) \varphi \rangle = \frac{1}{|T|} \sum_{g \in T} \chi^{-1}(g) Tr(\overline{\rho}(g) \pi(v)). \tag{3.1}$$

Note that $Tr(\overline{\rho}(g)\pi(v))$ is nothing other than the character $ch_{\overline{\tau}}(g \cdot v)$ of the Heisenberg–Weil representation $\overline{\tau}$ and that $|T| = q^N + 1$. Therefore the right-hand side of (3.1) is defined completely in terms of the two-dimension Heisenberg–Weil representation $\overline{\tau}$. Theorem 3.1 is then a particular case of the following theorem:

Theorem 3.2 ([14])**.** *Let* (V, ω) *be a two-dimensional symplectic vector space over the finite field* $k = \mathbb{F}_q$*, and* $(\tau, J, \mathcal{H})$ *the corresponding Heisenberg–Weil representation. Let* $T \subset Sp$ *be a maximal torus. We have the following estimate:*

$$\left| \sum_{g \in T} \chi(g) ch_\tau(g \cdot v) \right| \leq 2\sqrt{q}, \tag{3.2}$$

where χ *is a character of* T*, and* $0 \neq v \in V$ *is not an eigenvector of* T*.*

For the sake of completeness we write in Appendix B a new proof of Theorem 3.2.

Remark 3.3 (Alternative approach)**.** *We propose another approach to obtain the relation between the higher dimensional exponential sum over $k = \mathbb{F}_q$ and the one dimensional sum over $K = \mathbb{F}_{q^N}$. We use the existence of symplectic module structure (Theorem 2.6) and then—instead of using the self-reducibility property—we invoke the explicit formula (1.3) for the character of the Heisenberg–Weil representation. Denote by $(\tau, J, \mathcal{H})$ the Heisenberg–Weil representation associated with a $2N$-dimensional symplectic vectors space over k and by $(K, V, \overline{\omega})$ the symplectic module structure associated with the torus T. We deduce that*

$$\begin{aligned}\sum_{g\in T}\chi(g)ch_{\tau}(g\cdot v) &= \sum_{g\in T}\chi(g)\sigma((-1)^N\det(g-I))\psi(\tfrac{1}{2}\omega(\tfrac{v}{g-I},v))\\ &= \sum_{g\in T}\chi(g)\overline{\sigma}(-\det{}_K(g-I))\overline{\psi}(\tfrac{1}{2}\overline{\omega}(\tfrac{v}{g-I},v)),\end{aligned}$$

where $\overline{\sigma} = \sigma \circ Norm_{K/k}$, $\overline{\psi} = \psi \circ Tr_{K/k}$, and $\det_K(g - I)$ is the determinant of $g - I$ acting on V as a vector space over K. The last sum above is over a torus in $Sp(V, \overline{\omega})$ which we evaluate using Theorem 3.2.

3.2. **General case.** In this subsection we state and prove the analogue of Theorem 3.1, where we drop the assumption of T being completely inert. In what follows, we use the results of Subsections 2.2.2 and 2.3.2.

Let $(K, V, \overline{\omega})$ be the symplectic module structure associated with the torus T. The algebra K is no longer a field, but decomposes into a direct sum of fields $K = \oplus_{\alpha\in\Xi}K_\alpha$. We have canonical decompositions $(V, \omega) = \oplus(V_\alpha, \omega_\alpha)$ and $(V, \overline{\omega}) = \oplus(V_\alpha, \overline{\omega}_\alpha)$. Recall that V_α is a two-dimensional vector space over the field K_α. The Jacobi group $\overline{J}$ decomposes into $\overline{J} = \Pi\overline{J}_\alpha$, where $\overline{J}_\alpha = \overline{Sp}_\alpha \ltimes \overline{H}_\alpha$ is the Jacobi group associated to $(V_\alpha, \overline{\omega}_\alpha)$. The pullback $(\overline{\tau} = \iota^*\tau, \overline{J}, \mathcal{H})$ decomposes into a tensor product $(\otimes\overline{\tau}_\alpha, \Pi\overline{J}_\alpha, \otimes\mathcal{H}_\alpha)$ where $\overline{\tau}_\alpha$ is the Heisenberg–Weil representation of $\overline{J}_\alpha$. The torus T decomposes into $T = \Pi T\alpha$ where T_α is a maximal torus in $\overline{Sp}_\alpha$. Consequently, the character $\chi : T \to \mathbb{C}^*$ decomposes into a product $\chi = \Pi\chi_\alpha : \Pi T_\alpha \to \mathbb{C}^*$ and the space $\mathcal{H}_\chi$ decomposes into a tensor product over the character $\chi_\alpha : T_\alpha \to \mathbb{C}^*$.

$$\mathcal{H}_\chi = \bigotimes \mathcal{H}_{\chi_\alpha}. \tag{3.3}$$

It follows from the above decomposition that it is enough to estimate matrix coefficients with respect to "pure tensor" character vector φ of the form $\varphi = \otimes\varphi_\alpha$, where $\varphi_\alpha \in \mathcal{H}_{\chi_\alpha}$. For a vector of the form $v = \otimes v_\alpha$ we have

$$\langle\bigotimes\varphi_\alpha|\pi(v)\bigotimes\varphi_\alpha\rangle = \prod\langle\varphi_\alpha|\pi(v_\alpha)\varphi_\alpha\rangle. \tag{3.4}$$

Hence, we are reduced to estimate the matrix coefficients $\langle\varphi_\alpha|\pi(v_\alpha)\varphi_\alpha\rangle$, but these are defined in terms of the two-dimensional Heisenberg–Weil representation $\overline{\tau}_\alpha$. In addition, we recall the assumption that the vector $v \in V$ is not contained in any proper T-invariant subspace. This condition in turn implies that no summand v_α is an eigenvector of T_α. Hence, we can use Theorem 3.2 and the fact that $|T_\alpha|$ is of order of $q^{[K_\alpha:\mathbb{F}_q]} \pm 1$ to get

$$|\langle\varphi_\alpha|\pi(v_\alpha)\varphi_\alpha\rangle| \le (2 + o(1))/\sqrt{q}^{[K_\alpha:\mathbb{F}_q]}. \tag{3.5}$$

Consequently, using (3.4) and (3.5) we obtain

$$|\langle\bigotimes\varphi_\alpha|\pi(v)\bigotimes\varphi_\alpha\rangle| \le (2 + o(1))^{|\Xi|}/\sqrt{q}^{\sum[K_\alpha:\mathbb{F}_q]} = (2 + o(1))^{|\Xi|}/\sqrt{q}^N,$$

where $r_p = |\Xi|$ is the symplectic rank of the torus T. Let us summarize:

Theorem 3.4 (Bound on Wigner distributions—general case)**.** *Let (V,ω) be a $2N$-dimensional vector space over the finite field $\mathbb{F}_q$ and $(\tau, J, \mathcal{H})$ the corresponding Heisenberg–Weil representation. Let $\varphi \in \mathcal{H}_\chi$ be a unit χ-eigenstate with respect to a maximal torus $T \subset Sp$. We have the following estimate:*

$$|\langle\varphi|\pi(v)\varphi\rangle| \leq \frac{m_\chi \cdot (2+o(1))^{r_p}}{\sqrt{q}^N},$$

where $1 \leq r_p \leq N$ is the symplectic rank of T, $m_\chi = \dim \mathcal{H}_\chi$, and $v \in V$ is not contained in any T-invariant subspace.

4. The Hannay–Berry model

We shall proceed to describe the higher-dimensional Hannay–Berry model of quantum mechanics on toral phase spaces. This model plays an important role in the mathematical theory of quantum chaos as it serves as a model where general phenomena, which are otherwise treated only on a heuristic basis, can be rigorously proven.

4.1. **The phase space.** Our phase space is the $2N$-dimensional symplectic torus $(\mathbb{T}, \omega)$. We denote by Γ the group of linear symplectomorphisms of $\mathbb{T}$. Note that $\Gamma \simeq Sp(2N, \mathbb{Z})$. On the torus $\mathbb{T}$ we consider the algebra $\mathcal{A}$ of observables—trigonometric polynomials. The algebra $\mathcal{A}$ has as a natural basis the lattice Λ of characters (exponents) of $\mathbb{T}$. The form ω induces a skew-symmetric form on Λ, which we denote also by ω, and we assume it takes integral values on Λ and is normalized so that $\int_{\mathbb{T}} |\omega|^N = 1$.

4.2. **The mechanical system.** Our mechanical system is of a very simple nature, consists of an automorphism $A \in \Gamma$ which we assume to be generic element (see Definition 0.2), i.e., A is regular and admits no invariant co-isotropic sub-tori. The last condition can be equivalently restated in dual terms, namely, requiring that A admits no invariant isotropic subvectorspaces in $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. The element A generates, via its action as an automorphism $A : \mathbb{T} \longrightarrow \mathbb{T}$, a discrete time dynamical system.

4.3. **Quantization via the non-commutative torus model.** In this paper we employ a quantization model, that we call the *non-commutative torus* model, developed in [13, 14, 19]. This is a certain one-parameter family of "protocols" parameterized by a parameter $\hbar$ called the Planck constant. For each $\hbar$ the protocol associates to observables from $\mathcal{A}$ and automorphisms from Γ certain operators acting on a finite dimensional Hilbert space $\mathcal{H}$.

Let $\hbar = \frac{1}{p}$, where p is an odd prime number, and consider the additive character $\psi : \mathbb{F}_p \longrightarrow \mathbb{C}^*$, $\psi(t) = e^{\frac{2\pi i}{p} t}$. Define the non-commutative torus [6, 33] $\mathcal{A}_\hbar$ to be the free non-commutative $\mathbb{C}$ -algebra generated by the symbols $s(\xi)$, $\xi \in \Lambda$, and the relations

$$\psi(\tfrac{1}{2}\omega(\xi,\eta))s(\xi+\eta) = s(\xi)s(\eta). \tag{4.1}$$

Here we consider ω as a map $\omega : \Lambda \times \Lambda \longrightarrow \mathbb{F}_p$.

Note that $\mathcal{A}_\hbar$ satisfies the following properties:

- As a vector space $\mathcal{A}_\hbar$ is equipped with a natural basis $s(\xi)$, $\xi \in \Lambda$. Hence we can identify the vector space $\mathcal{A}_\hbar$ with the vector space $\mathcal{A}$ for each value of $\hbar$,

$$\mathcal{A} \simeq \mathcal{A}_\hbar. \tag{4.2}$$

- Substituting $\hbar = 0$ we have $\mathcal{A} = \mathcal{A}_0$. Hence, we see that indeed $\mathcal{A}_\hbar$ is a deformation of the algebra of trigonometric polynomials on $\mathbb{T}$.
- The group Γ acts by automorphisms on the algebra $\mathcal{A}_\hbar$, via $\gamma \cdot s(f) = s(\gamma f)$, where $\gamma \in \Gamma$ and $f \in \mathcal{A}_\hbar$. This action induces an action of Γ on the category of representations of $\mathcal{A}_\hbar$, taking a representation π and sending it to the representation π^γ, where $\pi^\gamma(f) = \pi(\gamma f)$.

We use the identification (4.2) and a distinguished representation of the algebra $\mathcal{A}_\hbar$ to describe the quantization of the functions. All the irreducible algebraic representations of $\mathcal{A}_\hbar$ are classified [13] and each of them is of dimension p^N.

Theorem 4.1 (Invariant representation [13])**.** *Let $\hbar = \frac{1}{p}$ where p is a prime number. There exists a unique (up to isomorphism) irreducible representation $\pi : \mathcal{A}_\hbar \to End(\mathcal{H}_\hbar)$ which is fixed by the action of Γ. Namely, π^γ is isomorphic to π for every $\gamma \in \Gamma$.*

Let $(\pi, \mathcal{A}_\hbar, \mathcal{H})$ be a representative of the the special representation defined in Theorem 4.1. For every element $\gamma \in \Gamma$ we have an isomorphism $\widetilde{\rho}(\gamma) : \mathcal{H} \to \mathcal{H}$ intertwining the representations π and π^γ, namely, it satisfies

$$\widetilde{\rho}(\gamma)\pi(f)\widetilde{\rho}(\gamma)^{-1} = \pi(\gamma f), \tag{4.3}$$

for every $f \in \mathcal{A}_\hbar$ and $\gamma \in \Gamma$. The isomorphism $\widetilde{\rho}(\gamma)$ is not unique but unique up to a scalar (this is a consequence of Schur's lemma). It is easy to realize that the collection $\{\widetilde{\rho}(\gamma)\}$ constitutes a projective representation $\widetilde{\rho} : \Gamma \to PGL(\mathcal{H})$. Assume now that $\hbar = \frac{1}{p}$ where p is an odd prime $\neq 3$.We have the following linearization theorem:

Theorem 4.2 (Linearization [14, 19])**.** *There exist a unique representation $\rho : \Gamma \to GL(\mathcal{H})$ that that satisfies (4.3) and factors through the quotient group $Sp \simeq Sp(2N, \mathbb{F}_p)$.*

Concluding, we established a distinguished pair of representations

$$\rho : \Gamma \to GL(\mathcal{H}) \quad \text{and} \quad \pi : \mathcal{A}_\hbar \to End(\mathcal{H}),$$

satisfying the compatibility condition (4.3).

Remark 4.3. *The representation $\rho : Sp \to GL(\mathcal{H})$ defined by Theorem 4.2 is the Weil representation associated with the additive character $\psi(t) = e^{\frac{2\pi i}{p}t}$—here obtained in a different manner via quantization of the torus.*

4.4. **The quantum dynamical system.** Recall that we started with a dynamic on $\mathbb{T}$, generated by a generic (i.e., regular with no non-trivial invariant co-isotropic sub-tori) element $A \in \Gamma$. Using the Weil representation we can associate to A the unitary operator $\rho(A) : \mathcal{H} \to \mathcal{H}$, which constitutes the generator of discrete time quantum dynamics. We would like to study $\rho(A)$-eigenstates

$$\rho(A)\varphi = \lambda\varphi, \quad \varphi \in \mathcal{H}, \tag{4.4}$$

which satisfy addition arithmetic symmetries.

5. Hecke quantum unique ergodicity

It turns out that the operator $\rho(A)$ has degeneracies—its eigenspaces might be extremely large. This is manifested in the existence of a group of hidden[4] symmetries commuting with $\rho(A)$. These symmetries can be computed using the Weil representation. Indeed, let $T_A = Z(A, Sp)$ be the centralizer of the element A in the group Sp. Clearly T_A contains the cyclic group $\langle A \rangle$ generated by the element A, but it often happens that T_A contains additional elements. The assumption that A is generic implies that for sufficiently large p (so that p does not divides the discriminant of A) the group T_A consists of the $\mathbb{F}_p$-rational points of a maximal torus $\mathbf{T}_A \subset \mathbf{Sp}$, i.e., $T_A = \mathbf{T}_A(\mathbb{F}_p)$. We will call the group T_A the *Hecke* torus. It acts semisimply on $\mathcal{H}$, decomposing it into a direct sum of character spaces $\mathcal{H} = \oplus\, \mathcal{H}_\chi$ where χ runs in the group of characters of T_A. We shall study common eigenstates $\varphi \in \mathcal{H}_\chi$, which we will call in this setting *Hecke eigenstates* and will be assumed to be normalized so that $\|\varphi\|_{\mathcal{H}} = 1$. In particular, we will bound the Wigner distributions $\langle \varphi | \pi(f) \varphi \rangle$, where $f \in \mathcal{A}$ is an observable on the torus $\mathbb{T}$. We will call these matrix coefficients *Hecke–Wigner distributions*. It will be convenient for us to treat two cases.

5.1. The strongly generic case.

Let us assume first that the automorphism A acts on $\mathbb{T}$ with no invariant sub-tori. In dual terms, this means that the element A acts irreducibly on the $\mathbb{Q}$-vector space $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote by r_p the symplectic rank of T_A, i.e., $r_p = |\Xi|$ where $\Xi = \Xi(T_A)$ is the symplectic type of T_A (see Definition 2.8). By definition we have $1 \leq r_p \leq N$.

Theorem 5.1. *Consider a non-trivial exponent $0 \neq \xi \in \Lambda$ and a sufficiently large prime number p. Then for every unit Hecke eigenstate $\varphi \in \mathcal{H}_\chi$ the following bound holds:*

$$|\langle \varphi | \pi(\xi) \varphi \rangle| \leq \frac{m_\chi \cdot (2 + o(1))^{r_p}}{\sqrt{p}^N}, \tag{5.1}$$

where $m_\chi = \dim \mathcal{H}_\chi$.

The lattice Λ constitutes a basis for $\mathcal{A}$; hence, using the bound (5.1) we obtain:

Theorem 5.2 (Hecke quantum unique ergodicity—strongly generic case)**.** *Consider an observable $f \in \mathcal{A}$ and a sufficiently large prime number p. Then for every normalized Hecke eigenstate φ we have*

$$\left| \langle \varphi | \pi(f) \varphi \rangle - \int_{\mathbb{T}} f d\mu \right| \leq \frac{C_f}{\sqrt{p}^N},$$

where $\mu = |\omega|^N$ is the corresponding volume form and C_f is an explicit computable constant which depends only on the function f.

Remark 5.3. *In Subsection 5.2 we will elaborate on the distribution of the symplectic rank r_p (5.1).*

[4] The group $Z(A, \Gamma)$, of the elements in Γ that commute with A, does not contribute much to the harmonic analysis of $\rho(A)$.

5.1.1. *Proof of Theorem 5.1.* The proof is by reduction to the bound on the Hecke–Wigner distributions obtained in Section 3, i.e., reduction to Theorem 3.4. Our first goal is to interpret the Hecke–Wigner distribution $\langle\varphi|\pi(\xi)\varphi\rangle$ in terms of the Heisenberg–Weil representation.

Step 1. *Replacing the non-commutative torus by the finite Heisenberg group.* Note that the Hilbert space $\mathcal{H}$ is a representation space of both the algebra $\mathcal{A}_\hbar$ and the group Sp via π and ρ, respectively. We will show next that the representation $(\pi, \mathcal{A}_\hbar, \mathcal{H})$ is "equivalent" to the Heisenberg representation of some finite Heisenberg group. The representation π is determined by its restriction to the lattice Λ. However, the restriction $\pi_{|\Lambda} : \Lambda \to GL(\mathcal{H})$ is not an homomorphism and in fact constitutes (see formula (4.1)) a projective representation of the lattice given by $\psi(\frac{1}{2}\omega(\xi,\eta))\pi(\xi+\eta) = \pi(\xi)\pi(\eta)$. It is evident from this formula that the map $\pi_{|\Lambda}$ factors through the quotient $\mathbb{F}_p$-vector space $V = \Lambda/p\Lambda$, i.e., $\Lambda \to V = \Lambda/p\Lambda \to GL(\mathcal{H})$. The vector space V is equipped with a symplectic structure ω obtained via specialization of the form on Λ. Let $H = H(V)$ be the Heisenberg group associated with (V,ω). So the map $\pi : V \to GL(\mathcal{H})$ lifts into an honest representation of the Heisenberg group $\pi : H \to GL(\mathcal{H})$. Finally, the Heisenberg representation π and the Weil representation ρ glue into a single representation $\tau = \rho \ltimes \pi$ of the Jacobi group $J = Sp \ltimes H$, which is of course nothing other than the Heisenberg–Weil representation

$$\tau : J \to GL(\mathcal{H}). \tag{5.2}$$

Step 2. *Reformulation.* Let $\mathbf{V}$ and $\mathbf{T}_A$ be the algebraic group scheme defined over $\mathbb{Z}$ so that $\Lambda = \mathbf{V}(\mathbb{Z})$ and for every prime p we have $V = \mathbf{V}(\mathbb{F}_p)$ and $T_A = \mathbf{T}_A(\mathbb{F}_p)$. In this setting for every prime number p we can consider the lattice element $\xi \in \Lambda$ as a vector in the $\mathbb{F}_p$-vector space V.

Let $(\tau, J, \mathcal{H})$ be the Heisenberg–Weil representation (5.2) and consider a unit Hecke eigenstate $\varphi \in \mathcal{H}_\chi$. We need to verify that for a sufficiently large prime number p we have

$$|\langle\varphi|\pi(\xi)\varphi\rangle| \leq \frac{m_\chi \cdot (2+o(1))^{r_p}}{\sqrt{p}^N}, \tag{5.3}$$

where $m_\chi = \dim \mathcal{H}_\chi$ and r_p is the symplectic rank of T_A.

Step 3. *Verification.* We need to show that we meet the conditions of Theorem 3.4. What is left to check is that for sufficiently large prime number p the vector $\xi \in V$ is not contained in any T_A-invariant subspace of V. Let us denote by O_ξ the orbit $O_\xi = T_A \cdot \xi$. We need to show that for a sufficiently large p we have

$$Span_{\mathbb{F}_p}\{O_\xi\} = V. \tag{5.4}$$

The condition (5.4) is satisfied since it holds globally. In more details, our assumption on A guarantees that it holds for the corresponding objects over the field of rational numbers $\mathbb{Q}$, i.e., $Span_{\mathbb{Q}}\{\mathbf{T}_A(\mathbb{Q})\cdot\xi\} = \mathbf{V}(\mathbb{Q})$. Hence (5.4) holds for sufficiently large prime number p.

This completes the proof of Theorem 5.1.

5.2. **The distribution of the symplectic rank.** We would like to compute the asymptotic distribution of the symplectic rank r_p (5.3) in the set $\{1,...,N\}$, i.e.,

$$\delta(r) = \lim_{x\to\infty} \frac{\#\{r_p = r \ ; \ p \leq x\}}{\pi(x)}, \tag{5.5}$$

where $\pi(x)$ denotes the number of prime numbers up to x.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$, and denote by G the Galois group $G = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Consider the vector space $\mathbf{V} = \mathbf{V}(\overline{\mathbb{Q}})$. By extension of scalars the symplectic form ω on $\mathbf{V}(\mathbb{Q})$ induces a $\overline{\mathbb{Q}}$-linear symplectic form on $\mathbf{V}$ which we will also denote by ω. Let $\mathbf{T}$ denote the algebraic torus $\mathbf{T} = \mathbf{T}_A(\overline{\mathbb{Q}})$. The action of $\mathbf{T}$ on $\mathbf{V}$ is completely reducible, decomposing it into one-dimensional character spaces $\mathbf{V} = \oplus_{\chi\in\mathfrak{X}} \mathbf{V}_\chi$. Let Θ be the restriction of the symplectic transpose $(\cdot)^t : End(\mathbf{V}) \to End(\mathbf{V})$ to $\mathbf{T}$. The involution Θ acts on the set of characters $\mathfrak{X}$ by $\chi \mapsto \Theta(\chi) = \chi^{-1}$ and this action is compatible with the action of the Galois group G on $\mathfrak{X}$ by conjugation $\chi \mapsto g\chi g^{-1}$, where $\chi \in \mathfrak{X}$ and $g \in G$. This means (recall that A is strongly generic) that we have a transitive action of G on the set $\mathfrak{X}/\Theta$. Consider the kernel $\mathrm{K} = \ker(G \to Aut(\mathfrak{X}/\Theta))$ and the corresponding finite Galois group $Q = G/\mathrm{K}$. Considering Q as a subgroup of $Aut(\mathfrak{X}/\Theta)$ we define the cycle number $c(C)$ of a conjugacy class $C \subset Q$ to be the number of irreducible cycles that compose a representative of C. A direct application of the Chebotarev density theorem [35] is the following:

Proposition 5.4 (Chebotarev's theorem)**.** *The distribution δ (5.5) obeys*

$$\delta(r) = \frac{|C_r|}{|Q|},$$

where $C_r = \underset{\substack{C\subset Q \\ c(C)=r}}{\cup} C.$

For a proof see Appendix A.10.

5.3. **The general generic case .** Let us now treat the more general case where the automorphism A acts on $\mathbb{T}$ in a generic way (Definition 0.2). In dual terms, this means that the torus $\mathbf{T}(\mathbb{Q}) = \mathbf{T}_A(\mathbb{Q})$ acts on the symplectic vector space $\mathbf{V}(\mathbb{Q})$ $= \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ decomposing it into an orthogonal symplectic direct sum

$$(\mathbf{V}(\mathbb{Q}), \omega) = \bigoplus_{\alpha\in\Xi} (\mathbf{V}_\alpha(\mathbb{Q}), \omega_\alpha), \tag{5.6}$$

with an irreducible action of $\mathbf{T}(\mathbb{Q})$ on each of the spaces $\mathbf{V}_\alpha(\mathbb{Q})$. For an element $\xi \in \Lambda$ define its support with respect to the decomposition (5.6) by $S_\xi = Supp(\xi) = \{\alpha;\ P_\alpha\xi \neq 0\}$, where $P_\alpha : \mathbf{V}(\mathbb{Q}) \to \mathbf{V}(\mathbb{Q})$ is the projector onto the space $\mathbf{V}_\alpha(\mathbb{Q})$ and denote by d_ξ the dimension $d_\xi = \sum_{\alpha\in S_\xi} \dim \mathbf{V}_\alpha(\mathbb{Q})$. The decomposition (5.6) induces a decomposition of the torus $\mathbf{T}(\mathbb{Q})$ into a product of completely inert tori

$$\mathbf{T}(\mathbb{Q}) = \prod_{\alpha\in\Xi} \mathbf{T}_\alpha(\mathbb{Q}). \tag{5.7}$$

Considering now a sufficiently large prime number p and specialize all the object involved to the finite field $\mathbb{F}_p$. The Hecke torus $T = \mathbf{T}(\mathbb{F}_p)$ acts via the Weil representation on the Hilbert space $\mathcal{H}$ decomposing it into an orthogonal direct sum $\mathcal{H} = \oplus\, \mathcal{H}_\chi$. The decomposition (5.7) induces decompositions on the level of groups of points $T = \prod T_\alpha$, where $T_\alpha = \mathbf{T}_\alpha(\mathbb{F}_p)$, on the level of characters $\chi = \Pi\chi_\alpha : \prod T_\alpha \to \mathbb{C}^*$, and on the level of character spaces $\mathcal{H}_\chi = \oplus\mathcal{H}_{\chi_\alpha}$. For each torus T_α we denote by $r_{p,\alpha} = r_p(T_\alpha)$ its symplectic rank (see Definition 2.8) and we consider the integer $|S_\xi| \leq r_{p,\xi} \leq d_\xi$ given by $r_{p,\xi} = \underset{\alpha\in S_\xi}{\Pi} r_{p,\alpha}$. Let us denote by m_{χ_ξ} the dimension $m_{\chi_\xi} = \sum_{\alpha\in S_\xi} \dim \mathcal{H}_{\chi_\alpha}$.

Theorem 5.5. *Consider a non-trivial exponent $0 \neq \xi \in \Lambda$ and a sufficiently large prime number p. Then for every unit Hecke eigenstate $\varphi \in \mathcal{H}_\chi$ we have*

$$|\langle \varphi | \pi(\xi) \varphi \rangle| \leq \frac{m_{\chi_\xi} \cdot (2 + o(1))^{r_{p,\xi}}}{\sqrt{p}^{d_\xi}}.$$

We consider the decomposition (5.6) and denote by $d = \min_\alpha \mathbf{V}_\alpha(\mathbb{Q})$. Using the lattice Λ as a basis for the algebra $\mathcal{A}$ we obtain:

Theorem 5.6 (Hecke quantum unique ergodicity—generic case)**.** *Consider an observable $f \in \mathcal{A}$ and a sufficiently large prime number p. Then for every unit Hecke eigenstate φ we have*

$$\left| \langle \varphi | \pi(f) \varphi \rangle - \int_{\mathbb{T}} f d\mu \right| \leq \frac{C_f}{\sqrt{p}^d},$$

where $\mu = |\omega|^N$ is the corresponding volume form and C_f is an explicit computable constant which depends only on the function f.

The proof of Theorem 5.5 is a straightforward application of Theorem 5.1. In more details, considering the "global" decomposition (5.6) of the torus $\mathbf{T}(\mathbb{Q})$ to a product of completely inert tori $\mathbf{T}_\alpha(\mathbb{Q})$ we may apply the theory developed for the strongly generic case in Subsection (5.1) to each of the tori $\mathbf{T}_\alpha(\mathbb{Q})$ to deduce Theorem 5.5.

Remark 5.7. *As explained in Subsection (5.2) the distribution of the symplectic rank $r_{p,\xi}$ is determined by the Chebotarev theorem applied to (now a product of) suitable finite Galois groups Q_α attached to the tori $\mathbf{T}_\alpha$, $\alpha \in S_\xi$ (5.7).*

Remark 5.8. *The corresponding quantum unique ergodicity theorem for statistical states of generic automorphism A of $\mathbb{T}$ (cf. Theorem 0.7) follows directly from Theorem 5.5 .*

Appendix A. Proofs

A.1. **Proof of Theorem 1.3.** Fix a character χ of T and denote by P_χ the orthogonal projector on the character space $\mathcal{H}_\chi$. The projector P_χ can be written in terms of the Weil representation $P_\chi = |T|^{-1} \sum_{g \in T} \chi^{-1}(g) \rho(g)$; therefore $m_\chi = \dim \mathcal{H}_\chi = Tr(P_\chi) = |T|^{-1} \sum_{g \in T} \chi^{-1}(g) Tr(\rho(g))$ which by the character formula (1.2) is equal to

$$\frac{1}{|T|} \left(\left(\sum_{g \in T \smallsetminus I} \chi^{-1}(g) \sigma(-\det(g - I)) \right) + q \right).$$

From the orthogonality relations for characters, and the fact that $|T| = q - 1$ if T splits and equal $q + 1$ if T is inert, we see that it is enough to prove the following claim:

Claim A.1. *On $T \smallsetminus I$ we have $\sigma(-\det(g - I)) = \pm \sigma_T(g)$ where the $+$ and $-$ are attained when the torus T is split or inert, respectively.*

A.1.1. *Proof of Claim A.1.* The equality is easily satisfied in the split case. Let us assume that T is inert and identify $Sp \simeq SL(2,\mathbb{F}_q) \subset SL(2,\mathbb{F}_{q^2})$. There exists a matrix $S \in SL(2,\mathbb{F}_{q^2})$ so that

$$STS^{-1} = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix};\ c \in C \right\},$$

where C is the kernel of the norm map $C = Ker(N : \mathbb{F}_{q^2} \to \mathbb{F}_q) \subset \mathbb{F}_{q^2}$. In these coordinates our claim reduced to the identity $(\frac{(c-1)^2}{c})^{\frac{q-1}{2}} = -c^{\frac{q+1}{2}}$. This completes the proof of the claim and of Theorem 1.3.

A.2. **Proof of Proposition 2.3, and Claim 2.2.** Consider a symplectic vector space (V,ω) over k and a maximal torus $T \subset Sp(V,\omega)$. Assume that T acts irreducibly on V. Denote by $A = Z(T, End(V))$ the centralizer of T in the algebra of all linear endomorphisms of V. We need to show that A is commutative and that $\dim_K V = 2$, where $K = A^{\Theta}$, with Θ being the canonical quadratic element in $Gal(A/k)$ obtained by restricting the symplectic transpose defined by ω on $End(V)$ to A.

Let $\overline{k}$ denote an algebraic closure of the field k and denote by G the Galois group $G = Gal(\overline{k}/k)$. Consider the vector space $\mathbf{V} = \overline{k} \otimes V$. By extension of scalars the symplectic structure ω induces a $\overline{k}$-linear symplectic structure on $\mathbf{V}$, which we denote also by ω. Let $\mathbf{T}$ denote the algebraic torus, i.e., $\mathbf{T} = \overline{k} \otimes T$. Consider the algebra $\mathbf{A} = Z(\mathbf{T}, End(\mathbf{V}))$. Note that in this setting $\mathbf{A}$ is not necessarily a division algebra. Let Θ be the restriction of the symplectic transpose $(\cdot)^t : End(\mathbf{V}) \to End(\mathbf{V})$, to the algebra $\mathbf{A}$ and denote by $\mathbf{K} = \mathbf{A}^{\Theta}$ the subalgebra consisting of elements $a \in \mathbf{A}$ which are fixed by Θ. The group G acts on all structures involved. We have $V = \mathbf{V}^G$, $T = \mathbf{T}^G$, $A = \mathbf{A}^G$ and $K = \mathbf{K}^G$.

The action of $\mathbf{T}$ on $\mathbf{V}$ is completely reducible, decomposing it into one-dimensional character spaces

$$\mathbf{V} = \bigoplus_{\chi \in \mathfrak{X}} \mathbf{V}_{\chi}. \tag{A.1}$$

The set $\mathfrak{X}$, consists of N pairs of characters, $\chi, \chi^{-1} \in \mathfrak{X}$. The algebra $\mathbf{A}$ consists of operators which are diagonal with respect to the decomposition (A.1) $\mathbf{A} = diag(a_{\chi} \in \overline{k} : \chi \in \mathfrak{X})$. In particular, this implies that $A = \mathbf{A}^G$ is commutative as was claimed. The involution Θ can be described as $a = diag(a_{\chi}) \longmapsto \Theta(a) = diag(\Theta(a)_{\chi} = a_{\chi^{-1}})$. Therefore, we obtain that $\mathbf{K} = \left\{a = diag(a_{\chi}) : a_{\chi} = a_{\chi^{-1}}\right\}$.

Finally, we observe that $\mathbf{V}$ is a free module of rank 2 over the algebra $\mathbf{K}$. By Hilbert's Theorem 90, this property continues to hold after taking Galois invariants, that is we obtain that $\dim_K V = 2$.

A.3. **Proof of Corollary 2.5.** We need to show that $T = \left\{a \in A : N_{A/K}(a) = 1\right\}$. Because T is a maximal torus in Sp, it coincides with its own centralizer $Z(T, Sp)$. We have $Z(T, Sp) = A \cap Sp$. An element $a \in A$ lies in the group Sp if it satisfies $\omega(au, av) = \omega(u, v)$ for every $u, v \in V$. This is equivalent to $\omega(u, \Theta(a)av) = \omega(u, v)$ which in turn implies that $N_{A/K}(a) = \Theta(a)a = 1$.

A.4. **Proof of Proposition 2.6.** We should prove that there exists a T-invariant K-linear symplectic form $\overline{\omega} : V \times V \to K$, satisfying the property $Tr \circ \overline{\omega} = \omega$. Consider the decomposition (A.1) $\mathbf{V} = \oplus \mathbf{V}_{\chi}$. Define $\overline{\omega} : \mathbf{V} \times \mathbf{V} \to \mathbf{K}$ as follows

$$\overline{\omega}(\sum v_{\chi}, \sum u_{\chi}) = diag(a_{\chi} = \omega(v_{\chi}, u_{\chi^{-1}}) + \omega(v_{\chi^{-1}}, u_{\chi})).$$

We have $Tr\circ\overline{\varpi}=\omega$. Clearly, the form $\overline{\varpi}$ is invariant under the action of the torus $\mathbf{T}$. Finally, the form $\overline{\varpi}$ commutes with the Galois action, hence it restricts to give a desired form $\overline{\varpi}:V\times V\rightarrow K,$ which is T-invariant and satisfies $Tr\circ\overline{\varpi}=\omega$.

A.5. **Proof of Lemma 2.7 .** We use the notation of Subsection A.2. Consider the decomposition $\mathbf{V}=\oplus_{\chi\in\mathfrak{X}}\mathbf{V}_{\chi}$. The Galois group $G=Gal(\overline{k}/k)$ acts on the set of characters $\mathfrak{X}$. The action is given by conjugation $\chi\mapsto g\chi g^{-1}$, where $\chi\in\mathfrak{X}$ and $g\in G$. The set $\mathfrak{X}$ decomposes into a union of G orbits, $\mathfrak{X}=\bigcup\mathcal{O}_{\beta}$. To each orbit $\mathcal{O}_{\beta}$ there exists a unique dual orbit $\mathcal{O}_{\widehat{\beta}}$ so that $\chi\in\mathcal{O}_{\beta}$ if and only if $\chi^{-1}\in\mathcal{O}_{\widehat{\beta}}$ (note that sometimes $\mathcal{O}_{\widehat{\beta}}=\mathcal{O}_{\beta}$). Let $\mathcal{O}_{\alpha}$ denote the union $\mathcal{O}_{\alpha}=\mathcal{O}_{\beta}\cup\mathcal{O}_{\widehat{\beta}}$. We denote by Ξ the set of $\mathcal{O}_{\alpha}$'s. We use the following terminology: If $\mathcal{O}_{\widehat{\beta}}=\mathcal{O}_{\beta}$, we say that $\mathcal{O}_{\alpha}$ is of type I, otherwise we say that $\mathcal{O}_{\alpha}$ is of type II.

The decomposition $\mathfrak{X}=\bigcup\limits_{\alpha\in\Xi}\mathcal{O}_{\alpha}$ induces a decomposition

$$(\mathbf{V},\boldsymbol{\omega})=\bigoplus_{\alpha\in\Xi}(\mathbf{V}_{\alpha},\omega_{\alpha}),\tag{A.2}$$

where $\mathbf{V}_{\alpha}=\oplus_{\chi\in\mathcal{O}_{\alpha}}\mathbf{V}_{\chi}$. This in turn induces the following decompositions:

$$\mathbf{T}{=}\prod\mathbf{T}_{\alpha},\ \ \mathbf{A}{=}\bigoplus\mathbf{A}_{\alpha},\tag{A.3}$$

where $\mathbf{T}_{\alpha}$ is the sub-torus consisting of elements $g\in\mathbf{T}$ such that $g_{|\mathbf{V}_{\beta}}=Id$ for every $\beta\neq\alpha$. In the same spirit $\mathbf{A}_{\alpha}$ is the subalgebra consisting of elements $a\in\mathbf{A}$ such that $a_{|\mathbf{V}_{\beta}}=Id$, for every $\beta\neq\alpha$. It is easy to verify that each algebra $\mathbf{A}_{\alpha}$ is closed under the involution Θ. Let $\mathbf{K}_{\alpha}$ denote the invariant subalgebra $\mathbf{K}_{\alpha}=\mathbf{A}_{\alpha}^{\Theta}$. We have the following decomposition:

$$\mathbf{K}=\bigoplus\mathbf{K}_{\alpha}.\tag{A.4}$$

Finally, following the same construction as in Subsection A.4, we can lift the symplectic form ω_{α} to a $\mathbf{K}_{\alpha}$-linear $\mathbf{T}_{\alpha}$-invariant symplectic form

$$\overline{\varpi}_{\alpha}:\mathbf{V}_{\alpha}\times\mathbf{V}_{\alpha}\rightarrow\mathbf{K}_{\alpha},\tag{A.5}$$

satisfying, $Tr\circ\overline{\varpi}_{\alpha}=\omega_{\alpha}$.

Now it is clear that all the decompositions (A.2) ,(A.3),(A.4), and (A.5) are compatible with the action of the Galois group G, hence, they induce corresponding decompositions on the level of invariants, which establish the content of the lemma.

We are left to show that for each $\alpha\in\Xi$, the algebra K_{α} is a field, and, in addition, $\dim_{K_{\alpha}}V_{\alpha}=2$. The second claim follows from the (very easy to verify) fact that the vector space $\mathbf{V}_{\alpha}$ is a free module of rank 2 over the algebra $\mathbf{A}_{\alpha}$. In proving the first claim, we individually analyze two cases

- The orbit $\mathcal{O}_{\alpha}$ is of type I. In this case T_{α} acts irreducibly on V_{α} and we are back in the completely inert situation of Subsection 2.2.1.
- The orbit $\mathcal{O}_{\alpha}$ is of type II. In this case $V_{\alpha}=V_{\beta}\oplus V_{\widehat{\beta}}$ where $V_{\beta},V_{\widehat{\beta}}$ are irreducible T_{α} invariant Lagrangian subspaces. Respectively, the algebra A_{α} decomposes into a direct sum of two fields $A_{\alpha}=A_{\beta}\oplus A_{\widehat{\beta}}$. Note that $A_{\beta}=Z(T_{\alpha},End(V_{\beta}))$ and $A_{\widehat{\beta}}=Z(T_{\alpha},End(V_{\widehat{\beta}}))$, which implies that $A_{\beta},A_{\widehat{\beta}}$ are indeed fields. Moreover, the involution Θ induces an isomorphism $A_{\beta}\simeq A_{\widehat{\beta}}$ and K_{α} is the diagonal field with respect to this isomorphism.

This concludes the proof of Lemma 2.7.

A.6. **Proof of Proposition 2.9.** The proof is immediate. Clearly, the representation $\overline{\pi}$ is irreducible (the homomorphism ι_H is surjective). Therefore, by the Stone–von Neumann theorem (Theorem 1.1) it is determined by its central character. The homomorphism ι restricts to an homomorphism $K = Z(\overline{H}) \to Z(H) = k$ between the centers, which is given by $Tr = Tr_{K/k}$. Consequently, this implies that $\overline{\pi}_{|Z(\overline{H})} = \overline{\psi} \cdot Id$. Concluding the proof of the proposition.

A.7. **Proof of Theorem 2.10.** It is enough to show that the representations $\overline{\rho}$ and $\overline{\pi}$ satisfy the compatibility condition (1.1). Indeed, for $g \in \overline{Sp}$ and $h \in \overline{H}$

$$\begin{aligned} \overline{\rho}(g)\overline{\pi}(h)\overline{\rho}(g)^{-1} &= \rho(\iota_S(g))\pi(\iota_H(h))\rho(\iota_S(g))^{-1} \\ &= \pi(\iota_S(g) \cdot \iota_H(h)) = \pi(\iota_H(g \cdot h) = \overline{\pi}(g \cdot h), \end{aligned}$$

concluding the proof.

A.8. **Proof of Proposition 2.12.** The representation $\overline{\pi}$ is irreducible (ι_H is surjective), this immediately implies that $(\overline{\pi}, \overline{H}, \mathcal{H}) \simeq (\otimes \overline{\pi}_\alpha, \Pi \overline{H}_\alpha, \otimes \mathcal{H}_\alpha)$, where $\overline{\pi}_\alpha$ is irreducible, for every α. By the Stone–von Neumann theorem a representation of the Heisenberg group $\overline{H}_\alpha$ is characterized by the action of the center, which clearly in this case acts via the character $\overline{\psi}_\alpha = \psi \circ Tr_{K_\alpha/k}$. This completes the proof.

A.9. **Proof of Theorem 2.13.** The representation $\overline{\rho}$ is characterized by the identity (1.1) with respect to the representation $\overline{\pi}$. Using Proposition 2.12, it is enough to show that $\otimes\overline{\rho}_\alpha$ satisfies the identity (1.1) with respect to the representation $\otimes\overline{\pi}_\alpha$, which is principally reduced to the case analyzed in Subsection A.7. This completes the proof.

A.10. **Proof of Proposition 5.4.** Given the realization of the symplectic rank $r_p = |\Xi|$ that appears in Subsection A.5, the proof of Proposition 5.4 is an immediate consequence of the Chebotarev density theorem [35].

Appendix B. Proof of Theorem 3.2

We need to show that $\left|\sum_{g\in T} \chi^{-1}(g) ch_\tau(g,v)\right| \leq 2\sqrt{q}$. It is easy to verify that for $g = 1$, $v \neq 0$ we have $ch_\tau(g,v) = 0$. Moreover, for $g \neq 1$ using the character formula (1.3) and Corollary 1.4 we have $ch_\tau(g,v) = \pm\sigma_T(g)\psi(\frac{1}{2}\omega(\frac{v}{g-I}, v))$ where σ_T is the unique quadratic character of T. We may assume $\chi \neq \sigma_T$ (the case $\chi = \sigma_T$ being trivial). Concluding, for a non-trivial character χ of T we consider the function $f_\chi(g) = \chi(g)\psi(\frac{1}{2}\omega(\frac{v}{g-I}, v))$ on the set $X = T \smallsetminus I$ and we would like to show that

$$\left| c_\chi = \sum_{g\in X} f_\chi(g) \right| \leq 2\sqrt{q}. \tag{B.1}$$

B.1. **Solution via geometrization.** Our problem fits nicely into Grothendieck's geometrization methodology: replacing sets by varieties; functions be sheaves; and reducing the proof of the estimate (B.1) to a topological statement on certain cohomology groups.

B.1.1. *Replacing sets by varieties.* We denote by $\overline{k}$ an algebraic closure of the field $k = \mathbb{F}_q$. We use bold-face letters to denote a variety $\mathbf{Y}$ and normal letters Y to denote its corresponding set of rational points $Y = \mathbf{Y}(k)$. By a variety $\mathbf{Y}$ over k we mean a quasi-projective algebraic variety, such that the defining equations are given by homogeneous polynomials with coefficients in the finite field k. In this situation, there exists a (geometric) *Frobenius* endomorphism $Fr : \mathbf{Y} \to \mathbf{Y}$, which is a morphism of algebraic varieties. We denote by Y the set of points fixed by Fr, i.e., $Y = \mathbf{Y}(k) = \{y \in \mathbf{Y} : Fr(y) = y\}$.

Important example for us are k and k^* which are the sets of rational points of the algebraic varieties $\mathbb{G}_a$ and $\mathbb{G}_m$, respectively, and T and $X = T \setminus I$ which are the sets of rational points of the algebraic varieties $\mathbf{T} \subset \mathbf{Sp}$ and $\mathbf{X} = \mathbf{T} \setminus \mathbf{I}$.

B.1.2. *Replacing functions by sheaves.* Let $\mathsf{D}^b(\mathbf{Y})$ denote the bounded derived category of constructible ℓ-adic sheaves on $\mathbf{Y}$ [4, 7, 29]. A Weil structure associated to an object $\mathcal{F} \in \mathsf{D}^b(\mathbf{Y})$ is an isomorphism $\theta : Fr^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. A pair $(\mathcal{F}, \theta)$ is called a Weil object. By an abuse of notation we often denote θ also by Fr. We choose once an identification $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, hence all sheaves are considered over the complex numbers. Given a Weil object $(\mathcal{F}, Fr^*\mathcal{F} \simeq \mathcal{F})$ one can associate to it a function $f^{\mathcal{F}} : Y \to \mathbb{C}$ to $\mathcal{F}$ as follows

$$f^{\mathcal{F}}(y) = \sum_i (-1)^i Tr(Fr_{|H^i(\mathcal{F}_y)}). \tag{B.2}$$

This procedure is called *Grothendieck's sheaf-to-function correspondence* [11].

Important examples for us are the additive character $\psi : k \longrightarrow \mathbb{C}^\times$ which is associated via the sheaf-to-function correspondence to the Artin–Schreier sheaf $\mathcal{L}_\psi$ on the variety $\mathbb{G}_a$, i.e., we have $f^{\mathcal{L}_\psi} = \psi$; the Legendre character σ on k^* which is associated to the Kummer sheaf $\mathcal{L}_\chi$ on the variety $\mathbb{G}_m$, i.e., $f^{\mathcal{L}_\chi} = \chi$; and the function f_χ on X which is associated with the Weil sheaf $\mathcal{F}_\chi = \mathcal{L}_{\chi(g)} \otimes \mathcal{L}_{\psi(1/2\omega(\frac{v}{g-1}, v))}$ on $\mathbf{X}$. Finally, we have the Weil object

$$\mathcal{C}_\chi = \int \mathcal{F}_\chi \in \mathsf{D}^b(\mathbf{pt}),$$

where $\int = \int_!$ denotes integration with compact support [7]. The Grothendieck–Lefschetz trace formula [11] implies that $f^{\mathcal{C}_\chi} = c_\chi$.

B.1.3. *Geometric statement.* The sheaf $\mathcal{F}_\chi$ is a non-trivial rank 1 irreducible local system of pure weight zero $w(\mathcal{F}_\chi) = 0$. The deep theorem of Deligne [7] say that integration with compact support does not increase weight; therefore $\mathcal{C}_\chi$ is of mixed weight $w(\mathcal{C}_\chi) \leq 0$, i.e., we have the following bound on each eigenvalue λ of Fr :

$$|\lambda(Fr_{|H^i(\mathcal{C}_\chi)})| \leq \sqrt{q}^i.$$

This means by (B.2) that in order to get the bound (B.1) it is enough to prove the following geometric statement:

Lemma B.1 (Vanishing lemma)**.** *The object $\mathcal{C}_\chi$ is cohomologically supported at degree* 1 *and in addition* $\dim H^1(\mathcal{C}_\chi) = 2$.

B.1.4. *Proof of the Vanishing lemma.* The fact that only $H^1(\mathcal{C}_\chi)$ does not vanishes follows from the fact that the local system $\mathcal{F}$ is of rank 1 and non-trivial, i.e., it admits non-trivial monodromy. We are left to compute the dimension of $H^1(\mathcal{C}_\chi)$. Because $\mathcal{C}_\chi$ is cohomologically supported at degree 1 we have

$$\dim H^1(\mathcal{C}_\chi) = -\chi_{Fr}(\mathcal{C}_\chi),$$

where χ_{Fr} denotes the Euler characteristic. This is a topological invariant defined by $\chi_{Fr}(\mathcal{C}_\chi) = \sum_i (-1)^i \dim H^i(\mathcal{C}_\chi)$.

The actual computation of the Euler characteristic $\chi_{Fr}(\mathcal{C}_\chi)$ is done using the *Ogg–Shafarevich–Grothendieck formula* [25, 29]. Recall that $\mathcal{C}_\chi = \int \mathcal{F}_\chi$. We have

$$\chi_{Fr}(\textstyle\int \overline{\mathbb{Q}}_\ell) - \chi_{Fr}(\int \mathcal{F}_\chi) = \displaystyle\sum_{y \in \mathbf{Y} \setminus \mathbf{X}} \mathrm{Swan}_y(\mathcal{F}_\chi), \tag{B.3}$$

where $\overline{\mathbb{Q}}_\ell$ denotes the constant sheaf on $\mathbf{X}$ and $\mathbf{Y}$ is some compact curve containing $\mathbf{X}$. This formula expresses the difference of $\chi_{Fr}(\int\limits_{\mathbf{X}} \overline{\mathbb{Q}}_\ell)$ from $\chi_{Fr}(\int\limits_{\mathbf{X}} \mathcal{F})$ as a sum of local contributions. We take $\mathbf{Y} = \mathbb{P}^1$; therefore under this choice the complement $\mathbf{Y} \setminus \mathbf{X}$ consists of 3 points $\mathbf{Y} \setminus \mathbf{X} = \{0, 1, \infty\}$. Using the standard properties of the Swan conductors (see [25, 29]), and the well known values of the Swan conductors of the standard sheaves $\mathcal{L}_\chi$ and $\mathcal{L}_\psi$, we obtain $Swan_0(\mathcal{F}) = Swan_0(\mathcal{L}_\chi) = 0$, $Swan_1(\mathcal{F}) = Swan_\infty(\mathcal{L}_\psi) = 1$, and $Swan_\infty(\mathcal{F}) = Swan_\infty(\mathcal{L}_\chi) = 0$. In addition, we have that $\chi_{Fr}(\int \overline{\mathbb{Q}}_\ell) = -1$. Concluding, using formula (B.3) we get $\chi_{Fr}(\int\limits_{\mathbf{X}} \mathcal{F}) = -2$.

This completes the proof of the Vanishing lemma and the proof of Theorem 3.2.

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Current address: Department of Mathematics, University of California, Berkeley, CA 94720, USA. E-mail: shamgar@math.berkeley.edu

Current address: Department of Mathematics, University of Chicago, IL 60637, USA. E-mail: hadani@math.uchicago.edu