HEEGAARD FLOER HOMOLOGY, LECTURES 1–4

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Introduction

Heegaard Floer homology, defined by Ozsváth-Szabo [OS04d], and knot Floer homology, defined by Ozsváth-Szabó [OS04b] and independently Rasmussen [Ras03], are invariants of 3-manifolds and knots inside of them. These notes aim to provide an overview of these invariants and the relationship between them.

We will describe our manifolds and knots via Heegaard diagrams, which we define in Section 1. From such diagrams, we will construct chain complexes for 3-manifolds, respectively knots, in Section 2, respectively Section 3, whose chain homotopy type (and in particular, homology) is independent of the choice of Heegaard diagram. Moreover, from the knot invariant associated to a knot K in S^3, one can compute the 3-manifold invariant for any Dehn surgery along K; we discuss this relationship in the case of integral surgery in Section 4.

1. Heegaard splittings and diagrams

1.1. **Heegaard splittings.** Our goal is to define an invariant of closed 3-manifolds and knots inside of them. In order to do this, we will need a way to describe our manifolds and knots. We will do this using Heegaard diagrams. Throughout, we will assume that all of our 3-manifolds are closed and oriented. Much of what follows comes from [Sav99, Lecture 1] and [OS06b, Sections 2 and 3].

Definition 1.1. A *handlebody of genus g* is a closed regular neighborhood of a wedge of g circles in $\mathbb{R}^3$.

Definition 1.2. Let Y be a 3-manifold. A *Heegaard splitting* of Y is a decomposition of $Y = H_1 \cup_f H_2$ where H_1, H_2 are handlebodies and f is an orientation reversing homeomorphism from ∂H_1 to ∂H_2. The *genus* of the Heegaard splitting is the genus of the surface ∂H_1 or equivalently ∂H_2.

Example 1.3. Note that $S^3 = B^3 \cup B^3$. This is a genus 0 Heegaard splitting of S^3.

Example 1.4. Consider S^3 as the 1-point compactification of $\mathbb{R}^3$. Consider the circle consisting of the z-axis and the point at infinity. A regular neighborhood of this union is a handlebody H_1 of genus 1. The complement of H_1 is also a handlebody of genus 1. Together, these two handlebodies form a genus 1 Heegaard splitting of S^3.

Theorem 1.5. *Any closed, orientable 3-manifold Y admits a Heegaard splitting.*

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Proof. Consider a triangulation of Y. Let H_1 be a closed regular neighborhood of the 1-skeleton of the triangulation; since H_1 is a neighborhood of a graph, it is a handlebody. The complement $H_2 = Y - H_1$ is also a handlebody; namely it is a neighborhood of the dual 1-skeleton, that is, the graph whose vertices are the centers of the tetrahedra and whose edges are segments perpendicular to the faces of the tetrahedra. □

Remark 1.6. Alternatively, a self-indexing Morse function on Y gives rise to a Heegaard splitting of Y; see [OS06b, Section 3], in particular, Exercise 3.5.

Definition 1.7. Two Heegaard splittings $Y = H_1 \cup_f H_2$ and $Y = H_1' \cup_{f'} H_2'$ of Y are *homeomorphic* if there exists a homeomorphism $\phi\colon Y \to Y$ taking H_i to H_i'.

Remark 1.8. One may also consider the following stricter notion of equivalence: two Heegaard splittings $Y = H_1 \cup_f H_2$ and $Y = H_1' \cup_{f'} H_2'$ of Y are *isotopic* if there exists a map $\psi\colon Y \times [0,1] \to Y$ such that

(1) $\psi|_{Y\times\{0\}} = \mathrm{id}_Y$,
(2) $\psi|_{Y\times t}$ is a homeomorphism for all t,
(3) $\psi|_{Y\times\{1\}}$ sends H_i to H_i'.

Note that if $Y = H_1 \cup_f H_2$ and $Y = H_1' \cup_{f'} H_2'$ are isotopic, then they are homeomorphic (via $\psi|_{Y\times\{1\}}$). The converse is false; indeed, the homeomorphism $\phi\colon Y \to Y$ need not be isotopic to the identity.

Definition 1.9. Let $Y = H_1 \cup_f H_2$ be a genus g Heegaard splitting of Y. A *stablization* of $Y = H_1' \cup_{f'} H_2'$ is the genus $g+1$ Heegaard splitting of Y where H_1' consists of H_1 together with a neighborhood N of a properly embedded unknotted arc γ in H_2, and H_2' consists of $H_2 - N$.

Exercise 1.10. Prove that the homeomorphism type (in fact, isotopy type) of a stabilization of $Y = H_1 \cup_f H_2$ is independent of the choice of γ.

The following theorem of Reidemeister and Singer highlights the importance of stabilizations.

Theorem 1.11 ([Rei33, Sin33])**.** *Any two Heegaard splittings of Y become isotopic after sufficiently many stablizations.*

1.2. **Heegaard diagrams.** We will describe a Heegaard splitting via a Heegaard diagram $\mathcal{H}$, as defined below.

Definition 1.12. Let H be a handlebody of genus g. A *set of attaching circles* for H is a set $\{\gamma_1, \dots, \gamma_g\}$ of simple closed curves in $\Sigma = \partial H$ such that

(1) the curves are pairwise disjoint,
(2) $\Sigma - \gamma_1 - \cdots - \gamma_g$ is connected,
(3) each γ_i bounds a disk in H.

Exercise 1.13. Show that $\Sigma - \gamma_1 - \cdots - \gamma_g$ is connected if and only if $[\gamma_1], \dots, [\gamma_g]$ are linearly independent in $H_1(\Sigma; \mathbb{Z})$.

See Figure 1 for an example of a set of attaching circles.

Definition 1.14. A *Heegaard diagram* compatible with $Y = H_1 \cup_f H_2$ is a triple $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ such that

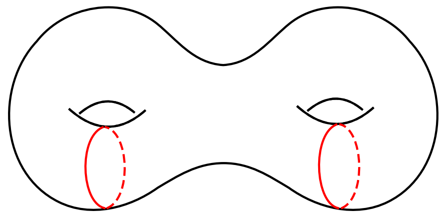

FIGURE 1. A set of attaching circles for the "obvious" handlebody in $\mathbb{R}^3$ bounded by Σ.

(1) Σ is closed oriented surface of genus g,
(2) $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$ is a set of attaching circles for H_1,
(3) $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ is a set of attaching circles for H_2.

We call $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ a *Heegaard diagram for* Y.

See Figure 2 for examples of Heegaard diagrams for $\mathbb{R}P^3$. (Another example of a Heegaard diagram is given in Figure 8 below.)

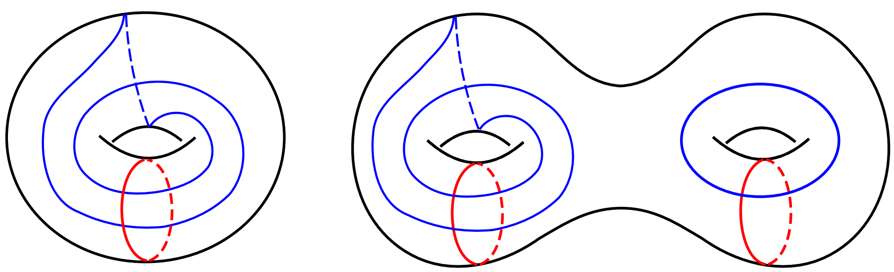

FIGURE 2. Left, a Heegaard diagram for $\mathbb{R}P^3$. Right, a stabilization.

Given a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for Y, we can build a 3-manifold as follows. Thicken Σ to $\Sigma \times [0,1]$. Attach thickened disks along $\alpha_i \times \{0\}$, $1 \le i \le g$, and along $\beta_i \times \{1\}$, $1 \le i \le g$.

Exercise 1.15. Verify that since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are sets of attaching circles, the boundary of the resulting 3-manifold is homeomorphic to $S^2 \sqcup S^2$.

Now fill in each of these boundary components with a copy of B^3; there is a unique way to do so, up to isotopy. The resulting 3-manifold is homeomorphic to Y.

Exercise 1.16. Show that $H_1(Y, \mathbb{Z}) \cong H_1(\Sigma; \mathbb{Z})/\langle [\alpha_1], \dots [\alpha_g], [\beta_1], \dots, [\beta_g] \rangle$.

Exercise 1.17. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram for Y. Compute $H_1(Y; \mathbb{Z})$ from $\mathcal{H}$ as follows. Choose an order and orientation on the α- and β-circles and form the matrix $M = (M_{ij})$ where M_{ij} is the algebraic intersection number between the i^{th} α-circle and the j^{th} β-circle. Show that M is a presentation matrix for $H_1(Y; \mathbb{Z})$.

We now consider three ways to alter a Heegaard diagram, called *Heegaard moves*: isotopies, handleslides, and stabilizations/destabilizations. Isotopies and handleslides do not change the genus of the Heegaard diagram, while stabilizations, respectively destablizations, increase the genus by one, respectively decrease the genus by one.

Let $\{\gamma_1, \dots, \gamma_g\}$ be a set of attaching circles for a handlebody H where $\partial H = \Sigma$. The set $\{\gamma_1, \dots, \gamma_g\}$ is *isotopic* to $\{\gamma'_1, \dots, \gamma'_g\}$ if there is a 1-parameter family of disjoint simple closed curves starting at $\{\gamma_1, \dots, \gamma_g\}$ and ending at $\{\gamma'_1, \dots, \gamma'_g\}$.

A *handleslide* of, say, γ_1 over γ_2 produces a new set of attaching circles $\{\gamma'_1, \gamma_2, \dots, \gamma_g\}$ where γ'_1 is any simple closed curve disjoint from $\gamma_1, \dots, \gamma_g$ such that γ'_1, γ_1, and γ_2 cobound an embedded pair of pants in $\Sigma - \gamma_3 - \cdots - \gamma_g$. See Figure 3.

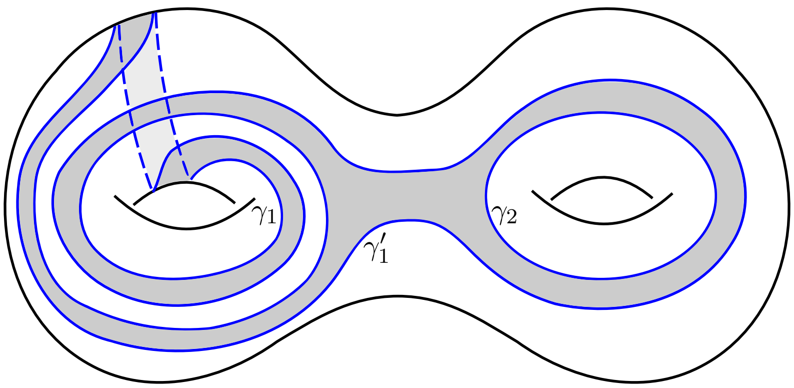

FIGURE 3. A handleslide.

Here is another way to think of a handleslide. Suppose that γ_1 and γ_2 can be connected by an arc δ in $\Sigma - \gamma_3 - \cdots - \gamma_g$. Let γ'_1 be the connected sum of γ_1 with a parallel copy of γ_2, where the connected sum is taken along a neigborhood of δ. See Figure 4.

Exercise 1.18. Prove that these two descriptions of handleslides agree (up to isotopy).

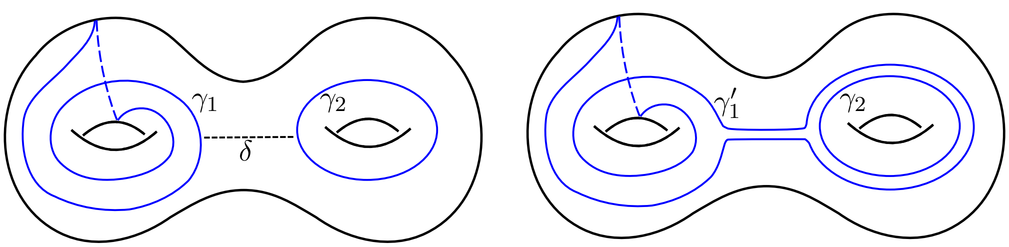

FIGURE 4. Another way to view a handleslide.

A *stabilization* of a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ results in a Heegaard diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ where

(1) $\Sigma' = \Sigma \# T^2$, where $T^2 = S^1 \times S^1$,
(2) $\boldsymbol{\alpha}' = \boldsymbol{\alpha} \cup \{\alpha_{g+1}\}$ and $\boldsymbol{\beta}' = \boldsymbol{\beta} \cup \{\beta_{g+1}\}$, where α_{g+1} and β_{g+1} are two simple closed curves supported in T^2 intersecting transversally in a single point.

We say that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a *destablilization* of $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$.

Exercise 1.19. Show that a stabilization of a Heegaard diagram corresponds to a stabilization of the corresponding Heegaard splitting.

We have the following standard fact about Heegaard diagrams (see, for example, [OS04d, Proposition 2.2]).

Theorem 1.20. *Let* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *and* $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ *be two Heegaard diagrams for* Y*. Then after applying a sequence of isotopies, handleslides, and stabilizations to each of them, the two diagrams become homeomorphic (i.e., there is a homeomorphism* $\Sigma \to \Sigma'$ *taking* $\boldsymbol{\alpha}$ *to* $\boldsymbol{\alpha}'$ *and* $\boldsymbol{\beta}$ *to* $\boldsymbol{\beta}'$*, setwise).*

We will be interested in *pointed Heegaard diagrams*, that is, tuples $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ where w is a basepoint in $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$. We now consider pointed isotopies, where the isotopies are not allowed to pass over w, and pointed handleslides, where w is not allowed to be in the pair of pants involved in the handleslide. We have the following upgraded version of Theorem 1.20.

Theorem 1.21 ([OS04d, Proposition 7.1])**.** *Let* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ *and* $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', w')$ *be two pointed Heegaard diagrams for* Y*. Then after applying a sequence of pointed isotopies, pointed handleslides, and stabilizations to each of them, the two diagrams become homeomorphic.*

1.3. Doubly pointed Heegaard diagrams.

We will also be interested in describing knots inside of our 3-manifolds. For simplicity, we will focus on the case where the ambient 3-manifold is S^3.

Definition 1.22. A *doubly pointed Heegaard diagram* for a knot $K \subset S^3$ is a tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, where w, z are basepoints in $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$, such that

1. $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram for S^3,
2. K is the union of arcs a and b where a is an arc in $\Sigma - \boldsymbol{\alpha}$ connecting w to z, pushed slightly into H_1 and b is an arc in $\Sigma - \boldsymbol{\beta}$ connecting z to w, pushed slightly into H_2.

See Figure 5 for an example of a doubly pointed Heegaard diagram for the left-handed trefoil.

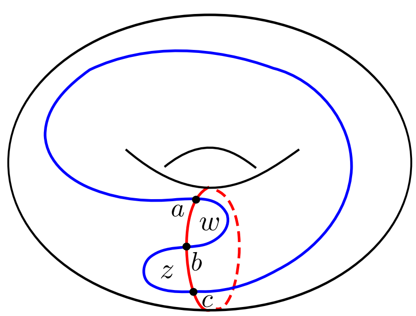

FIGURE 5. A doubly pointed Heegaard diagram for the left-handed trefoil, $-T_{2,3}$. (The labelled points a, b, and c will be used in Section 3.)

Given a knot diagram D for a knot K, one can obtain a doubly pointed Heegaard diagram for K as follows. Suppose that D has c crossings. Forgetting the crossing data of the diagram D yields an immersed curve C in the plane. The complement of C is $c+2$ regions in the plane, one of which is unbounded. Let Σ be the boundary of a regular neighborhood of C in $\mathbb{R}^3$; note that Σ is a surface of genus $c+1$. For each of the bounded regions in the complement of C, we put a β-circle on Σ. For each crossing of D, we put an α-circle on Σ as in Figure 6. Lastly, we add an α-circle, say α_{c+1}, corresponding to a meridan of K. Place a w-basepoint on one side of α_{c+1} and a z-basepoint on the other side. (Note that which side one choose for w determines the orientation of K.) See Figure 7 for an example.

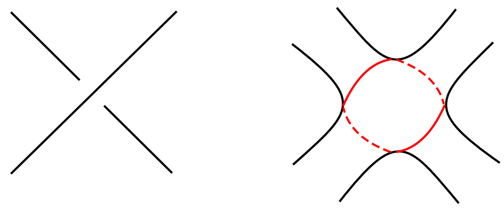

FIGURE 6. A knot crossing and the corresponding portion of the associated doubly pointed Heegaard diagram.

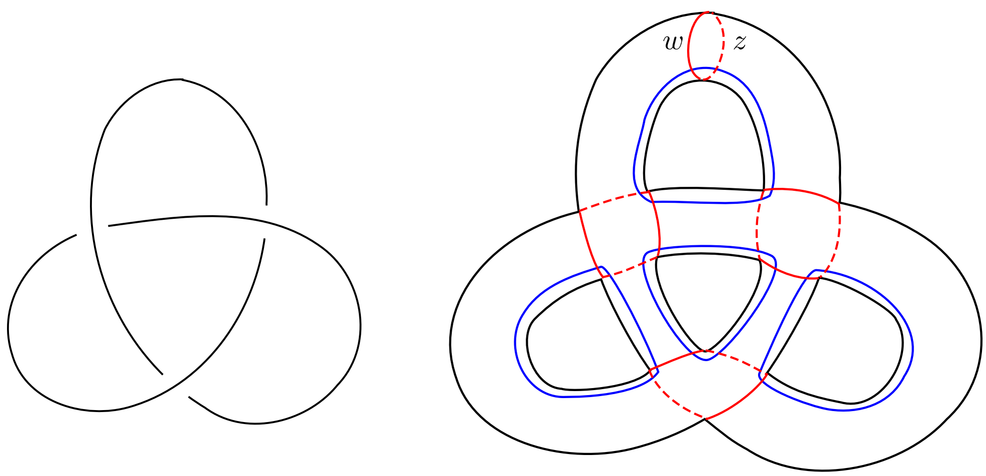

FIGURE 7. Another doubly pointed Heegaard diagram for the left-handed trefoil, $-T_{2,3}$.

Exercise 1.23. Show that the above construction yields a doubly pointed Heegaard diagram for $K \subset S^3$.

In the aforementioned construction, if we replace the circle α_{c+1} with an n-framed longitude and remove the basepoint z, we obtain a pointed Heegaard diagram for $S^3_n(K)$. See Figure 8 for an example.

We now consider doubly pointed isotopies, which are required to miss both w and z, and doubly pointed handleslides, where neither w nor z are allowed to be in the pair of pants involved in the handleslide.We have the following analog of Theorem 1.21.

Theorem 1.24 ([OS04b, Proposition 3.5])**.** *Let* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ *and* $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', w', z')$ *be two doubly pointed Heegaard diagrams for* $K \subset S^3$*. Then after applying a sequence*

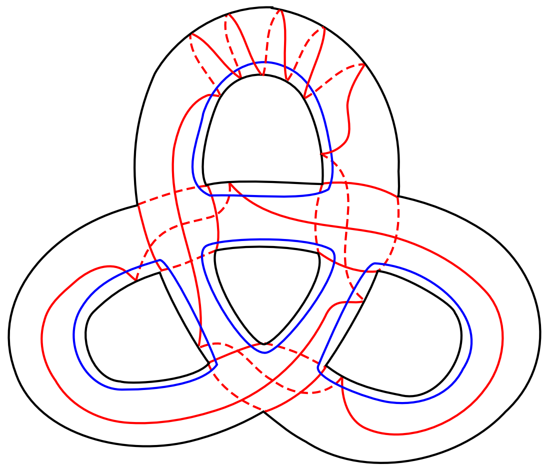

FIGURE 8. A Heegaard diagram for $S^3_{+5}(-T_{2,3})$.

of doubly pointed isotopies, doubly pointed handleslides, and stabilizations to each of them, the two diagrams become homeomorphic.

Exercise 1.25. Find a sequence of doubly pointed Heegaard moves from Figure 7 to Figure 5.

In Sections 2 and 3, we will define invariants of 3-manifolds and knots in S^3. These invariants will be defined in terms of pointed and doubly pointed Heegaard diagrams, and invariance will follow from the fact that the invariants remain unchanged under pointed and doubly pointed Heegaard moves.

We conclude this section with some additional exercises.

Exercise 1.26. Find a genus 1 doubly pointed Heegaard diagram for the figure eight knot.

Exercise 1.27. Find a genus 1 doubly pointed Heegaard diagram for the torus knot $T_{p,q}$.

Exercise 1.28. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram for Y. What is the manifold described by $(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$? By $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$?

Exercise 1.29. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ be a Heegaard diagram for a knot K in S^3. What is the knot described by $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$? By $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)$?

2. HEEGAARD FLOER HOMOLOGY

2.1. **Overview.** From a Heegaard diagram $\mathcal{H}$ for Y, we will build chain complexes $\widehat{CF}(\mathcal{H})$ and $CF^-(\mathcal{H})$ whose chain homotopy types are invariants of Y; the former is a finitely generated chain complex over $\mathbb{F}$ and the latter is a finitely generated graded chain complex over $\mathbb{F}[U]$. Throughout, $\mathbb{F}$ denotes the field $\mathbb{Z}/2\mathbb{Z}$ and U is a formal variable of degree -2. We denote their homologies by $\widehat{HF}(Y)$ and $HF^-(Y)$ respectively; the former is a graded vector space over $\mathbb{F}$ and the latter is a graded module over $\mathbb{F}[U]$. When discussing properties that apply to either flavor of Heegaard Floer homology, will write HF° rather than $\widehat{HF}$, HF^-, HF^+, or HF^∞

(the latter two of which are defined below). The material in this section draws from [OS06b, Sections 4-8] and [OS04d, Sections 3 and 4].

For simplicity, we will consider the case where Y is a rational homology sphere. The case $b_1(Y) > 0$ requires some additional admissibility assumptions on $\mathcal{H}$; see [OS04d, Section 4.2]. Before defining these invariants, we discuss certain aspects of their formal structure.

The Heegaard Floer homology of Y splits as a direct sum over spin^c structures on Y:

$$HF^\circ(Y) = \bigoplus_{\mathfrak{s}\in \mathrm{spin}^c(Y)} HF^\circ(Y,\mathfrak{s})$$

See [OS06b, Section 6] for a discussion of spin^c structures in terms of homotopy classes of non-vanishing vector fields. Note that spin^c structures on Y are in (non-canonical) bijection with elements in $H^2(Y;\mathbb{Z})$. The above splitting on the level of homology comes from a splitting on the chain level.

Both $\widehat{HF}(Y)$ and $HF^-(Y)$ are finitely generated and graded. Finitely generated graded vector spaces are simply a direct sum of graded copies of $\mathbb{F}$. Finitely generated graded modules over a PID are also completely characterized. Since the only homogenously graded polynomials in $\mathbb{F}[U]$ are the monomials U^n, any finitely generated graded module over $\mathbb{F}[U]$ is isomorphic to

$$\bigoplus_i \mathbb{F}[U]_{(d_i)} \oplus \bigoplus_j \mathbb{F}[U]_{(c_j)}/(U^{n_j}), \tag{2.1}$$

where $\mathbb{F}[U]_{(d)}$ denotes the ring $\mathbb{F}[U]$ where the element 1 has grading d.

Moreover, by [OS04c, Theorem 10.1], we have that for a rational homology sphere Y, for all $\mathfrak{s} \in \mathrm{spin}^c(Y)$,

$$HF^-(Y,\mathfrak{s}) \cong \mathbb{F}[U]_{(d)} \oplus \bigoplus_j \mathbb{F}[U]_{(c_j)}/U^{n_j}, \tag{2.2}$$

that is, there is exactly one free summand. We define the *d-invariant* of $(Y,\mathfrak{s})$ to be

$$d(Y,\mathfrak{s}) = \max\{\mathrm{gr}(x) \mid x \in HF^-(Y,\mathfrak{s}), U^N x \neq 0 \ \forall\ N > 0\}.$$

The U-torsion part, denoted $\bigoplus_j \mathbb{F}[U]_{(c_j)}/U^{n_j}$ above, is called $HF_{\mathrm{red}}(Y,\mathfrak{s})$. A rational homology sphere Y with $HF_{\mathrm{red}}(Y,\mathfrak{s}) = 0$ for all $\mathfrak{s} \in \mathrm{spin}^c(Y)$ is called an *L-space*.

Remark 2.1. Different grading conventions exist in the literature. We have chosen our grading convention so that $HF^-(S^3) \cong \mathbb{F}[U]_{(0)}$, as opposed to the perhaps more common $\mathbb{F}[U]_{(-2)}$. Our grading convention choice simplifies certain formulas, such as the Künneth formula [OS04c, Theorem 1.5]:

$$CF^-(Y_1\#Y_2,\mathfrak{s}_1\#\mathfrak{s}_2) \simeq CF^-(Y_1,\mathfrak{s}_1) \otimes_{\mathbb{F}[U]} CF^-(Y_2,\mathfrak{s}_2).$$

Note that our choice of grading convention also impacts the gradings on $HF_{\mathrm{red}}(Y,\mathfrak{s})$.

The chain complexes $\widehat{CF}(\mathcal{H},\mathfrak{s})$ and $CF^-(\mathcal{H},\mathfrak{s})$ fit in the following U-equivariant exact sequence:

$$0 \to CF^-(\mathcal{H},\mathfrak{s}) \xrightarrow{\cdot U} CF^-(\mathcal{H},\mathfrak{s}) \to \widehat{CF}(\mathcal{H},\mathfrak{s}) \to 0,$$

yielding the U-equivariant exact triangle:

$$\begin{array}{ccc} HF^-(Y,\mathfrak{s}) & \xrightarrow{\cdot U} & HF^-(Y,\mathfrak{s}) \\ \nwarrow & & \swarrow \\ & \widehat{HF}(Y,\mathfrak{s}). & \end{array}$$

Remark 2.2. Equation (2.2) and the above exact triangle imply that for a rational homology sphere Y, we have $\dim \widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$; cf. Exercise 2.9. It follows that $\dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$ if and only if Y is an L-space.

We may also consider $HF^\infty(Y,\mathfrak{s}) = H_*(CF^\infty(Y,\mathfrak{s})$ where

$$CF^\infty(Y,\mathfrak{s}) = CF^-(Y,\mathfrak{s}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U,U^{-1}].$$

Note that $CF^-(Y,\mathfrak{s}) \subset CF^\infty(Y,\mathfrak{s})$ and define $CF^+(Y,\mathfrak{s})$ to be the quotient

$$CF^\infty(Y,\mathfrak{s})/CF^-(Y,\mathfrak{s}).$$

We have the following short exact sequence:

$$0 \to CF^-(Y,\mathfrak{s}) \to CF^\infty(Y,\mathfrak{s}) \to CF^+(Y,\mathfrak{s}) \to 0.$$

The above short exact sequence on the chain level induces the following U-equivariant exact triangle:

$$\begin{array}{ccc} HF^-(Y,\mathfrak{s}) & \longrightarrow & HF^\infty(Y,\mathfrak{s}) \\ \nwarrow & & \swarrow \\ & HF^+(Y,\mathfrak{s}). & \end{array}$$

Exercise 2.3. When Y is a rational homology sphere, $HF^\infty(Y,\mathfrak{s}) \cong \mathbb{F}[U,U^{-1}]$. Use this fact to give a description of $HF^+(Y,\mathfrak{s})$ in terms of $d(Y,\mathfrak{s})$ and $HF_{\mathrm{red}}(Y,s)$.

Remark 2.4. In light of Exercise 2.3, one may define $HF_{\mathrm{red}}(Y,\mathfrak{s})$ in terms of $HF^-(Y,\mathfrak{s})$ or $HF^+(Y,\mathfrak{s})$; this choice also affects the grading of $HF_{\mathrm{red}}(Y,\mathfrak{s})$.

A smooth cobordism W from Y_0 to Y_1 (that is, a compact smooth 4-manifold W with $\partial W = -Y_0 \sqcup Y_1$) induces a map from the Heegaard Floer homology of Y_0 to Y_1. More specifically, given a spin^c-structure $\mathfrak{t}$ on W, we have a homomorphism

$$F^\circ_{W,\mathfrak{t}} \colon HF^\circ(Y_0,\mathfrak{t}|_{Y_0}) \to HF^\circ(Y_1,\mathfrak{t}|_{Y_1}),$$

which is defined via a handle decomposition of W (but does not depend on the choice of handle decomposition). The cobordism map $F^\circ_{W,\mathfrak{t}}$ has a grading shift depending only on W and $\mathfrak{t}$. See [OS06a] for further details or [OS06c, Section 3.2] for an expository overview.

Let Z be a compact 3-manifold with torus boundary, and γ_0, γ_1, and γ_∞ three simple closed curves in ∂Z such that

$$\#(\gamma_0 \cap \gamma_1) = \#(\gamma_1 \cap \gamma_\infty) = \#(\gamma_\infty \cap \gamma_0) = -1.$$

Let Y_i be the result of Dehn filling Z along γ_i for $i = 0, 1, \infty$; that is, Y_i is the union of Z and a solid torus, where γ_i is meridian of the solid torus. Then by [OS04c, Theorem 9.12], we have an exact triangle

$$\begin{array}{ccc} \widehat{HF}(Y_0) & \xrightarrow{\widehat{F}_0} & \widehat{HF}(Y_1) \\ & \nwarrow \widehat{F}_\infty \quad \widehat{HF}(Y_\infty) \quad \widehat{F}_1 \swarrow & \end{array} \tag{2.3}$$

where $\widehat{F}_i$ is the cobordism map associated to the corresponding 2-handle cobordism. The analogous exact triangle also holds for HF^+.

Remark 2.5. For the analogous exact triangle for the minus or infinity flavors, one must work over the formal power series ring $\mathbb{F}[[U]]$ and semi-infinite Laurent polynomials $\mathbb{F}[[U, U^{-1}]$ respectively; see [MO10, Section 2].

2.2. **The Heegaard Floer chain complex.** Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ be a pointed Heegaard diagram for Y where Σ has genus g, and as usual $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$. We further require that the α- and β-circles intersect transversally.

Consider the g-fold symmetric product

$$\mathrm{Sym}^g(\Sigma) = \Sigma^{\times g}/S_g,$$

where S_g denotes the symmetric group on g-elements. Points in $\mathrm{Sym}^g(\Sigma)$ consist of unordered g-tuples of points in Σ.

Exercise 2.6. Even though the action of S_g on $\Sigma^{\times g}$ is not free, show that the quotient $\mathrm{Sym}^g(\Sigma)$ is a smooth manifold. (Hint: Use the Fundamental Theorem of Algebra to define a map between ordered and unordered g-tuples of complex numbers.)

We have two half-dimensional subspaces of $\mathrm{Sym}^g(\Sigma)$:

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g \qquad \text{and} \qquad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g.$$

The chain complex $\widehat{CF}(\mathcal{H})$ is freely generated over $\mathbb{F}$ by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, that is, intersection points between $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. Note that points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ can be viewed in Σ as g-tuples of intersection points between the α- and β-circles such that each α-circle, respectively β-circle is used exactly once. We will also in interested in the following subspace of $\mathrm{Sym}^g(\Sigma)$:

$$V_w = \{w\} \times \mathrm{Sym}^{g-1}(\Sigma).$$

The differential $\partial\colon \widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H})$ will count certain holomorphic disks in $\mathrm{Sym}^g(\Sigma)$. Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$, and let e_α, respectively e_β, denote the arc in $\partial\mathbb{D}$ with $\mathrm{Re}(z) \geq 0$, respectively $\mathrm{Re}(z) \leq 0$. A *Whitney disk* from $\mathbf{x}$ to $\mathbf{y}$ is a continuous map $\phi\colon \mathbb{D} \to \mathrm{Sym}^g(\Sigma)$ such that

(1) $\phi(-i) = \mathbf{x}$,
(2) $\phi(i) = \mathbf{y}$,
(3) $\phi(e_\alpha) \subset \mathbb{T}_\alpha$,
(4) $\phi(e_\beta) \subset \mathbb{T}_\beta$.

See Figure 9. Let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the set of homotopy classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$.

Exercise 2.7. There is a rather straightforward obstruction to the existence of a Whitney disk from $\mathbf{x}$ to $\mathbf{y}$.

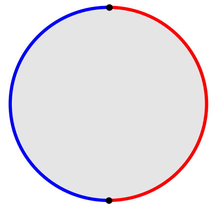

FIGURE 9. A schematic of a Whitney disk.

(1) Viewing $\mathbf{x}$, respectively $\mathbf{y}$, as an unordered g-tuple $\{x_1, \ldots, x_g\}$, respectively $\{y_1, \ldots, y_g\}$, of points in $\boldsymbol{\alpha} \cap \boldsymbol{\beta}$, we may choose a collection of arcs $a \subset \boldsymbol{\alpha}$ such that $\partial a = y_1 + \cdots + y_g - x_1 - \cdots - x_g$ and a collection of arcs $b \subset \boldsymbol{\beta}$ such that $\partial a = x_1 + \cdots + x_g - y_1 - \cdots - y_g$. Then $a - b$ is a 1-cycle in Σ. Verify that the image of $\epsilon(\mathbf{x}, \mathbf{y}) = [a - b] \in H_1(Y; \mathbb{Z})$ is well-defined.

(2) Show that $H_1(\mathrm{Sym}^g(\Sigma); \mathbb{Z}) \cong H_1(\Sigma; \mathbb{Z})$. (Hint: See [OS04d, Lemma 2.6].) Combined with Exercise 1.16, conclude that

$$\frac{H_1(\mathrm{Sym}^g(\Sigma); \mathbb{Z})}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta; \mathbb{Z})} \cong H_1(Y; \mathbb{Z}),$$

and that if $\epsilon(\mathbf{x}, \mathbf{y}) \neq 0 \in H_1(Y; \mathbb{Z})$, then there cannot exist a Whitney disk in $\mathrm{Sym}^g(\Sigma)$ from $\mathbf{x}$ to $\mathbf{y}$.

A choice of complex structure on Σ induces one on $\mathrm{Sym}^g(\Sigma)$. Given $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, let $\mathcal{M}(\phi)$ denote the moduli space of holomorphic representatives of ϕ. ([OS04d, Proposition 3.9] ensures that, under generic perturbations, $\mathcal{M}(\phi)$ is smooth.) The expected dimension of $\mathcal{M}(\phi)$ is called the *Maslov index* and denoted $\mu(\phi)$. There is an $\mathbb{R}$-action on $\mathcal{M}(\phi)$ coming from complex automorphisms of $\mathbb{D}$ that preserve i and $-i$. Let $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$. If $\mu(\phi) = 1$, then $\widehat{\mathcal{M}}(\phi)$ is a compact zero dimensional manifold [OS04d, Theorem 3.18]. Let $n_w(\phi)$ denote the algebraic intersection between $\phi(\mathbb{D})$ and V_w.

We define a relative $\mathbb{Z}$-grading, called the Maslov grading, on $\widehat{CF}(\mathcal{H})$ as follows. Let $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. Define

$$\mathrm{gr}(\mathbf{x}) - \mathrm{gr}(\mathbf{y}) = \mu(\phi) - 2n_w(\phi)$$

By [OS04d, Proposition 2.15], this relative grading is well-defined.

We may use the function ϵ defined in Exercise 2.7 to partition the intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. These equivalence classes are in bijection with $H_1(Y; \mathbb{Z})$ and hence are in bijection with $\mathrm{spin}^c(Y)$; see [OS04d, Section 2.6] for more details. In particular, we have a splitting $\widehat{CF}(\mathcal{H}) = \oplus_{\mathfrak{s} \in \mathrm{spin}^c(Y)} \widehat{CF}(\mathcal{H}, \mathfrak{s})$.

The differential $\partial \colon \widehat{CF}(\mathcal{H}, \mathfrak{s}) \to \widehat{CF}(\mathcal{H}, \mathfrak{s})$ is defined to be

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1 \\ n_w(\phi) = 0}} \# \widehat{\mathcal{M}}(\phi)\, \mathbf{y}.$$

Note that in the first summation, it suffices to only consider $\mathbf{y}$ with $\epsilon(\mathbf{x},\mathbf{y})=0$; that is, the differential respects the splitting $\widehat{CF}(\mathcal{H}) = \oplus_{\mathfrak{s}\in \text{spin}^c(Y)} \widehat{CF}(\mathcal{H},\mathfrak{s})$. It follows from the definition of the relative Maslov grading that the differential ∂ lowers the Maslov grading by one. By [OS04d, Theorem 4.1], $\partial^2 = 0$. Let $\widehat{HF}(\mathcal{H},\mathfrak{s}) = H_*(\widehat{CF}(\mathcal{H},\mathfrak{s}))$.

Remark 2.8. We may define a relative $\mathbb{Z}/2\mathbb{Z}$ grading on $\widehat{CF}(\mathcal{H})$ that agrees with the mod 2 reduction of the relative Maslov grading as follows; see [OS04c, Section 5]. Orient $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ and define the relative $\mathbb{Z}/2\mathbb{Z}$ grading between $\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ to be the product of their local intersection numbers. (Here, we are identifying $\mathbb{Z}/2\mathbb{Z}$ with $\{\pm1\}$ under multiplication.)

Exercise 2.9. Let $\mathcal{H}$ be a Heegaard diagam for a rational homology sphere Y. Use Remark 2.8 to prove that $\chi(\widehat{HF}(\mathcal{H})) = \pm|H_1(Y;\mathbb{Z})|$ and conclude that $\dim \widehat{HF}(\mathcal{H}) \geq |H_1(Y;\mathbb{Z})|$.

We now define the chain complex $CF^-(\mathcal{H})$, which is freely generated over $\mathbb{F}[U]$ by $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$. Here, U is a formal variable with $\text{gr}(U) = -2$. We no longer require that Whitney disks miss the basepoint (i.e., we remove the $n_w(\phi)=0$ requirement), and instead use the variable U to count the algebraic intersection number $n_w(\phi)$ of $\phi(\mathbb{D})$ and V_w:

$$\partial\mathbf{x} = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi)U^{n_w(\phi)}\,\mathbf{y}.$$

The definition of ∂ is extended to all elements of $CF^-(\mathcal{H})$ by $\mathbb{F}[U]$-linearity. As in the case of $\widehat{CF}$, the chain complex $CF^-(\mathcal{H})$ splits over $\text{spin}^c(Y)$, that is, $CF^-(\mathcal{H}) = \oplus_{\mathfrak{s}\in\text{spin}^c(Y)} CF^-(\mathcal{H},\mathfrak{s})$. By [OS04d, Theorem 4.3], $\partial^2 = 0$. Let $HF^-(\mathcal{H},\mathfrak{s}) = H_*(CF^-(\mathcal{H},\mathfrak{s})$.

Theorem 2.10 ([OS04d, Theorem 1.1])**.** *Let $\mathcal{H}$ be a pointed Heegaard diagram for Y. Then the isomorphism type of $\widehat{HF}(\mathcal{H},\mathfrak{s})$ and $HF^-(\mathcal{H},\mathfrak{s})$ is an invariant of Y and $\mathfrak{s}$.*

In order to prove the above theorem, one must show that $\widehat{HF}(\mathcal{H},\mathfrak{s})$ is independent of the choice of Heegaard diagram, basepoint, and complex structure. Indeed, Ozsváth-Szabó show that a Heegaard move induces a homotopy equivalence on the associated chain complexes.

In [OS06a, Theorem 7.1], Ozsváth-Szabó prove that the relative $\mathbb{Z}$-grading may be lifted to a well-defined absolute $\mathbb{Q}$-grading; this is done by considering a cobordism from S^3 to Y and considering holomorphic triangles associated to a Heegaard triple.

Exercise 2.11. Compute $\widehat{HF}(L(p,q),\mathfrak{s})$ and $HF^-(L(p,q),\mathfrak{s})$ for all $\mathfrak{s}\in\text{spin}^c(L(p,q))$.

In general, computing $\widehat{HF}(Y)$ and $HF^-(Y)$ from a Heegaard diagram is not easy. In Section 4, we will see how to compute $\widehat{HF}(Y)$ and $HF^-(Y)$ when Y is surgery on a knot in S^3.

Exercise 2.12. Recall that $pq\pm1$ surgery on the torus knot $T_{p,q}$ is a lens space. Use Exercise 2.11 and Equation (2.3) to compute $\widehat{HF}(S^3_n(T_{p,q}))$ for $n\geq pq-1$.

The following exercise will be of use in Section 4.

Exercise 2.13. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ be a Heegaard diagram for Y. Let $\mathcal{H}' = (-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, w)$. Show that the chain complex $CF^-(\mathcal{H})$ is isomorphic to $CF^-(\mathcal{H}')$.

3. Knot Floer homology

3.1. **Overview.** Let K be a knot in S^3. The simplest version of the knot invariant is $\widehat{HFK}(K)$, a bigraded vector space over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, that is

$$\widehat{HFK}(K) = \bigoplus_{m,s} \widehat{HFK}_m(K,s).$$

Knot Floer homology categorifies the Alexander polynomial [OS04b, Equation (1)] in the sense that the graded Euler characteristic of $\widehat{HFK}(K)$ is $\Delta_K(t)$:

$$\Delta_K(t) = \sum_{m,s} (-1)^m \dim \widehat{HFK}_m(K,s)\, t^s.$$

Moreover, knot Floer homology strengthens two key properties of the Alexander polynomial. Let

$$\Delta_K(t) = a_0 + \sum_{s>0} a_s(t^s + t^{-s})$$

denote the symmetrized Alexander polynomial. While the Alexander polynomial gives a lower bound on the genus of K in the following manner:

$$g(K) \geq \max\{s \mid a_s \neq 0\},$$

knot Floer homology actually detects $g(K)$ [OS04a]:

$$g(K) = \max\{s \mid \widehat{HFK}(K,s) \neq 0\}.$$

Similarly, while the Alexander polynomial obstructs fiberedness in that

$$K \text{ fibered } \Rightarrow a_{g(K)} = \pm 1,$$

knot Floer homology actually detects fiberedness [Ghi08, Ni07]:

$$K \text{ is fibered } \Leftrightarrow \widehat{HFK}(K, g(K)) = \mathbb{F}.$$

3.2. **The knot Floer complex.** We now modify the constructions in Section 2 to the case of doubly pointed Heegaard diagrams in order to define knot invariants. As mentioned above, for simplicity, we restrict ourselves to knots in S^3. (With mild modifications, the constructions described here apply to any null-homologous knot in a rational homology sphere.) Some of this material comes from [OS06b, Section 10]; see [OS04b] for more details and proofs. Many of our conventions and notations come from [Zem17], see especially [Zem17, Section 1.5]. Knot Floer homology was independently defined by Rasmussen in [Ras03].

We will work over the ring $\mathbb{F}[U,V]$, where as before $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. We endow this ring with a bigrading $\mathrm{gr} = (\mathrm{gr}_U, \mathrm{gr}_V)$. We call gr_U the *U-grading* and gr_V the *V-grading*. The variables U and V have grading

$$\mathrm{gr}(U) = (-2, 0) \qquad \text{and} \qquad \mathrm{gr}(V) = (0, -2).$$

It will often be convenient to consider the following linear combination of gr_U and gr_V,

$$A = \frac{1}{2}(\mathrm{gr}_U - \mathrm{gr}_V),$$

called the *Alexander grading*. Note that $A(U) = -1$ and $A(V) = 1$.

Let $\mathcal{H}$ be a doubly pointed Heegaard diagram for a knot $K \subset S^3$. Let $\mathit{CFK}_{\mathbb{F}[U,V]}(\mathcal{H})$ be the free $\mathbb{F}[U,V]$-module generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. This module is relatively bigraded as follows. Let $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ and define

$$\begin{aligned}
\mathrm{gr}_U(\mathbf{x}) - \mathrm{gr}_U(\mathbf{y}) &= \mu(\phi) - 2n_w(\phi) \\
\mathrm{gr}_V(\mathbf{x}) - \mathrm{gr}_V(\mathbf{y}) &= \mu(\phi) - 2n_z(\phi).
\end{aligned}$$

By [OS04c, Proposition 7.5] (see also [Zem17, Section 5.1]), this relative grading is well-defined. The relative gradings gr_U and gr_V can be lifted to absolute gradings, using the absolute grading on $\mathit{HF}^-(S^3)$; we describe this process below.

The differential $\partial\colon \mathit{CFK}_{\mathbb{F}[U,V]}(\mathcal{H}) \to \mathit{CFK}_{\mathbb{F}[U,V]}(\mathcal{H})$ is defined to be

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi) U^{n_w(\phi)} V^{n_z(\phi)}\, \mathbf{y},$$

and is extended to all elements of $\mathit{CFK}_{\mathbb{F}[U,V]}(\mathcal{H})$ by $\mathbb{F}[U,V]$-linearity. Note that the differential preserves the Alexander grading.

Setting $V = 1$ and forgetting gr_V (that is, only considering the grading gr_U), we recover $\mathit{CF}^-(S^3)$, whose homology is isomorphic to $\mathbb{F}[U]_{(0)}$, where the subscript (0) now denotes gr_U; this determines the absolute U-grading. Note that setting $V = 1$ corresponds to forgetting the z-basepoint. To determine the absolute V-grading, we simply reverse the roles of U and V in the above construction. That is, we set $U = 1$, forget gr_U, and only consider gr_V, recovering $\mathit{CF}^-(S^3)$, whose homology is isomorphic to $\mathbb{F}[V]$ where $\mathrm{gr}_V(1) = 0$. This corresponds to forgetting the w-basepoint.

Theorem 3.1 ([OS04b, Theorem 3.1])**.** *Let $\mathcal{H}$ be a doubly pointed Heegaard diagram for a knot $K \subset S^3$. The chain homotopy type of $\mathit{CFK}_{\mathbb{F}[U,V]}(\mathcal{H})$ is an invariant of $K \subset S^3$.*

Note that [OS04b, Theorem 3.1] is phrased in terms of filtered chain complexes; see [Zem17, Section 1.5] for a description of the translation between filtered chain complexes and modules over $\mathbb{F}[U,V]$. We will often abuse notation and write $\mathit{CFK}_{\mathbb{F}[U,V]}(K)$ rather than $\mathit{CFK}_{\mathbb{F}[U,V]}(\mathcal{H})$.

Example 3.2. Figure 5 shows a doubly pointed Heegaard diagram for the left-handed trefoil. We have

$$\begin{aligned}
\partial a &= Ub \\
\partial b &= 0 \\
\partial c &= Vb.
\end{aligned}$$

Setting $V = 1$, we see that the homology, which is isomorphic to $\mathbb{F}[U]$, is generated by $[a + Uc]$, implying that $\mathrm{gr}_U(a) = \mathrm{gr}_U(Uc) = 0$. Setting $U = 1$, we see that the homology, which is isomorphic to $\mathbb{F}[V]$, is generated by $[c + Va]$, implying that

$\mathrm{gr}_V(c) = \mathrm{gr}_V(Va) = 0$. It follows that the generators a, b, c have the folllowing gradings:

	gr_U	gr_V	A
a	0	2	-1
b	1	1	0
c	2	0	1

Then $H_*(CFK_{\mathbb{F}[U,V]}(-T_{2,3})) \cong \mathbb{F}[U,V]_{(0,0)} \oplus \mathbb{F}_{(1,1)}$, where the subscript denotes $\mathrm{gr} = (\mathrm{gr}_U, \mathrm{gr}_V)$ of 1. The $\mathbb{F}[U,V]$ summand is generated by $Va + Ub$ and the $\mathbb{F}$ summand by b.

Exercise 3.3. Let $\mathcal{H}$ be a doubly pointed Heegaard diagram for $K \subset S^3$.

(1) Show that

$$H_*(CFK_{\mathbb{F}[U,V]}(\mathcal{H}) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}], s) \cong HF^-(S^3),$$

where the left-hand side denotes the part of the homology in Alexander grading s, thought of as an $\mathbb{F}[W]$-module, where $W = UV$, and the right-hand side is viewed as a module over $\mathbb{F}[W]$, rather than $\mathbb{F}[U]$. (Hint: Consider the map $CF^-(\mathcal{H}) \to (CFK_{\mathbb{F}[U,V]}(\mathcal{H}) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}], s)$ given by $U^n\mathbf{x} \mapsto U^n V^{n+s-A(\mathbf{x})}\mathbf{x}$.) Conclude that

$$H_*(CFK_{\mathbb{F}[U,V]}(\mathcal{H}) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}]) \cong \mathbb{F}[U, V, V^{-1}].$$

(2) Repeat part (1) reversing the roles of U and V.

The knot Floer complex behaves nicely under connected sum, reversal, and mirroring. By [OS04b, Theorem 7.1], we have that

$$CFK_{\mathbb{F}[U,V]}(K_1 \# K_2) \simeq CFK_{\mathbb{F}[U,V]}(K_1) \otimes_{\mathbb{F}[U,V]} CFK_{\mathbb{F}[U,V]}(K_2).$$

Let K^r denote the reverse of K and mK the mirror. By [OS04b, Section 3], we have that

$$CFK_{\mathbb{F}[U,V]}(mK) \simeq CFK_{\mathbb{F}[U,V]}(K)^*, \tag{3.1}$$

where $C^* = \mathrm{Hom}_{\mathbb{F}[U,V]}(C, \mathbb{F}[U,V])$ and

$$CFK_{\mathbb{F}[U,V]}(K^r) \simeq CFK_{\mathbb{F}[U,V]}(K). \tag{3.2}$$

Exercise 3.4. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ be a doubly pointed Heegaard diagram for K. Show that $\mathcal{H}_1 = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ and $\mathcal{H}_2 = (-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, w, z)$ are both diagrams for K^r. Show that $CFK_{\mathbb{F}[U,V]}(\mathcal{H})$ is isomorphic to $CFK_{\mathbb{F}[U,V]}(\mathcal{H}_2)$ (cf. Exercise 2.13). Conclude that Equation (3.2) holds.

Remark 3.5. It follows from Equation (3.2) and Exercise 3.4 that $CFK_{\mathbb{F}[U,V]}(K)$ is chain homotopy equivalent to complex C' obtained from $CFK_{\mathbb{F}[U,V]}(K)$ by exchanging the roles of U and V. (Note that one should then also exchange the roles of gr_U and gr_V.)

Exercise 3.6. Compute $CFK_{\mathbb{F}[U,V]}(T_{2,3})$ two ways: from a doubly pointed Heegaard diagram and by applying Equation (3.1) to Example 3.2, and confirm that the two answer agree.

3.3. **Algebraic variations.** As usual, let $\mathcal{H}$ be a doubly pointed Heegaard diagram for a knot $K \subset S^3$. There are several algebraic modifications we may make to $CFK_{\mathbb{F}[U,V]}(\mathcal{H})$. Since the chain homotopy type of $CFK_{\mathbb{F}[U,V]}(\mathcal{H})$ is an invariant of the knot K, it follows that these algebraic modifications also yield knot invariants.

The first modification we consider is setting both $U = 0$ and $V = 0$, resulting in a bigraded chain complex $CFK_{\mathbb{F}}(\mathcal{H})$ over the field $\mathbb{F}$. Setting $U = V = 0$ is equivalent to requiring that $n_w(\phi) = n_z(\phi) = 0$ in the definition of the differential. We denote the homology of $CFK_{\mathbb{F}}(\mathcal{H})$ by $\widehat{HFK}(K)$, which is a bigraded vector space.

It is common to use gr_U and A as the bigrading on $\widehat{HFK}(K)$. (Of course, this is the same information as gr_U and gr_V, since $A = \frac{1}{2}(\mathrm{gr}_U - \mathrm{gr}_V)$.) We write $\widehat{HFK}_m(K, s)$ to denote the summand of $\widehat{HFK}(K)$ with $\mathrm{gr}_U = m$ and $A = s$. The grading gr_U is often called the *Maslov grading*.

Example 3.7. Setting $U = V = 0$ in Example 3.2 results in

$$\partial a = \partial b = \partial c = 0.$$

Thus, $\widehat{HFK}(K)$ has dimension 3, generated by a, b, and c, with gradings given in the table in Example 3.2. That is,

$$\widehat{HFK}_m(K, s) = \begin{cases} \mathbb{F} & \text{if } (m, s) = (0, -1), (1, 0), \text{ or } (2, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.8 ([OS04b, Equation (1)]). *The graded Euler characteristic of $\widehat{HFK}(K)$ is equal to the Alexander polynomial of K:*

$$\Delta_K(t) = \sum_{m,s} (-1)^m \dim \widehat{HFK}_m(K, s)\, t^s.$$

Recall from [Kau83] (see also [OS06b, Theorem 11.3]) that the Alexander polynomial of K can be computed in terms of the Kauffman states of a diagram for K. Note that the Kauffman states of the left diagram in Figure 7 are in bijection with the Heegaard Floer generators of the right diagram. This observation, together with a computation of the bigradings, is at the heart of the proof of Theorem 3.8; see [OS06b, Sections 11-13] for details.

Another algebraic modification is to set a single variable, say V, equal to zero, resulting in a chain complex $CFK_{\mathbb{F}[U]}$ over the PID $\mathbb{F}[U]$. This corresponds to requiring that $n_z(\phi) = 0$ in the definition of the differential. The homology of $CFK_{\mathbb{F}[U]}$ is a $\mathbb{F}[U]$-module, denoted $HFK^-(K)$. As a finitely generated graded module over a PID, $HFK^-(K)$ is isomorphic to a direct sum of free summands and U-torsion summands as in Equation (2.1).

It is common to view $HFK^-(K)$ as bigraded by gr_U and A. The action of U lowers gr_U by 2 and A by 1.

Exercise 3.9. Prove that $H_*(CFK_{\mathbb{F}[U]} \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]) \cong \widehat{HF}(S^3) \otimes_{\mathbb{F}} \mathbb{F}[U, U^{-1}] \cong \mathbb{F}[U, U^{-1}]$ and conclude that there is a unique free summand in $HFK^-(K)$.

Example 3.10. Setting $V = 0$ in Example 3.2 results in the free $\mathbb{F}[U]$-module generated by a, b, and c with differential

$$\partial a = Ub$$
$$\partial b = 0$$
$$\partial c = 0.$$

Hence

$$HFK^-(K) \cong \mathbb{F}[U]_{(2)} \oplus \mathbb{F}_{(1)}$$

where $[c]$ is a generator for the $\mathbb{F}[U]$-summand, $[b]$ is a generator for the $\mathbb{F}$-summand, and the subscript denotes gr_U.

There are other algebraic modifications one may consider, such as setting $U^n = 0$ or $UV = 0$; the latter modification will be of use in Section 4.

3.4. **Computations.** How does one compute $CFK_{\mathbb{F}[U,V]}$ in practice? For small crossing knots, $CFK_{\mathbb{F}[U,V]}$ can be computed via grid diagrams [MOS09, MOST07]; see [BG12] for a table of $\widehat{HFK}$ for knots up to 12 crossings computed using grid diagrams. For an excellent textbook on the subject of grid diagrams, see [OSS15]. An invariant that is conjecturally equivalent to $CFK_{\mathbb{F}[U,V]/(UV=0)}$ can be algorithmically computed following [OS17]; such computations are significantly faster than computations with grid diagrams. At the time of writing, a computer implementation of this algorithm is available at `https://web.math.princeton.edu/~szabo/HFKcalc.html`.

For certain special families of knots, we can compute $CFK_{\mathbb{F}[U,V]}$ directly from the definition (in the case of $(1,1)$-knots) or from other easier to compute knot invariants such as the Alexander polynomial and signature (in the case of alternating knots and knots admitting L-space surgeries).

A knot in S^3 that admits a genus 1 double pointed Heegaard diagram is called a $(1,1)$-*knot*. For a $(1,1)$-knot K, the complex $CFK_{\mathbb{F}[U,V]}(K)$ can be computed by counting embedded disks in the universal cover of $\Sigma = T^2$, similar to Example 3.2; see [GMM05].

If K is alternating (or more generally, quasi-alternating; see [OS05b, Definition 3.1]), then [OS03, Theorem 1.3] states that $\widehat{HFK}(K)$ is completely determined by the Alexander polynomial and signature of K. Moreover, [Pet13, Lemma 7], which is completely algebraic, states that if K is alternating, then $\widehat{HFK}(K)$ completely determines the chain homotopy type of $CFK_{\mathbb{F}[U,V]}(K)$.

Exercise 3.11. Let K be an alternating knot. A key ingredient in the proof of [OS03, Theorem 1.3] is that if $\widehat{HFK}_m(K,s) \neq 0$, then $m = s + \frac{\sigma(K)}{2}$, where $\sigma(K)$ denotes the signature of K. (If $\widehat{HFK}(K)$ is supported on a single diagonal with respect to the Maslov and Alexander gradings, we say K is *homologically thin*.) Show that this fact combined with Theorem 3.8 completely determines the bigraded vector space $\widehat{HFK}(K)$ when K is an alternating knot.

If K admits a lens space surgery (or more generally, an L-space surgery), it follows from [OS05a, Theorem 1.2] that $CFK_{\mathbb{F}[U,V]}(K)$ is completely determined by the Alexander polynomial of K, as follows. If K admits an L-space surgery, then the non-zero coefficients in $\Delta_K(t)$ are all ± 1 and they alternate in sign. Let

$$\Delta_K(t) = \sum_{i=0}^{n} (-1)^i t^{a_i},$$

for some decreasing sequence (a_i) and even n. Let $b_i = a_i - a_{i-1}$. If K admits a positive L-space surgery, then $\mathit{CFK}_{\mathbb{F}[U,V]}(K)$ is generated by $x_0, \dots, x_n$ where for i odd,

$$\partial x_i = U^{b_i} x_{i-1} + V^{b_{i+1}} x_{i+1},$$

and for i even, $\partial x_i = 0$. The absolute grading is determined by $\mathrm{gr}_U(x_0) = 0$ and $\mathrm{gr}_V(x_n) = 0$. (There is no loss of generality in considering only positive L-space surgeries. Indeed, if K admits a negative L-space surgery, then mK admits a positive L-space surgery and one can apply Equation (3.1).)

Exercise 3.12. Suppose K admits a positive L-space surgery. Express $\widehat{\mathit{HFK}}(K)$ in terms of $\Delta_K(t)$, and verify that $\widehat{\mathit{HFK}}(K)$ satisfies Theorem 3.8.

(For the relationship between $(1,1)$-knots and L-space knots, see [GLV18].)

4. Heegaard Floer homology of knot surgery

In this section, we discuss the relationship between the knot Floer complex $\mathit{CFK}_{\mathbb{F}[U,V]}(K)$ and $\mathit{HF}^-(S^3_n(K))$, where $S^3_n(K)$ denotes n-surgery on $K \subset S^3$.

We begin with some observations about $\mathit{CFK}_{\mathbb{F}[U,V]}(K)$, which is a chain complex over $\mathbb{F}[U,V]$. Let $W = UV$. Note that multiplication by W preserves the Alexander grading. Hence as an $\mathbb{F}[W]$-module, the complex $\mathit{CFK}_{\mathbb{F}[U,V]}(K)$ splits as a direct sum over the Alexander grading. (However, note that neither multiplication by U nor by V respects this splitting.)

Following [OS04b, Section 4], one may identify spin^c-structures on $S^3_n(K)$ with $\mathbb{Z}/n\mathbb{Z}$. We have that $\mathit{HF}^-(S^3_n(K))$ is a module over a polynomial ring in a single variable; we denote this variable by W (rather than U, as in Section 2).

Theorem 4.1 ([OS04b, Theorem 4.4], cf. [Ras03, Section 4])**.** *Let $n \geq 2g(K) - 1$ and $|s| \leq \lfloor \frac{n}{2} \rfloor$. Then*

$$\mathit{HF}^-(S^3_n(K), [s]) \cong H_*(\mathit{CFK}_{\mathbb{F}[U,V]}(K, s))$$

as relatively $\mathbb{Z}$-graded $\mathbb{F}[W]$-modules. That is, on the left-hand side, $W = U$ while on the right-hand side, $W = UV$. The relative grading on the right-hand side may be taken to be either gr_U or gr_V.

Remark 4.2. See [OS04b, Corollary 4.2] for the absolutely graded version of Theorem 4.1.

Example 4.3. Let $Y = S^3_{+1}(-T_{2,3})$. (It follows from [Mos71, Proposition 3.1] that $S^3_{+1}(-T_{2,3}) \cong -\Sigma(2,3,7)$.) We will use Theorem 4.1 to compute $\mathit{HF}^-(Y)$. Since Y is an integer homology sphere, there is a unique spin^c-structure on Y. From Example 3.2, we have that $\mathit{CFK}_{\mathbb{F}[U,V]}(-T_{2,3}, 0)$ is generated over $\mathbb{F}[W]$, where $W = UV$, by

$$Va, b, Uc.$$

The differential is given by

$$\partial(Va) = W \cdot b$$
$$\partial b = 0$$
$$\partial(Uc) = W \cdot b.$$

Note that the elements Va and Uc are in the same relative grading, while the relative grading of b is one greater than the relative grading of Va. We have that

$$HF^-(Y) \cong H_*(CFK_{\mathbb{F}[U,V]}(-T_{2,3}, 0)) \cong \mathbb{F}[W]_{(0)} \oplus \mathbb{F}_{(1)},$$

where the $\mathbb{F}[W]$-summand is generated by $Va+Uc$ and the $\mathbb{F}$-summand is generated by b. (The absolute gradings are computed following [OS04b, Corollary 4.2].)

Example 4.4. Let $Y = S^3_{+3}(-T_{2,3})$. By Theorem 4.1,

$$HF^-(Y, [s]) \cong H_*(CFK_{\mathbb{F}[U,V]}(-T_{2,3}, s)$$

for $s = -1, 0, 1$. By Example 4.3, we have

$$HF^-(Y, [0]) \cong H_*(CFK_{\mathbb{F}[U,V]}(-T_{2,3}, 0)) \cong \mathbb{F}[W] \oplus \mathbb{F}.$$

From Example 3.2, we have that $CFK_{\mathbb{F}[U,V]}(-T_{2,3}, -1)$ is generated over $\mathbb{F}[W]$ by

$$a, Ub, U^2c.$$

The differential is given by

$$\partial a = Ub$$
$$\partial(Ub) = 0$$
$$\partial(U^2c) = W \cdot Ub.$$

Hence $H_*(CFK_{\mathbb{F}[U,V]}(-T_{2,3}, -1)) \cong \mathbb{F}[W]$, generated by $U^2c + Wa$. Similarly, $H_*(CFK_{\mathbb{F}[U,V]}(-T_{2,3}, 1)) \cong \mathbb{F}[W]$, generated by $V^2a + Wc$; we leave this calculation to the reader.

The proof of Theorem 4.1 relies on relating the Heegaard diagrams for (S^3, K) and $S^3_n(K)$; see, for example, Figures 7 and 8. The rough idea is that for each generator of the Heegaard diagram for (S^3, K) and for each spin^c-structure on $S^3_n(K)$, there is a canonical "nearest" generator obtained by replacing the intersection point on the meridian with a nearby intersection point on the n-framed longitude. The remainder of the proof relies on the relationship between spin^c-structures and the Alexander grading, as well as a count of holomorphic triangles in a Heegaard triple.

Note that Theorem 4.1 requires that surgery coefficient n to be greater than or equal to $2g(K) - 1$. In [OS08, Theorem 1.1], Ozsváth-Szabó provide a recipe for computing the Heegaard Floer homology of any integer surgery along $K \subset S^3$; they improve this to a formula for rational surgery in [OS11].

In these notes, we will work with the minus flavor, as in [MO10, Theorem 1.1]. One disadvantage to working with the minus flavor is that one must work with completed coefficients (cf. Remark 2.5), that is, we work over the power series rings $\mathbb{F}[[U]]$ and $\mathbb{F}[[U, V]]$. To this end, for a pointed Heegaard diagram $\mathcal{H}$ for Y, let

$$\mathbf{CF}^-(\mathcal{H}) = CF^-(\mathcal{H}) \otimes_{\mathbb{F}[U]} \mathbb{F}[[U]]$$

and let $\mathbf{HF}^-(Y) = H_*(\mathbf{CF}^-(\mathcal{H}))$. Note that since $\mathbb{F}[[U]]$ is flat over $\mathbb{F}[U]$, we have that $\mathbf{HF}^-(Y) \cong HF^-(Y) \otimes_{\mathbb{F}[U]} \mathbb{F}[[U]]$. Similarly, let

$$\mathbf{CFK}(K) = CFK_{\mathbb{F}[U,V]}(K) \otimes_{\mathbb{F}[U,V]} \mathbb{F}[[U, V]].$$

We again write $\mathbf{CFK}(K, s)$ to denote the part of $\mathbf{CFK}(K)$ in Alexander grading s. (Note that this is not quite a grading in the usual sense, as $\mathbf{CFK}(K)$ is a direct product rather than direct sum of its homogenously graded pieces. We will abuse notation and still refer to s as the Alexander grading, and similarly for gr_U and gr_V.)

By Remark 3.5, we have that $\mathbf{CFK}(K) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}]$ is chain homotopy equivalent to $\mathbf{CFK}(K) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$ after exchanging the roles of U and V. (Note that this chain homotopy equivalence reverses the Alexander grading.) Moreover, in any fixed Alexander grading, both are homotopy equivalent to $\mathbf{CF}^-(\mathcal{H})$ (cf. Exercise 3.3), and so let ϕ denote this Alexander grading-preserving chain homotopy equivalence of the $\mathbb{F}[UV]$-modules.

Let $\mathbf{v}\colon \mathbf{CFK}(K) \to \mathbf{CFK}(K) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}]$ denote inclusion. Let $\mathbf{h}\colon \mathbf{CFK}(K) \to \mathbf{CFK}(K) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}]$ denote the composition

$$\mathbf{CFK}(K) \to \mathbf{CFK}(K) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}] \xrightarrow{\phi} \mathbf{CFK}(K) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}],$$

where the first map is inclusion. Moreover, since multiplication by V is invertible in $\mathbf{CFK}(K) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}]$, we have that

$$V^n\colon \mathbf{CFK}(K) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}] \to \mathbf{CFK}(K) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}]$$

is a (relatively graded) isomorphism.

Consider

$$\mathbf{D}_n\colon \mathbf{CFK}(K) \to \mathbf{CFK}(K) \otimes_{\mathbb{F}[V]} \mathbb{F}[V, V^{-1}]$$

where $\mathbf{D}_n = \mathbf{v} + V^n\mathbf{h}$.

Recall that given two chain complexes $(X, \partial_X), (Y, \partial_Y)$ and a chain map $f\colon X \to Y$, the *mapping cone* of f is the chain complex $\mathrm{Cone}(f) = X \oplus Y$ with the differential $\partial(x, y) = (\partial_X x, f(x) + \partial_Y y)$.

Theorem 4.5 ([MO10, Theorem 1.1], cf. [OS08, Theorem 1.1])**.** *We have the following isomorphism of $\mathbb{F}[W]$-modules*

$$\mathbf{HF}^-(S^3_n(K)) \cong H_*(\mathrm{Cone}(\mathbf{D}_n)),$$

where on the left-hand side $W = U$ and on the right-hand side, $W = UV$.

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