# Sequences of height 1 primes in Z[X]

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Abstract: For each partition $J \cup K$ of $\{1, 2, \ldots, n\}$ ($n \geq 2$) with $|J| \geq 2$, let $\overline{M}(J \mid K)$ be a finite set of maximal ideals in the polynomial ring Z[X], and suppose that if $J \cup K$ and $J' \cup K'$ are two different partitions, then $\overline{M}(J \mid K)$ and $\overline{M}(J' \mid K')$ are disjoint. Then there is a sequence $p_1, p_2, \ldots, p_n$ of height 1 primes in Z[X] such $\overline{M}(J \mid K)$ equals the set of maximal ideals that contain $p_j$ for each $j \in J$, but do not contain $p_k$ for any $k \in K$. A version involving infinite sequences is also given.

Introduction: In [3], Roger Wiegand shows that spec Z[X] is characterized among countable posets by the following five axioms.

1) It has a unique minimal prime.

2) It has dimension 2.

3) Each height 1 prime is contained in infinitely many maximal primes.

4) If $p_1 \neq p_2$ are height 1 primes, then only finitely many maximal ideals contain both $p_1$ and $p_2$.

RW) Given a finite set S of height 1 primes and a finite set T of maximal primes, there is a height 1 prime p contained in each maximal in T, and such that if a maximal q contains both p and some prime in S, then $q \in T$.

Remarks: a) In [2], Wiegand shows that two countable posets, both of which satisfy the above five axioms, are isomorphic.

b) One of the most interesting consequences of Wiegand's work is that spec Z[X] is not isomorphic to spec Q[X, Y]. The latter does not satisfy axiom RW, although it does satisfy the first four axioms. The nature of spec Q[X, Y] appears to still be rather mysterious. That illustrates the difficulty of characterizing what posets arise as the spectrum of a Noetherian ring, a problem that has been under scrutiny for several decades. See [1] for a progress report.

c) Our work here is really about any poset satisfying those axioms. Nonetheless, since spec Z[X] is the most interesting example, we will always refer to that, knowing that our results apply equally well to any isomorphic poset. For instance, [3] shows that if D is an order in an algebraic number field, then spec D[X] is isomorphic to spec Z[X].

This paper might be considered an homage to RW (both the axiom and the man).

NOTATION: We work within Spec Z[X]. Let $p_1, p_2, p_3, \ldots p_n$ $(n \geq 2)$ be a finite sequence of distinct height one primes. Let $J \cup K$ be a partition of $\{1, 2, \ldots, n\}$. Let $M(J \mid K)$ be the set of maximal primes that contain $p_j$ for each $j \in J$ but do not contain $p_k$ for each $k \in K$.

LEMMA 1: Let $J \cup K$ be a partition of $\{1, 2, \ldots, n\}$.

i) If $J' \cup K'$ is a partition of $\{1, 2, \ldots, n\}$, then $M(J \mid K)$ and $M(J' \mid K')$ are disjoint unless $J = J'$ and $K = K'$.

ii) $M(J \mid K)$ is finite if $|J| \geq 2$.

iii) $M(J \mid K)$ is infinite if $|J| < 2$.

Proof: (i) is obvious, (ii) follows from axiom 4, and (iii) follows from axioms 3 and 4.

The purpose of this paper is to determine to what extent we can pre-determine the various $M(J \mid K)$. More specifically, suppose that for any partition $J \cup K$ of $\{1, 2, \ldots, n\}$ we specify a set $\overline{M}(J \mid K)$ of maximal ideals of $Z[X]$ in such a way that the three statements of lemma 1 hold for the $\overline{M}$ sets. Must there be a sequence $p_1, p_2, \ldots, p_n$ of height 1 primes in $Z[X]$ such that $M(J \mid K) = \overline{M}(J \mid K)$ for all $J \cup K$? We give an example showing that the answer is no. However, we then show that the problem only arises for those $M(J \mid K)$ that are infinite. Indeed, all of the finite $M(J \mid K)$ (i.e., have $|J| \geq 2$) can be pre-determined (and the infinite ones can be partially pre-determined).

EXAMPLE: Consider the case $n = 2$. We take $\overline{M}(\{1\} \mid \{2\})$ to be the set of maximal ideals that contain $2Z[X]$, $\overline{M}(\{2\} \mid \{1\})$ to be the set of maximal ideals that contain $3Z[X]$ with the exception of $(3, X)Z[X]$, $\overline{M}(\{1, 2\} \mid \varnothing)$ to be empty, and $\overline{M}(\varnothing \mid \{1, 2\})$ to be the set of all maximal ideals not in one of the previous three sets. (Notice that the three conditions of Lemma 1 hold.)

Suppose there does exist a sequence $p_1, p_2$ of height 1 primes in $Z[X]$ such that $\overline{M}(J \mid K) = M(J \mid K)$ in each of those four cases. Since $p_1$ is contained in every maximal ideal in $\overline{M}(\{1\} \mid \{2\})$, we must have that $p_1 = 2Z[X]$. Similarly, $p_2 = 3Z[X]$. However, $M(\{2\} \mid \{1\})$ contains $(3, X)Z[X]$, and that is not in $\overline{M}(\{2\} \mid \{1\})$. (Also $\overline{M}(\varnothing \mid \{1, 2\}) \neq M(\varnothing \mid \{1, 2\})$.

THEOREM A: Suppose $n \geq 2$, and for every partition $J \cup K$ of

$\{1, 2, \ldots., n\}$, there is a *finite* set $\overline{M}(J \mid K)$ of maximal ideals of $Z[X]$, such that if $J \cup K$ and $J' \cup K'$ are two different partitions of $\{1, 2, \ldots, n\}$, then $\overline{M}(J \mid K)$ and $\overline{M}(J' \mid K')$ are disjoint. Then there is a sequence $p_1, p_2, \ldots, p_n$ of height 1 primes in $Z[X]$ such that when $M(J \mid K)$ is defined in terms of that sequence, we have $\overline{M}(J \mid K) \subseteq M(J \mid K)$ for all $J \cup K$, and furthermore, equality holds whenever $|J| \geq 2$.

We will prove the theorem by induction. We will first do the initial step, when $n = 2$, as a lemma. We will then present the inductive step as Theorem B. Later, we will show that Theorem B is essentially equivalent to axiom RW.

LEMMA 2: If T and W are disjoint finite sets of maximal ideals of $Z[X]$, then there is a height 1 prime p such that p is contained in every maximal in U but is not contained in any of the maximals in W.

Proof: Let S be a finite set of height 1 primes such that every prime in W contains a prime in S. Let p be a height 1 prime satisfying axiom RW for that S and T. That axiom shows p is contained in every maximal in T. Now suppose $p \subset q \in W$. Then since q also contains a prime in S, RW tells us $q \in T$. That contradicts disjointness of T and W.

We are ready for the initial step of our induction.

LEMMA 3: For each partition $J \cup K$ of $\{1, 2\}$, let $\overline{M}(J \mid K)$ be a finite set of maximal ideals of $Z[X]$, and suppose that if $J \cup K$ and $J' \cup K'$ are two different partitions of $\{1, 2\}$, then $\overline{M}(J \mid K)$ and $\overline{M}(J' \mid K')$ are disjoint. Then there are height 1 prime ideals $p_1$ and $p_2$ such that $\overline{M}(J \mid K) \subseteq M(J \mid K)$, and equality holds when $J = \{1, 2\}$ (the only J for which $|J| \geq 2$).

Proof: We have four finite sets of maximal ideals, $\overline{M}(\varnothing \mid \{1, 2\})$, $\overline{M}(\{1\} \mid \{2\})$, $\overline{M}(\{2\} \mid \{1\})$, and $\overline{M}(\{1, 2\} \mid \varnothing\}$, and they are pairwise disjoint. By Lemma 2, there is a prime $p_1$ contained in every maximal in $\overline{M}(\{1\} \mid \{2\}) \cup \overline{M}(\{1, 2\} \mid \varnothing\}$ and in no maximal in $\overline{M}(\varnothing \mid \{1, 2\}) \cup \overline{M}(\{2\} \mid \{1\})$. Also, there is a height 1 prime p contained in every maximal in $\overline{M}(\varnothing \mid \{1, 2\})$.

Let $S = \{p_1, p\}$ and let $T = \overline{M}(\{2\} \mid \{1\}) \cup \overline{M}(\{1, 2\} \mid \varnothing\}$. Select $p_2$ so that it satisfies axiom RW with respect to that S and T. We need to show a total of five inclusions, (two of them giving an equality).

i) $\overline{M}(\varnothing \mid \{1, 2\}) \subseteq M(\varnothing \mid \{1, 2\})$: Let $q \in \overline{M}(\varnothing \mid \{1, 2\})$. The choice of $p_1$ shows $p_1 \not\subset q$. Since $p \subset q \notin T$, we must have $p_2 \not\subset q$.
Thus $q \in M(\varnothing \mid \{1, 2\})$.

ii) $\overline{M}(\{1\} \mid \{2\}) \subseteq M(\{1\} \mid \{2\})$: Let $q \in \overline{M}(\{1\} \mid \{2\})$. We already know $p_1 \subset q$. Since $q \notin T$, we must have $p_2 \not\subset q$.

iii) $\overline{M}(\{2\} \mid \{1\}) \subseteq M(\{2\} \mid \{1\})$: Let $q \in \overline{M}(\{2\} \mid \{1\}) \subseteq T$. Thus $p_2 \subset q$, and we already know $p_1 \not\subset q$.

iv) $\overline{M}(\{1, 2\} \mid \varnothing) \subseteq M(\{1, 2\} \mid \varnothing)$: Let $q \in \overline{M}(\{1, 2\} \mid \varnothing)$. We already know $p_1 \subset q$, and since we have $q \in T$, we also have $p_2 \subset q$.

v) $M(\{1, 2\} \mid \varnothing) \subseteq \overline{M}(\{1, 2\} \mid \varnothing)$: Let $q \in M(\{1, 2\} \mid \varnothing)$. Then q contains both $p_1$ and $p_2$, and so by axiom RW,
$q \in T = \overline{M}(\{2\} \mid \{1\}) \cup \overline{M}(\{1, 2\} \mid \varnothing\}$. Since $p_1 \subset q$, q cannot be in $M(\{2\} \mid \{1\})$.

NOTATION: Let $n \geq 2$, and let F be a subset of $\{1, 2, \ldots, n\}$.
Let $F^* = F \cup \{n + 1\}$.

The next lemma is trivial, but very useful.

LEMMA 4: Let $J \cup K$ be a partition of $\{1, 2, \ldots, n, n + 1\}$. Let $H = J \cap \{1, 2, \ldots, n\}$ and $G = K \cap \{1, 2, \ldots, n\}$. Then $H \cup G$ is a partition of $\{1, 2, \ldots, n\}$, and $J \cup K$ is either $H^* \cup G$ or $H \cup G^*$.

NOTATION: $H \cup G$ will always represent a partition of $\{1, 2, \ldots, n\}$ and $J \cup K$ will always represent a partition of $\{1, 2, .., n + 1\}$. If $H = J \cap \{1, 2, \ldots, n\}$ and $G = K \cap \{1, 2, \ldots, n\}$, then we will call $H \cup G$ the restriction of $J \cup K$.

THEOREM B: Let $p_1, p_2, .., p_n$ $(n \geq 2)$ be a sequence of height 1 primes of Z[X]. For each partition $J \cup K$ of $\{1, 2, \ldots, n, n + 1\}$, let $\overline{M}(J \mid K)$ be a finite set of maximal ideals, and assume that if $H \cup G$ is the restriction of $J \cup K$, then $\overline{M}(J \mid K) \subseteq M(H \mid G)$ (that last set defined via the sequence $p_1, p_2, \ldots, p_n$). Also assume that $\overline{M}(H^* \mid G)$ and $\overline{M}(H \mid G^*)$ are disjoint. Finally, assume that if $|H| \geq 2$, then $M(H \mid G) = \overline{M}(H^* \mid G) \cup \overline{M}(H \mid G^*)$. Then there is a height 1 prime $p_{n+1}$ such that for each partition $J \cup K$ of $\{1, 2, \ldots, n, n + 1\}$, $\overline{M}(J \mid K) \subseteq M(J \mid K)$, (those sets defined via $p_1, p_2, \ldots, p_n, p_{n+1}$), and equality holds when $|J| \geq 2$.

Proof: Let T be the union of all the sets $\overline{M}(H^* \mid G)$, over all partitions $H \cup G$ of $\{1, 2, .., n\}$. Using Lemma 2, let p be a height 1 prime contained in each maximal in the union of all sets of the form $\overline{M}(H \mid G^*)$. Let $S = \{p_1, p_2, \ldots, p_n, p\}$. Pick $p_{n+1}$ so as to satisfy axiom RW with respect to S and T.

Let $q \in \overline{M}(J \mid K)$. We must show that $q \in M(J \mid K)$. If $H \cup G$ is the restriction of $J \cup K$, then by an assumption in Theorem B, we have

$q \in \overline{M}(J \mid K) \subseteq M(H \mid G)$. Therefore, we must show that $p_{n+1}$ is contained in q if and only if $J \cup K$ has the form $H^* \cup G$ (as opposed to $H \cup G^*$).

Suppose $J \cup K$ has the form $H^* \cup G$. Then $q \in \overline{M}(J \mid K) = \overline{M}(H^* \mid G) \subseteq T$, and so that we know $p_{n+1} \subset q$, as required. Now suppose $J \cup K$ has the form $H \cup G^*$. Suppose $p_{n+1} \subset q$. Since $q \in \overline{M}(H \mid G^*)$, we know that p is also contained in q. By axiom RW, we have $q \in T$. Thus q is contained in some set of the form $\overline{M}(H'^* \mid G')$. By assumption, we have $\overline{M}(H \mid G^*) \subseteq M(H \mid G)$, and $\overline{M}(H'^* \mid G') \subseteq M(H' \mid G')$. Thus $q \in M(H \mid G) \cap M(H' \mid G')$. As that intersection is not empty, we must have $H = H'$ and $G = G'$. Thus q is contained in both $\overline{M}(H \mid G^*)$ and $\overline{M}(H^* \mid G)$. That contradicts an assumption of Theorem B.

It only remains to show that if $|J| \geq 2$, then $\overline{M}(J \mid K) = M(J \mid K)$, and we already have one inclusion. For the other, suppose $q \in M(J \mid K)$. First suppose $J \cup K$ has form $H^* \cup G$. Since $|J| = |H^*| \geq 2$, we have $|H| \geq 1$, and so for some i ($1 \leq i \leq n$), we have $i \in H$. We see that q contains both $p_i$ and $p_{n+1}$. By axiom RW, $q \in T$. Thus q is contained in some set of the form $\overline{M}(H'^* \mid G') \subseteq M(H' \mid G')$. As also $q \in M(J \mid K) = M(H^* \mid G) \subseteq M(H \mid G)$, we must have $H = H'$ and $G = G'$, showing $q \in \overline{M}(H^* \mid G)$.

The only remaining chore is to show $\overline{M}(J \mid K) = M(J \mid K)$ when $J \cup K$ has the form $H \cup G^*$, and $|H| = |J| \geq 2$. By two of the assumptions in Theorem B, we have a partition $M(H \mid G) = \overline{M}(H^* \mid G) \cup \overline{M}(H \mid G^*)$. Now obviously $M(H \mid G) = M(H^* \mid G) \cup M(H \mid G^*)$ is also a partition. As we just proved that $\overline{M}(H^* \mid G) = M(H^* \mid G)$, we must have $\overline{M}(H \mid G^*) = M(H \mid G^*)$.

We are ready to prove Theorem A.

Proof of Theorem A: By Lemma 3, Theorem A holds when $n = 2$. Now suppose it holds for some $n \geq 2$, and assume that for every partition $J \cup K$ of $\{1, 2, \ldots, n, n + 1\}$, there is a finite set $\overline{M}(J \mid K)$ of maximal ideals such that the hypothesis of Theorem A holds. Let $H \cup G$ be a partition of $\{1, 2, \ldots, n\}$, and define $\overline{M}(H \mid G)$ to be the finite set $\overline{M}(H^* \mid G) \cup \overline{M}(H \mid G^*)$. Then we easily see that the hypothesis of Theorem A holds for the various $\overline{M}(H \mid G)$, and so by induction, there is a sequence $p_1, p_2, \ldots, p_n$ of height 1 primes such that when $M(H \mid G)$ is defined via that sequence, we have $\overline{M}(H \mid G) \subseteq M(H \mid G)$, with equality holding whenever $|H| \geq 2$.

We now claim that the hypotheses of Theorem B hold. Let $H \cup G$ be the restriction of $J \cup K$, so that $J \cup K$ is either $H^* \cup G$ or $H \cup G^*$. In both cases, we have $\overline{M}(J \mid K) \subseteq \overline{M}(H \mid G) \subseteq M(H \mid G)$. Now $\overline{M}(H^* \mid G)$ and $\overline{M}(H \mid G^*)$ are disjoint by the hypothesis of Theorem A. Finally, suppose $|H| \geq 2$. Then we have $M(H \mid G) = \overline{M}(H \mid G) = \overline{M}(H^* \mid G) \cup \overline{M}(H \mid G^*)$. That proves the claim. Theorem B now shows that Theorem A is true.

We will now give two variants of Theorem A. In the first, we discuss $M(J \mid K)$ where we assume J and K are disjoint subsets of $\{1, 2, \ldots, n\}$, but do not assume $J \cup K$ is a partition of that set. In the second, we consider infinite sequences of height 1 primes.

Lemma 5: Suppose that for each pair of disjoint subsets J and K of $\{1, 2, \ldots, n\}$ we specify a set $\overline{M}(J \mid K)$ of maximal ideals of Z[X]. Then the following two statements are equivalent.

a) If $t \in \{1, 2, .., n\} - (J \cup K)$, then $\overline{M}(J \cup \{t\} \mid K) \cup \overline{M}(J \mid K \cup \{t\})$ is a partition of $\overline{M}(J \mid K)$.

b) Each $\overline{M}(J \mid K)$ can be partitioned as $\cup \overline{M}(J' \mid K')$, the union over all sets J′ and K′ such that $J' \cup K'$ is a partition of $\{1, 2, \ldots, n\}$ with $J \subseteq J'$ and $K \subseteq K'$.

Proof: (a) ⇒ (b): If $J \cup K = \{1, 2, \ldots, n\}$, the result is trivial. Otherwise, let $\{1, 2, \ldots, n\} - (J \cup K) = \{t_1, t_2, \ldots, t_m\}$.
We will induct on m. If m = 1, by (a) we have $\overline{M}(J \mid K)$ can be partitioned as $\overline{M}(J \cup \{t_1\} \mid K \cup \emptyset) \cup \overline{M}(J \cup \emptyset \mid K \cup \{t_1\})$, and are done. If $m \geq 1$, we apply induction to both $\overline{M}(J \cup \{t_1\} \mid K \cup \emptyset)$ and $\overline{M}(J \cup \emptyset \mid K \cup \{t_1\})$, and the set $\{t_2, t_3, \ldots, t_m\}$, and (b) follows.

(b) ⇒ (a): Suppose $t \in \{1, 2, \ldots, n\} - (J \cup K)$. With notation as in (b), we have $\overline{M}(J \mid K) = \cup \overline{M}(J' \mid K') = (\bigcup_{t \in J'} \overline{M}(J' \mid K')) \cup (\bigcup_{t \notin J'} \overline{M}(J' \mid K')) =$ (using (b) again) $\overline{M}(J \cup \{t\} \mid K) \cup \overline{M}(J \mid K \cup \{t\})$, and that last is clearly a partition.

THEOREM AA: Suppose $n \geq 2$, and for every pair J and K of disjoint subsets of $\{1, 2, \ldots., n\}$, there is a finite set $\overline{M}(J \mid K)$ of maximal ideals of Z[X], such that the following two assumptions hold.

(i) If $t \in \{1, 2, .., n\} - (J \cup K)$, then $\overline{M}(J \cup \{t\} \mid K) \cup \overline{M}(J \mid K \cup \{t\})$ is a partition of $\overline{M}(J \mid K)$.

(ii) If $J' \cup K'$ and $J'' \cup K''$ are two different partitions of $\{1, 2, \ldots, n\}$, then $\overline{M}(J' \mid K')$ and $\overline{M}(J'' \mid K'')$ are disjoint.

Then there is a sequence $p_1, p_2, \ldots, p_n$ of height 1 primes in Z[X] such that $\overline{M}$(J | K) $\subseteq$ M(J | K) for all such pairs J and K, and furthermore, equality holds whenever $|J| \geq 2$.

Proof: If $t \in \{1, 2, .., n\} - (J \cup K)$, then $M(J \cup \{t\} \mid K) \cup M(J \mid K \cup \{t\})$ is obviously a partition of M(J | K), and so condition (b) of Lemma 5 holds both for sets of the form $\overline{M}$(J | K) and for sets of the form M(J | K). It easily follows that to prove that the conclusion of the theorem holds for all our pairs J and K, it will suffice to show it holds for those pair J′, K′ for which $J' \cup K'$ is a partition of $\{1, 2, \ldots, n\}$. The truth of that fact follows from assumption (ii) and Theorem A.

THEOREM AAA: Suppose for every pair J and K of finite disjoint subsets of the positive integers, there is a finite set $\overline{M}$(J | K) of maximal ideals of Z[X], such that the following two assumptions hold.

(i) If t is a positive integer with $t \notin J \cup K$, then $\overline{M}(J \cup \{t\} \mid K) \cup \overline{M}(J \mid K \cup \{t\})$ is a partition of $\overline{M}$(J | K).

(ii) If either $J \cap K' \neq \varnothing$ or $J' \cap K \neq \varnothing$, then $\overline{M}$(J | K) and $\overline{M}$(J′ | K′) are disjoint. (Note that holds for M(J | K) and M(J′ | K′).)

Then there is an infinite sequence $p_1, p_2, p_3, \ldots$ of height 1 primes in Z[X] such that $\overline{M}$(J | K) $\subseteq$ M(J | K) for all such pairs J and K, and furthermore, equality holds whenever $|J| \geq 2$.

Proof: By Theorem AA, we can find $p_1$ and $p_2$ such that the conclusion holds for pairs J and K with $J \cup K \subseteq \{1, 2\}$. We claim that we can inductively extend that sequence to $p_1, p_2, \ldots, p_n$ for each $n \geq 2$ in such a way that the conclusion holds for all pairs J and K with

$J \cup K \subseteq \{1, 2, \ldots, n\}$.

Suppose we have done so for n, and wish to add another term, $p_{n+1}$. We will use Theorem B. Let $J \cup K$ by a partition of $\{1, 2, \ldots, n, n + 1\}$, with restriction $H \cup G$. Since $J \cup K$ is either $H^* \cup G$ or $H \cup G^*$, and since by assumption (i), $\overline{M}(H^* \mid G) \cup \overline{M}(H \mid G^*)$ is a partition of $\overline{M}(H \mid G)$, we see that $\overline{M}(J \mid K) \subseteq \overline{M}(H \mid G) \subseteq M(H \mid G)$ (the last inclusion by induction). That establishes the first assumption of Theorem B. The second assumption of Theorem B follows immediately from assumption (ii). Finally, assume $|H| \geq 2$, then by induction, we have $M(H \mid G) = \overline{M}(H \mid G) = \overline{M}(H^* \mid G) \cup \overline{M}(H \mid G^*)$, giving us the third assumption of Theorem B. That theorem supplies the desired $p_{n+1}$, and since we can repeat this process indefinitely, we are done.

We will show that Theorem B does not hold in Q[X, Y], by showing that the theorem is essentially equivalent to axiom RW. We first explain the use of the adjective 'essentially'.

Given the five poset axioms listed at the start of this paper, it is not hard to prove that every maximal ideal contains at least two height 1 primes. However, that is not true if we delete axiom RW from the list, since there are rings whose spec consists of a unique minimal prime, a unique height 1 prime, and infinitely many height 2 primes, and such a spec does satisfy the first four axioms. If we delete RW, we will have to add both Theorem B and that new 'two height 1 primes in each maximal' axiom, if we want to recover RW as a theorem. (That new axiom does hold for any height 2 prime in any Noetherian ring, and so holds in Q[X, Y].)

THEOREM C: Suppose Spec R satisfies poset axioms (1) through (4) from the list at the start of this paper, and also that every maximal ideal contains at least two height 1 primes. Furthermore, suppose that Theorem B holds for R. Then axiom RW holds.

Proof: Let T be a finite set of maximal ideals, and let S be a finite set of height 1 primes. To prove that axiom RW holds, it does no harm to add finitely many additional height 1 primes to S, and so (by our new axiom) we may assume that each maximal in T contains at least two primes in S.

Order the primes in S as $p_1, p_2, .., p_n$. Let $H \cup G$ be any partition of $\{1, 2, ..., n\}$. Let $\overline{M}(H^* \mid G) = M(H \mid G) \cap T$. Let $\overline{M}(H \mid G^*)$ be $\varnothing$ when $|H| < 2$, and be $M(H \mid G) - \overline{M}(H^* \mid G)$ when $|H| \geq 2$. Thus, we have defined $\overline{M}(J \mid K)$ for each partition $J \cup K$ of $\{1, 2, .., n, n + 1\}$, and it is straightforward to verify that the three hypotheses of Theorem B hold.

By that theorem, there is a height 1 prime $p_{n+1}$ such that given the sequence $p_1, p_2, .., p_n, p_{n+1}$, we have that $\overline{M}(J \mid K) \subseteq M(J \mid K)$, and equality holds whenever $|J| \geq 2$.

We claim that $p_{n+1}$ satisfies axiom RW with respect to S and T. First, let $q \in T$. (We must show $p_{n+1} \subset q$.) We know that there is an integer $1 \leq i \leq n$ such that $p_i$ is contained in q. Thus q is contained in some $M(H \mid G)$ with $H \cup G$ a partition of $\{1, 2, \ldots, n\}$, and with $i \in H$. Thus $q \in M(H \mid G) \cap T = \overline{M}(H^* \mid G) = M(H^* \mid G)$, the second equality since $|H^*| \geq 2$. As q is in that last set, we have $p_{n+1} \subset q$, as desired.

Now suppose that for some $1 \leq i \leq n$, we have both $p_i$ and $p_{n+1}$ contained in the maximal ideal q. (We must show $q \in T$.) By our added assumption, we know that there is also a $j \neq i$ $(1 \leq j \leq n)$ such that $p_j \subset q$. Thus q is contained in some $M(H \mid G)$ with $\{i, j\} \subset H$, so that $|H| \geq 2$. Now $M(H \mid G)$ is partitioned as $\overline{M}(H^* \mid G) \cup \overline{M}(H \mid G^*)$. As $p_{n+1} \subset q$, and as $\overline{M}(H \mid G^*) \subseteq M(H \mid G^*)$, we see that $q \notin \overline{M}(H \mid G^*)$, and so we must have $q \in \overline{M}(H^* \mid G)$. As that last set is a subset of T, we are done.

QUESTIONS: Note that just because Theorem B fails to hold in Q[X, Y], it does not necessarily follow that Theorem A fails there. We do not know. Nor do we know if Lemma 3 is also true or false in Q[X, Y].

A weak corollary of Theorem A says that in Z[X], if for every partition $J \cup K$ of {1, 2, …., n} such that J contains at least 2 integers, there is a non-negative integer N(J | K), then there a sequence $p_1, p_2, \ldots, p_n$ of height 1 primes such that for each such $J \cup K$, we have $|M(J \mid K)| = N(J \mid K)$. Does that hold in Q[X, Y]?

## References

[1] R. Wiegand and S. Wiegand, Prime ideals in Noetherian rings: a survey, in *Ring and Module Theory*, ed. By T. Albu, G. F. Birkenmeier, A. Erdu **u** gan, A. Tercan, (Birkauser, Boston, 210), pp.175-193.

[2] R. Wiegand, Homeomorpisms of affine surfaces over a finite field, J. London Math. Soc, (2), 18, (1978), 28-32.

[3] R. Wiegand, The prime spectrum of a two-dimensional affine domain, J. pure and applied Algebra, 40 (1986) 209-214.