Bond Options, Caps and the Black Model
Recall the **Black formula** for pricing options on futures:

\[
C(F, K, \sigma, r, T, \tau) = Fe^{-\tau T} N(d_1) - Ke^{-\tau T} N(d_2)
\]

where

\[
d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln\left( \frac{F}{K} \right) + \frac{1}{2} \sigma^2 T \right]
\]

\[
d_2 = d_1 - \sigma \sqrt{T}
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Options on Bonds: The set-up

- Consider a call option on a zero-coupon bond paying $1 at time $T + s$. The maturity of the option is $T$ and the strike is $K$.
- The payoff of the above option is

$$ (P(T, T + s) - K)^+ $$

where $P(T, T + s)$ denotes the price of the bond (maturing at $T + s$) at time $T$.
- Questions:
  How do we apply the Black-Scholes setting to the above option? What are the correct assumptions that are analogues of the lognormality we imposed on the prices of the underlying asset in the Black-Scholes pricing model?
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Exchange Options: The definition and set-up

- It turns out that the convenient tool for solving the above problem is to recast the set-up in terms of a particular family of exotic options, namely, exchange options.
- An exchange option pays off only if the underlying asset outperforms some other asset (benchmark). Hence, these options are also called out-performance options.
- Consider an exchange call option maturing $T$ periods from now which allows its holder to obtain 1 unit of risky asset #1 in return for one unit of risky asset #2.
- $S_t$ ... the price of the risky asset #1 at time $t$
- $K_t$ ... the price of the risky asset #1 at time $t$
- $\delta_S$ ... the dividend yield of the risky asset #1
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Exchange Options: The pricing formula

\[ C(S, K, (\sigma_S, \sigma_K), r, T, (\delta_S, \delta_K)) = S e^{-\delta S T} N(d_1) - K e^{-\delta K T} N(d_2) \]

where

\[ d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S e^{-\delta S T}}{K e^{-\delta K T}} \right) + \frac{1}{2} \sigma^2 T \right] \]

\[ d_2 = d_1 - \sigma \sqrt{T} \]

with

\[ \sigma^2 = \sigma_S^2 + \sigma_K^2 - 2 \rho \sigma_S \sigma_K \]

- In words, \( \sigma \) is the volatility of \( \ln(S/K) \) (over the life of the call).
- Note that if we take either \( S \) or \( K \) to be a riskless asset, the above formula collapses into the “ordinary” Black-Scholes formula.
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Exchange Options: Application to options on bonds

In our case, the two risky assets are

S  The bond
K  The strike - Note that the strike should not be seen as constant. Its time-value (in the long run) is dependent on the interest rate which is not even deterministic!

• \( S_t \) denotes the value at time \( t \) of the bond, i.e., it is the prepaid forward price of the bond
• \( K_t \) denotes the value at time \( t \) of the strike, i.e., it is the prepaid forward price of the strike whose nominal value at time \( T \) is \( K \)
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Forward Contracts Revisited

- Recall that a forward contract is an agreement to pay a specified delivery price $K$ at a delivery date $T$ in exchange for an asset.
- Let the asset’s price at time $t$ be denoted by $S_t$.
- Then, we denote the $T$-forward price of this asset at time $t$ by $F_{t,T}[S]$.
- It is defined as the value of the delivery price $K$ which makes the forward contract have the no-arbitrage price at time $t$ equal to zero.
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Forward Contracts:
Connection with bond-prices

• **Theorem:** Assume that zero-coupon bonds of all maturities are/can be traded. Then,

\[ F_{t,T}[S] = \frac{S_t}{P(t, T)}, \text{ for } 0 \leq t \leq T \]

• **The argument:**

Suppose that at time \( t \) you:

1. **Sell** the above forward contract - this is not a “real sale” as no income can be generated in doing so (by definition)
2. Also, you **short** \( \frac{S_t}{P(t, T)} \) zero-coupon bonds - doing so you get the income of \( S_t \)
3. With the above produced income \( S_t \), you by one share of the asset \( S \)
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Connection with bond-prices (cont’d)

You do nothing until time $T$

Then, at time $T$ you:

1. **Deliver** the one share of asset $S$ that you own
2. **Get** the delivery price $K$ in return
3. **Cover** the short bond position - recall that the bonds we shorted all had maturity $T$ at which time they are worth exactly $1$, i.e., the amount of the payment they produce at maturity

- The net-effect at time $T$ is that you have $K - \frac{S_t}{B(t,T)}$ - this value must be equal to zero, or else there is arbitrage
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Forward Contracts on Bonds

- In particular, if the asset $S$ is actually another bond with maturity $T + s$, we have that

$$ F_{t,T}[P(T, T + s)] = \frac{P(t, T + s)}{P(t, T)}, \text{ for } 0 \leq t \leq T, s \geq 0 $$

- So, the prepaid forward price at time $t$ on the bond is $S_t = F_{t,T}[P(T, T + s)]P(t, T) = P(t, T + s)$ in the exchange option setting.

- And, if the asset is just some nominal value given at time $T$, we can see this as $K$ bonds which deliver $1$ at maturity $T$.

- So, $K_t = KP(t, T)$ is the prepaid forward price we will use in the exchange option pricing formula.

- The volatility that enters the pricing formula for exchange options is:

$$ \text{Var} \left[ \ln \left( \frac{S_t}{K_t} \right) \right] = \text{Var} \left[ \ln \left( \frac{P(t, T + s)}{KP(t, T)} \right) \right] = \text{Var} \left[ F_{t,T}[P(T, T + s)] \right] $$
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$$= \text{Var} \left[ F_{t, T}[P(T, T + s)] \right]$$
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$$F_{t,T}[P(T, T + s)] = \frac{P(t, T + s)}{P(t, T)}$$, for $0 \leq t \leq T$, $s \geq 0$

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• And, if the asset is just some nominal value given at time $T$, we can see this as $K$ bonds which deliver $\$1$ at maturity $T$

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• The volatility that enters the pricing formula for exchange options is:

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= \text{Var} \left[ F_{t,T}[P(T, T + s)] \right]$$
Forward Contracts on Bonds

• In particular, if the asset $S$ is actually another bond with maturity $T + s$, we have that

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Black formula

- If we assume that the bond forward price process 
\( \{F_{t,T}[P(T, T + s)]\}_t \) agrees with the Black-Scholes assumptions 
and that its constant volatility is \( \sigma \), we obtain the Black formula for a bond option:

\[
C[F, P(0, T), \sigma, T] = P(0, T)[FN(d_1) - KN(d_2)]
\]

where

\[
d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln(\frac{F}{K}) + \frac{1}{2} \sigma^2 T \right]
\]

\[
d_2 = d_1 - \sigma \sqrt{T}
\]

with \( F = F_{0,T}[P(T, T + s)] \)
Forward (Implied) Interest Rate

- We are now at time 0. Assume that you would like to earn at the interest rate in the period between time $T$ and time $T + s$. Denote this forward interest rate by $R_0(T, T + s)$.

- The unit investment in the interest rate at time $T$ until time $T + s$ should be consistent (in the sense of no-arbitrage) with the strategy that includes a zero-coupon bond maturing at time $T$ and another with maturity at time $T + s$, i.e., we should have

$$1 + R_0(T, T + s) = \frac{P(0, T)}{P(0, T + s)}$$
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Forward Rate Agreements

- Consider a borrower (or, analogously, a lender) who wants to hedge against increases in the cost of borrowing a certain amount of money at a future date (that is, in the interest rate).
- Forward rate agreements (FRAs) are over-the-counter contracts that guarantee a borrowing or lending rate on a given principal amount.
- FRAs are, thus, a type of forward contracts based on the interest rate:
  - If the reference ("real") interest rate is above the rate agreed upon in the FRA, then the borrower gets paid.
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Forward Rate Agreements: Settlement time

- FRAs can be settled at maturity (in arrears), i.e., at the time the loan is repaid or at the initiation of the borrowing or lending transaction, i.e., at the time the loan is taken.
- Let $r$ denote the reference interest rate for the prescribed loan period.
- If in arrears, then the payment is
  \[ (r - r_{FRA}) \times \text{notional principal} \]
- If at the initiation, then the payment is
  \[ \frac{1}{1 + r} \times (r - r_{FRA}) \times \text{notional principal} \]
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• Alternatively, one (in this case a borrower) might buy a **call option on the FRA** - this should be a better strategy as there is no downfall in dropping interest rates
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- **Options on the FRA** are called caplets
  - This option, at time $T + s$ pays
    $$ (R_T(T, T + s) - K_R)^+ $$
    where $K_R$ denotes the strike
  - If settled at time $T$, then the above type of option has payoff
    $$ \frac{1}{1 + R_T(T, T + s)} (R_T(T, T + s) - K_R)^+ $$
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Forward Rate Agreements:
Pricing caplets through the Black formula

• Using simple algebra, we can transform the above payoff into

\[(1 + K_R) \left( \frac{1}{1 + K_R} - \frac{1}{1 + R_T(T, T + s)} \right)^+ \]

• Recalling the consistency equation

\[1 + R_0(T, T + s) = \frac{P(0, T)}{P(0, T + s)} \]

we see that the value \[\frac{1}{1 + R_T(T, T + s)}\] is the value at time \(t\) of a zero-coupon bond paying $1 at time \(T + s\)

• Setting the value \[\frac{1}{1 + K_R}\] as the new strike, we see that we can use the Black formula to price the above described caplet
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Forward Rate Agreements: Caps

• An interest rate **cap** is a collection of caplets
• Suppose a borrower has a floating rate loan with interest payments at times $t_i$, $i = 1, ..., n$. A cap would make the series of payments at times $t_{i+1}$ given by

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