Level Annuities with Payments More Frequent than Each Interest Period

1 Examples

- 2 Annuity-immediate
- 3 Annuity-due

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Annuity-due

- At what annual effective rate of interest is the present value of a series of payments of \$1 every six months forever, with the first payment made immediately, equal to \$10?
- ⇒ The equation of value is

$$10 = \sum_{k=0}^{\infty} v^{k/2} = \frac{1}{1 - v^{1/2}}$$

So,

$$\sqrt{v} = \sqrt{\frac{1}{1+i}} = 0.9$$

Thus. $i = 0.9^{-2} - 1 = 0.2346$

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- A loan of \$3,000 is to be repaid with quarterly installments at the end of each quarter for five years. If the rate of interest charged on the loan is 10% convertible semiannually, find the amount of each quarterly payment.
- \Rightarrow We are given that the effective interest rate per half-year equals 0.10/2 = 0.05.

Let j be the equivalent rate of interest per quarter, i.e., per payment period. Then

$$j = (1.05)^{1/2} - 1 = 0.024695$$

Let the quarterly payment be denoted by R. The equation of value then reads as

$$Ra_{\overline{20}|j} = 3000$$

Thus

$$R = \frac{3000}{15.6342} = 191.89$$



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- J . . . the effective interest rate per payment period, i.e.,

$$J = (1+i)^{1/m} - 1$$

where *i* denotes the interest rate per interest period

• In the context where the interest period is a year, i is the annual interest rate and $i^{(m)}$ is the nominal interest rated convertible m times per year, we have that

$$J = \frac{i^{(m)}}{m}$$

 Note that in the above set-up the sum of payments made during one interest period is equal to 1



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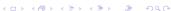
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The present value of the annuity-immediate described above is

$$a_{\overline{n}}^{(m)}{}_{i}=rac{1}{m}\cdot a_{\overline{n}\overline{m}}$$
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The accumulated value of the annuity-immediate described above is

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Value at issuance and accumulated value: Formulae

$$a_{\overline{n}|\ i}^{(m)} = \frac{1 - v^n}{i^{(m)}} = a_{\overline{n}|\ i} \cdot \frac{i}{i^{(m)}}$$

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- ⇒ Our basic time unit is a half-year.

Since each payment is equal to R, this means that the total amount paid during one interest period is 2R.

Using the notation we just developed, we get the following equation of value:

$$2Ra_{\overline{10}|\ 0.05}^{(2)} = 3000$$

$$R = \frac{1500}{a_{\frac{1}{10},0.05}^{(2)}} = \frac{1500}{\frac{i}{i^{(2)}} a_{\frac{10}{10},0.05}} = \frac{1500}{1.012348 \cdot 7.7217} = 191.89$$

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- Again, consider a basic annuity that lasts for n interest periods, and has nm payments
- This annuity has a payment at the beginning of each mth of the interest periods
- Then, the value at issuance of this annuity-due is $\ddot{a}_{\overline{n}|}^{(m)}$; and

$$\ddot{a}_{\overline{n}|}^{(m)}_{i} = \frac{1 - v^{n}}{d^{(m)}}$$

• Similarly, we get that the accumulated value $\ddot{s}_{nl}^{(m)}$ is

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- Assume compound interest. Payments of \$400 per month are made over a ten-year period.
- Find an expression for the value of these payments two years prior to the first payment.
- ⇒ The \$400 monthly amounts translate into the \$4800 annual amounts. So, the symbolic representation is

$$4800 \cdot v^2 \cdot \ddot{a}_{\overline{10}|}^{(12)} = 4800 \cdot \left(\ddot{a}_{\overline{12}|}^{(12)} - \ddot{a}_{\overline{2}|}^{(12)} \right)$$

 Find an expression for the accumulated value of these payments three years after the end of the annuity's term.

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Odds and Ends

• In general, we have that

$$\ddot{a}_{\overline{n}|i} > \ddot{a}_{\overline{n}|i}^{(m)} = a_{\overline{n}|i}^{(m)} = a_{\overline{n}|i}$$

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