Level Annuities with Payments More Frequent than Each Interest Period

1. Examples
2. Annuity-immediate
3. Annuity-due

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An Example

- At what annual effective rate of interest is the present value of a series of payments of $1 every six months forever, with the first payment made immediately, equal to $10?

$\Rightarrow$ The equation of value is

$$10 = \sum_{k=0}^{\infty} v^{k/2} = \frac{1}{1 - v^{1/2}}$$

So,

$$\sqrt{v} = \sqrt{\frac{1}{1+i}} = 0.9$$

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- A loan of \$3,000 is to be repaid with quarterly installments at the end of each quarter for five years. If the rate of interest charged on the loan is 10% convertible semiannually, find the amount of each quarterly payment.

⇒ We are given that the effective interest rate per half-year equals $0.10/2 = 0.05$.
Let j be the equivalent rate of interest per quarter, i.e., per payment period. Then

$$j = (1.05)^{1/2} - 1 = 0.024695$$

Let the quarterly payment be denoted by R. The equation of value then reads as

$$Ra_{\overline{20|}j} = 3000$$

Thus,

$$R = \frac{3000}{15.6342} = 191.89$$

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Set-up

- Consider a annuity-immediate that lasts for n **interest periods**, and has m payments during each of the interest periods with each payment being equal to $1/m$ and taking place at the end of an m^{th} of the interest period
- J ... the effective interest rate per payment period, i.e.,

$$J = (1+i)^{1/m} - 1$$

where i denotes the interest rate per interest period
- In the context where the interest period is a year, i is the annual interest rate and $i^{(m)}$ is the nominal interest rated convertible m times per year, we have that

$$J = \frac{i^{(m)}}{m}$$

- Note that in the above set-up the sum of payments made during one interest period is equal to 1

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Value at issuance and accumulated value

- The **present value** of the annuity-immediate described above is

$$a^{(m)}_{\overline{n}|\,i} = \frac{1}{m} \cdot a_{\overline{nm}|\,J}$$

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Value at issuance and accumulated value: Formulae

$$a^{(m)}_{\overline{n|}\,i} = \frac{1 - v^n}{i^{(m)}} = a_{\overline{n|}\,i} \cdot \frac{i}{i^{(m)}}$$

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An Example Revisited

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⇒ Our basic time unit is a half-year.

Since each payment is equal to R, this means that the total amount paid during one interest period is $2R$.

Using the notation we just developed, we get the following equation of value:

$$2Ra^{(2)}_{\overline{10|}\,0.05} = 3000$$

So,

$$R = \frac{1500}{a^{(2)}_{\overline{10|}\,0.05}} = \frac{1500}{\frac{i}{i^{(2)}}a_{\overline{10|}\,0.05}} = \frac{1500}{1.012348 \cdot 7.7217} = 191.89$$

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Value at issuance and accumulated value

- Again, consider a basic annuity that lasts for n **interest periods**, and has nm payments
- This annuity has a payment at the **beginning** of each m^{th} of the interest periods
- Then, the value at issuance of this annuity-due is $\ddot{a}^{(m)}_{\overline{n|}\,i}$ and

$$\ddot{a}^{(m)}_{\overline{n|}\,i} = \frac{1 - v^n}{d^{(m)}}$$

- Similarly, we get that the accumulated value $\ddot{s}^{(m)}_{\overline{n|}\,i}$ is

$$\ddot{s}^{(m)}_{\overline{n|}\,i} = \frac{(1+i)^n - 1}{d^{(m)}}$$

- *Assignment:* See Examples 4.3.8 and 4.3.13 for the calculator recipes . . .

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An Example

- Assume compound interest. Payments of $400 per month are made over a ten-year period.

I. Find an expression for the value of these payments two years prior to the first payment.

⇒ The $400 monthly amounts translate into the $4800 annual amounts. So, the symbolic representation is

$$4800 \cdot v^2 \cdot \ddot{a}^{(12)}_{\overline{10}|} = 4800 \cdot \left(\ddot{a}^{(12)}_{\overline{12}|} - \ddot{a}^{(12)}_{\overline{2}|} \right)$$

II. Find an expression for the accumulated value of these payments three years after the end of the annuity's term.

⇒

$$4800 \cdot \ddot{s}^{(12)}_{\overline{10}|} \cdot (1+i)^3 = 4800 \cdot \left(\ddot{s}^{(12)}_{\overline{13}|} - \ddot{s}^{(12)}_{\overline{3}|} \right)$$

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Odds and Ends

- In general, we have that

$$\ddot{a}_{\overline{n|}\,i} > \ddot{a}^{(m)}_{\overline{n|}\,i} > a^{(m)}_{\overline{n|}\,i} > a_{\overline{n|}\,i}$$

- For perpetuities, we have

$$a^{(m)}_{\overline{\infty|}\,i} = \frac{1}{i^{(m)}}$$

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