Loan Repayment Methods

1. Amortized Loans

2. The Sinking Fund Method
Loan Repayment Methods

1. Amortized Loans

2. The Sinking Fund Method
The Set-up

- When a loan is an **amortized loan**, each payment is understood to consist of:
  1. the interest due on the **outstanding loan balance**;
  2. the rest of the payment which goes towards reducing the outstanding loan balance and which is referred to as the **principal payment**.
- The chart (table) containing the payment amount, interest paid in each payment, principal repaid in each payment and the outstanding balance after each payment is called the **amortization schedule**.
The Set-up

- When a loan is an amortized loan, each payment is understood to consist of:
  1. the interest due on the outstanding loan balance;
  2. the rest of the payment which goes towards reducing the outstanding loan balance and which is referred to as the principal payment.
- The chart (table) containing the payment amount, interest paid in each payment, principal repaid in each payment and the outstanding balance after each payment is called the amortization schedule.
The Set-up

- When a loan is an amortized loan, each payment is understood to consist of:
  1. the interest due on the **outstanding loan balance**;
  2. the rest of the payment which goes towards reducing the outstanding loan balance and which is referred to as the **principal payment**.

- The chart (table) containing the payment amount, interest paid in each payment, principal repaid in each payment and the outstanding balance after each payment is called the **amortization schedule**.
When a loan is an amortized loan, each payment is understood to consist of:

1. the interest due on the outstanding loan balance;
2. the rest of the payment which goes towards reducing the outstanding loan balance and which is referred to as the principal payment.

The chart (table) containing the payment amount, interest paid in each payment, principal repaid in each payment and the outstanding balance after each payment is called the amortization schedule.
An Example

- Consider a loan for $1,000 which is to be repaid in four annual payments under the effective annual interest rate of 8%.

We assume that all payments are equal and get their value as

\[
\frac{1000}{a_4} = \frac{1000}{3.3121} = 301.92
\]

Year #1 Then, the amount of interest contained in the first payment is

\[
l_1 = i \cdot 1000 = 0.08 \cdot 1000 = 80
\]

Hence, the portion of the first payment that goes toward the reduction of the outstanding balance equals

\[
301.92 - 80 = 221.92
\]

The outstanding balance at the end of the first year is, then

\[
1000 - 221.92 = 778.08
\]
An Example

Consider a loan for $1,000 which is to be repaid in four annual payments under the effective annual interest rate of 8%. We assume that all payments are equal and get their value as

\[
\frac{1000}{a_{4|}} = \frac{1000}{3.3121} = 301.92
\]

Year #1 Then, the amount of interest contained in the first payment is

\[
l_1 = i \cdot 1000 = 0.08 \cdot 1000 = 80
\]

Hence, the portion of the first payment that goes toward the reduction of the outstanding balance equals

\[301.92 - 80 = 221.92\]

The outstanding balance at the end of the first year is, then

\[1000 - 221.92 = 778.08\]
An Example

- Consider a loan for $1,000 which is to be repaid in four annual payments under the effective annual interest rate of 8%. We assume that all payments are equal and get their value as

\[
\frac{1000}{a_{4}} = \frac{1000}{3.3121} = 301.92
\]

Year #1 Then, the amount of interest contained in the first payment is

\[
l_1 = i \cdot 1000 = 0.08 \cdot 1000 = 80
\]

Hence, the portion of the first payment that goes toward the reduction of the outstanding balance equals

\[
301.92 - 80 = 221.92
\]

The outstanding balance at the end of the first year is, then

\[
1000 - 221.92 = 778.08
\]
An Example

- Consider a loan for $1,000 which is to be repaid in four annual payments under the effective annual interest rate of 8%. We assume that all payments are equal and get their value as

\[
\frac{1000}{a_{4}} = \frac{1000}{3.3121} = 301.92
\]

**Year #1** Then, the amount of interest contained in the first payment is

\[I_1 = i \cdot 1000 = 0.08 \cdot 1000 = 80\]

Hence, the portion of the first payment that goes toward the reduction of the outstanding balance equals

\[301.92 - 80 = 221.92\]

The outstanding balance at the end of the first year is, then

\[1000 - 221.92 = 778.08\]
An Example

• Consider a loan for $1,000 which is to be repaid in four annual payments under the effective annual interest rate of 8%. We assume that all payments are equal and get their value as

\[
\frac{1000}{a_{4\vert}} = \frac{1000}{3.3121} = 301.92
\]

Year #1 Then, the amount of interest contained in the first payment is

\[I_1 = i \cdot 1000 = 0.08 \cdot 1000 = 80\]

Hence, the portion of the first payment that goes toward the reduction of the outstanding balance equals

\[301.92 - 80 = 221.92\]

The outstanding balance at the end of the first year is, then

\[1000 - 221.92 = 778.08\]
An Example: The amortization schedule

- If we continue the procedure we completed for the first year for the remaining 3 payments, we get the entire amortization schedule:

<table>
<thead>
<tr>
<th>Year</th>
<th>Pmt</th>
<th>Interest</th>
<th>Principal repaid</th>
<th>OLB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>1000</td>
</tr>
<tr>
<td>1</td>
<td>301.29</td>
<td>80.00</td>
<td>221.92</td>
<td>778.08</td>
</tr>
<tr>
<td>2</td>
<td>301.29</td>
<td>62.25</td>
<td>239.67</td>
<td>538.41</td>
</tr>
<tr>
<td>3</td>
<td>301.29</td>
<td>43.07</td>
<td>258.85</td>
<td>279.56</td>
</tr>
<tr>
<td>4</td>
<td>301.29</td>
<td>22.36</td>
<td>279.56</td>
<td>0</td>
</tr>
</tbody>
</table>
**An Example: The amortization schedule**

- If we continue the procedure we completed for the first year for the remaining 3 payments, we get the entire amortization schedule:

<table>
<thead>
<tr>
<th>Year</th>
<th>Pmt</th>
<th>Interest</th>
<th>Principal repaid</th>
<th>OLB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
<td>0</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>1</td>
<td>301.29</td>
<td>80.00</td>
<td>221.92</td>
<td>778.08</td>
</tr>
<tr>
<td>2</td>
<td>301.29</td>
<td>62.25</td>
<td>239.67</td>
<td>538.41</td>
</tr>
<tr>
<td>3</td>
<td>301.29</td>
<td>43.07</td>
<td>258.85</td>
<td>279.56</td>
</tr>
<tr>
<td>4</td>
<td>301.29</td>
<td>22.36</td>
<td>279.56</td>
<td>0</td>
</tr>
</tbody>
</table>
An Example: Smaller final payment

- A $1,000 loan is being repaid by payments of $100 (plus the final smaller payment) at the end of each quarter-year for as long as it is necessary. Assume that $i^{(4)} = 0.16$.

Find the amount of interest and the amount of principal repaid in the fourth payment.

⇒ Using the retrospective method (why??), we get that the outstanding loan balance at the beginning of the fourth quarter equals

$$1000(1.04)^3 - 100 \cdot s_3 = 1124.86 - 312.16 = 812.70$$

The interest that is to be repaid in the fourth payment is exactly the amount of interest that is accrued during the fourth quarter-year on the balance above, i.e.,

$$0.04 \cdot 812.70 = 32.51$$

Evidently, the fourth payment is not yet the final, smaller one. So, the principal payment contained in the fourth payment is

$$100 - 32.51 = 67.49$$
An Example: Smaller final payment

A $1,000 loan is being repaid by payments of $100 (plus the final smaller payment) at the end of each quarter-year for as long as it is necessary. Assume that \( i^{(4)} = 0.16 \).

Find the amount of interest and the amount of principal repaid in the fourth payment.

⇒ Using the retrospective method (why??), we get that the outstanding loan balance at the beginning of the fourth quarter equals

\[
1000(1.04)^3 - 100 \cdot s_3^1 = 1124.86 - 312.16 = 812.70
\]

The interest that is to be repaid in the fourth payment is exactly the amount of interest that is accrued during the fourth quarter-year on the balance above, i.e.,

\[
0.04 \cdot 812.70 = 32.51
\]

Evidently, the fourth payment is not yet the final, smaller one. So, the principal payment contained in the fourth payment is

\[
100 - 32.51 = 67.49
\]
An Example: Smaller final payment

- A $1,000 loan is being repaid by payments of $100 (plus the final smaller payment) at the end of each quarter-year for as long as it is necessary. Assume that $i^{(4)} = 0.16$.
Find the amount of interest and the amount of principal repaid in the fourth payment.

⇒ Using the retrospective method (why??), we get that the outstanding loan balance at the beginning of the fourth quarter equals

$$1000(1.04)^3 - 100 \cdot \ddot{s}_3 = 1124.86 - 312.16 = 812.70$$

The interest that is to be repaid in the fourth payment is exactly the amount of interest that is accrued during the fourth quarter-year on the balance above, i.e.,

$$0.04 \cdot 812.70 = 32.51$$

Evidently, the fourth payment is not yet the final, smaller one. So, the principal payment contained in the fourth payment is

$$100 - 32.51 = 67.49$$
An Example: Smaller final payment

- A $1,000 loan is being repaid by payments of $100 (plus the final smaller payment) at the end of each quarter-year for as long as it is necessary. Assume that $i^{(4)} = 0.16$

Find the amount of interest and the amount of principal repaid in the fourth payment.

⇒ Using the retrospective method (why??), we get that the outstanding loan balance at the beginning of the fourth quarter equals

$$1000(1.04)^3 - 100 \cdot s_3{\overline{1}} = 1124.86 - 312.16 = 812.70$$

The interest that is to be repaid in the fourth payment is exactly the amount of interest that is accrued during the fourth quarter-year on the balance above, i.e.,

$$0.04 \cdot 812.70 = 32.51$$

Evidently, the fourth payment is not yet the final, smaller one. So, the principal payment contained in the fourth payment is

$$100 - 32.51 = 67.49$$
An Example: Smaller final payment

- A $1,000 loan is being repaid by payments of $100 (plus the final smaller payment) at the end of each quarter-year for as long as it is necessary. Assume that $i^{(4)} = 0.16$. Find the amount of interest and the amount of principal repaid in the fourth payment.

⇒ Using the retrospective method (why?), we get that the outstanding loan balance at the beginning of the fourth quarter equals

$$1000(1.04)^3 - 100 \cdot s_3| = 1124.86 - 312.16 = 812.70$$

The interest that is to be repaid in the fourth payment is exactly the amount of interest that is accrued during the fourth quarter-year on the balance above, i.e.,

$$0.04 \cdot 812.70 = 32.51$$

Evidently, the fourth payment is not yet the final, smaller one. So, the principal payment contained in the fourth payment is

$$100 - 32.51 = 67.49$$
Loan Repayment Methods

1. Amortized Loans

2. The Sinking Fund Method
The Set-up

- We assume that the payments made prior to the end of the loan term do not contain any portion of the principal, i.e., they only go toward the interest.
- Hence, a single “lump-sum” payment should repay the entire loan at the end of the loan term.
- In order to finance this final payment, the borrower might wish to make deposits on a separate savings account during the life of the loan. This account is called the sinking fund account.
- This repayment method is referred to as the sinking fund method.
- Note that we need to differentiate between two accounts in this repayment schedule, i.e., there are two interest rates at play.
- We usually denote the interest rate governing the loan by $i$, and the interest rate of the sinking fund account by $j$.
- It is customary (but not necessary) that we assume that $j < i$. 


The Set-up

- We assume that the payments made prior to the end of the loan term do not contain any portion of the principal, i.e., they only go toward the interest.
- Hence, a single “lump-sum” payment should repay the entire loan at the end of the loan term.
- In order to finance this final payment, the borrower might wish to make deposits on a separate savings account during the life of the loan. This account is called the sinking fund account.
- This repayment method is referred to as the sinking fund method.
- Note that we need to differentiate between two accounts in this repayment schedule, i.e., there are two interest rates at play.
- We usually denote the interest rate governing the loan by $i$, and the interest rate of the sinking fund account by $j$.
- It is customary (but not necessary) that we assume that $j < i$. 
The Set-up

• We assume that the payments made prior to the end of the loan term do not contain any portion of the principal, i.e., they only go toward the interest.
• Hence, a single “lump-sum” payment should repay the entire loan at the end of the loan term.
• In order to finance this final payment, the borrower might wish to make deposits on a separate savings account during the life of the loan. This account is called the sinking fund account.
• This repayment method is referred to as the sinking fund method.
• Note that we need to differentiate between two accounts in this repayment schedule, i.e., there are two interest rates at play.
• We usually denote the interest rate governing the loan by $i$, and the interest rate of the sinking fund account by $j$.
• It is customary (but not necessary) that we assume that $j < i$. 
The Set-up

• We assume that the payments made prior to the end of the loan term do not contain any portion of the principal, i.e., they only go toward the interest.

• Hence, a single “lump-sum” payment should repay the entire loan at the end of the loan term.

• In order to finance this final payment, the borrower might wish to make deposits on a separate savings account during the life of the loan. This account is called the sinking fund account.

• This repayment method is referred to as the sinking fund method.

• Note that we need to differentiate between two accounts in this repayment schedule, i.e., there are two interest rates at play.

• We usually denote the interest rate governing the loan by $i$, and the interest rate of the sinking fund account by $j$.

• It is customary (but not necessary) that we assume that $j < i$. 
The Set-up

- We assume that the payments made prior to the end of the loan term do not contain any portion of the principal, i.e., they only go toward the interest.
- Hence, a single “lump-sum” payment should repay the entire loan at the end of the loan term.
- In order to finance this final payment, the borrower might wish to make deposits on a separate savings account during the life of the loan. This account is called the **sinking fund account**.
- This repayment method is referred to as the **sinking fund method**.
- Note that we need to differentiate between two accounts in this repayment schedule, i.e., there are two interest rates at play.
  - We **usually** denote the interest rate governing the loan by $i$, and the interest rate of the sinking fund account by $j$.
  - It is customary (but not necessary) that we assume that $j < i$. 
The Set-up

- We assume that the payments made prior to the end of the loan term do not contain any portion of the principal, i.e., they only go toward the interest.
- Hence, a single “lump-sum” payment should repay the entire loan at the end of the loan term.
- In order to finance this final payment, the borrower might wish to make deposits on a separate savings account during the life of the loan. This account is called the **sinking fund account**.
- This repayment method is referred to as the **sinking fund method**.
- Note that we need to differentiate between two accounts in this repayment schedule, i.e., there are **two interest rates** at play.
- We **usually** denote the interest rate governing the loan by $i$, and the interest rate of the sinking fund account by $j$.
- It is customary (but not necessary) that we assume that $j < i$. 

The Set-up

- We assume that the payments made prior to the end of the loan term do not contain any portion of the principal, i.e., they only go toward the interest.
- Hence, a single “lump-sum” payment should repay the entire loan at the end of the loan term.
- In order to finance this final payment, the borrower might wish to make deposits on a separate savings account during the life of the loan. This account is called the sinking fund account.
- This repayment method is referred to as the sinking fund method.
- Note that we need to differentiate between two accounts in this repayment schedule, i.e., there are two interest rates at play.
- We usually denote the interest rate governing the loan by $i$, and the interest rate of the sinking fund account by $j$.
- It is customary (but not necessary) that we assume that $j < i$. 
Some more notation

- Assume that the loan amount is denoted by $L$.
- Then, at the end of each period, one needs to pay the interest payment $L \cdot i$
  and the sinking fund deposit of $\frac{L}{s_{m|j}}$.
- So, the total payment at the end of each period is $L \cdot \left( i + \frac{1}{s_{m|j}} \right)$.
- We define $a_{m|ij} = \frac{1}{i + \frac{1}{s_{m|j}}} = \frac{a_{m|j}}{(i - j)a_{m|j} + 1}$.
- Note that if $i = j$, then we are back in the amortized loan setting!
Some more notation

- Assume that the loan amount is denoted by $L$.
- Then, at the end of each period, one needs to pay the interest payment $L \cdot i$
  and the sinking fund deposit of

$$
\frac{L}{s_{\overline{m}|j}}
$$

- So, the total payment at the end of each period is

$$
L \cdot \left( i + \frac{1}{s_{\overline{m}|j}} \right)
$$

- We define

$$
a_{\overline{m}|i\&j} = \frac{1}{i + \frac{1}{s_{\overline{m}|j}}} = \frac{a_{\overline{m}|j}}{(i - j)a_{\overline{m}|j} + 1}
$$

- Note that if $i = j$, then we are back in the amortized loan setting!
Some more notation

- Assume that the loan amount is denoted by \( L \).
- Then, at the end of each period, one needs to pay the interest payment \( L \cdot i \)
  and the sinking fund deposit of \( \frac{L}{s_{\bar{m}|j}} \)
- So, the total payment at the end of each period is \( L \cdot \left( i + \frac{1}{s_{\bar{m}|j}} \right) \)
- We define \( a_{\bar{m}|i\&j} = \frac{1}{i + \frac{1}{s_{\bar{m}|j}}} = \frac{a_{\bar{m}|j}}{(i - j)a_{\bar{m}|j} + 1} \)
- Note that if \( i = j \), then we are back in the amortized loan setting!
Some more notation

- Assume that the loan amount is denoted by $L$.
- Then, at the end of each period, one needs to pay the interest payment $L \cdot i$
  and the sinking fund deposit of

  \[ \frac{L}{s_{\overline{m}|j}} \]

- So, the total payment at the end of each period is

  \[ L \cdot \left( i + \frac{1}{s_{\overline{m}|j}} \right) \]

- We define

  \[ a_{\overline{m}|i\&j} = \frac{1}{i + \frac{1}{s_{\overline{m}|j}}} = \frac{a_{\overline{m}|j}}{(i - j)a_{\overline{m}|j} + 1} \]

- Note that if $i = j$, then we are back in the amortized loan setting!
Some more notation

• Assume that the loan amount is denoted by $L$.
• Then, at the end of each period, one needs to pay the interest payment $L \cdot i$
  and the sinking fund deposit of

$$\frac{L}{s_{\bar{m}|j}}$$

• So, the total payment at the end of each period is

$$L \cdot \left( i + \frac{1}{s_{\bar{m}|j}} \right)$$

• We define

$$a_{\bar{m}|i\&j} = \frac{1}{i + \frac{1}{s_{\bar{m}|j}}} = \frac{a_{\bar{m}|j}}{(i - j)a_{\bar{m}|j} + 1}$$

• Note that if $i = j$, then we are back in the amortized loan setting!