Brownian Motion and Ito's Lemma

1. The Sharpe Ratio

2. The Risk-Neutral Process

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The Sharpe Ratio

- Consider a portfolio of assets indexed by i.
  If asset i has expected return α_i, the risk premium is defined as

  $$\text{RiskPremium}_i = \alpha_i - r$$

  where r denotes the risk-free rate.
- The Sharpe ratio is defined as

  $$\text{SharpeRatio}_i = \frac{\text{RiskPremium}_i}{\sigma_i} = \frac{\alpha_i - r}{\sigma_i},$$

  where σ_i stands for the volatility of the asset i
- We can use the Sharpe ratio to compare two **perfectly correlated** claims, such as a derivative and its underlying asset
- Two assets that are perfectly correlated must have the **same** Sharpe ratio, or else there will be an arbitrage opportunity

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The Sharpe Ratio: Two stocks with the same source of uncertainty

- Consider two nondividend-paying stocks modeled as

$$dS_t = \alpha_1 S_t \, dt + \sigma_1 S \, dZ_t$$
$$d\tilde{S}_t = \alpha_2 \tilde{S}_t \, dt + \sigma_2 \tilde{S}_t \, dZ_t$$

  where Z is a standard Brownian motion
- The stock price processes S and $\tilde{S}$ are perfectly correlated since they have the same "driving" Brownian motion
- Let us suppose that they have different Sharpe ratios and demostrate that there is arbitrage opportunity in the market

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The Sharpe Ratio: Two stocks with the same source of uncertainty (cont'd)

- Without loss of generality assume that

$$\frac{\alpha_1 - r}{\sigma_1} > \frac{\alpha_2 - r}{\sigma_2}$$

- **Buy** $1/S\sigma_1$ shares of asset S
- **Short-sell** $1/\tilde{S}\sigma_2$ shares of asset $\tilde{S}$
- **Invest/borrow** the risk-free bond in the amount of the cost difference

$$\frac{1}{\sigma_1} - \frac{1}{\sigma_2}$$

- The return of the above strategy is

$$\frac{1}{\sigma_1 S}\, dS - \frac{1}{\sigma_2 \tilde{S}}\, dS_2 + \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) dt = \left(\frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2}\right) dt > 0$$

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The True Price Process

- The model

$$dS_t = S_t[(\alpha - \delta)\,dt + \sigma\,dZ_t]$$

where δ denotes the dividend yield on S

- The drift contains the "average appreciation" of the stock
- The "uncertainty" is driven by the stochastic process Z
- To facilitate calculations (recall the binomial model!) we look at the process S under a new probability measure which renders the price process to be a martingale.
- This is different from the "physical" measure - whatever that may be ...

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The Risk-Neutral Measure

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