Goodness-of-Fit

Looking at outcomes of a MULTINOMIAL EXPERIMENT, possibly w/ categorical descriptions.

Say, the possible outcomes of this experiment are described as mutually exclusive & exhaustive events

$$A_1, A_2, \dots, A_k$$

Let, in our probabilistic model,

$$\mathbb{P}[A_i] = p_i \quad , \quad i = 1, \dots, k$$

Note: $\boxed{p_1 + \dots + p_k = 1}$

Repeat the same multinomial experiment n times.

Let X_i ... the # of times A_i occurred, $i = 1..k$

$(X_1, \dots, X_k)$ .... multinomial dist'n.

Note: $\boxed{X_1 + \dots + X_k = n}$

Define:

$$Q^2 = \sum_{i=1}^{k} \frac{(X_i - n \cdot p_i)^2}{np_i} \text{ "}\sim\text{" } \chi^2(df = k-1)$$

Works for $np_i \geq 5$ for all $i = 1..k$

Test summary:

Testing $H_0: p_1 = p_{10},\ p_2 = p_{20}, \dots, p_k = p_{k0}$

vs.

H_a: At least one of the true probabilities is different from the null value

This χ^2-test is always the one-sided upper-tail one!.

Look at:
- O_i ... # observed counts
- $E_i = n\,p_{i0}$, i.e., the expected counts under the null hypothesis

$\Rightarrow$ The TEST STATISTIC is:

$$Q^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \text{ "}\sim\text{" } \chi^2(df = k-1)$$

With the significance level α:
Look @ the critical value of the test statistic
$(\chi^2_\alpha)^*(df = k-1)$

IF the observed value of Q^2 exceeds the critical value, then reject the null!

EXAMPLE 7.2.1

A plant geneticist grows 200 progeny from a cross that is hypothesized to result in a 3:1 phenotypic ratio of red-flowered to white-flowered plants. Suppose the cross produces 170 red- to 30 white-flowered plants. (a) Calculate Q^2 for this experiment. (b) Does the given data support the 3:1 ratio at $\alpha = 0.05$?

$n = 200$

$k = 2$ ... the number of categories

$H_0: p_1 = \frac{3}{4}, p_2 = \frac{1}{4}$

vs.

H_a: the color dist'n is different from the null.

Observed counts: $O_1 = 170$, $O_2 = 30$

Expected counts: $E_1 = 150$, $E_2 = 50$

(both $\geq 5 \Rightarrow$ we have the approximate χ^2 dist'n).

More precisely: $\chi^2(df = 1)$

the observed value of the TS is:

$$Q^2 = \frac{(170-150)^2}{150} + \frac{(30-50)^2}{50} = \underline{\underline{10.67}}$$

The critical value of $\chi^2(df=1)$ @ sign. level of 0.05:

3.84

$\Rightarrow$ Reject the null hypothesis!

EXAMPLE 7.2.3

A die is rolled 60 times and the face values are recorded. The results are as follows.

Up face	1	2	3	4	5	6
Frequency	8	11	5	12	15	9

Is the die balanced fair? Test this question using $\alpha = 0.05$.

$H_0: p_1 = \dots = p_6 = \frac{1}{6}$

vs

H_a: @ least one $p_i \neq \frac{1}{6}$

TS = ? O_i are in the table &

$E_i = 60 \cdot \frac{1}{6} = 10$ for every $i = 1..6$

The observed value of the test statistic is:

$$\frac{(8-10)^2}{10} + \frac{(11-10)^2}{10} + \frac{(5-10)^2}{10} + \frac{(12-10)^2}{10} + \frac{(15-10)^2}{10} + \frac{(9-10)^2}{10} =$$

$$= \frac{1}{10}(4+1+25+4+25+1) = 6$$

the critical value of $\chi^2(df=5)$ @ $\alpha = 0.05$ is: 11.07

$\Rightarrow$ Fail to Reject !

EXAMPLE 7.2.2

A TV station broadcasts a series of programs on the ill effects of smoking marijuana. After the series, the station wants to know whether people have changed their opinion about legalizing marijuana. Given in the following tables are the data based on a survey of 500 randomly chosen individuals:

Before the series was shown

For Legalization	Decriminalization	Existing Law (Fine or Imprisonment)	No Opinion
7%	18%	65%	10%

After the series was shown

For Legalization	Decriminalization	Existing Law (Fine or Imprisonment)	No Opinion
39%	9%	36%	16%

Here, $k=4$, and we wish to test the following hypothesis:

$$H_0: p_1 = 0.07;\ p_2 = 0.18;\ p_3 = 0.65;\ p_4 = 0.1$$

versus

$$H_a: \text{At least one of the probabilities is different from the hypothesized value}$$

The test is always an upper tail test. Test this hypothesis using $\alpha = 0.01$.

$$\begin{cases} O_1 = (500)(0.39) = 195,\ O_2 = (500)(0.09) = 45, \\ O_3 = (500)(0.36) = 180,\ O_4 = (500)(0.16) = 80. \end{cases}$$

$E_1 = (500)(0.07) = 35$, $E_2 = (500)\cdot(0.18) = 90$,
$E_3 = (500)(0.65) = 325$, $E_4 = (500)(0.10) = 50$

$$Q^2 = \frac{(195-35)^2}{35} + \frac{(45-90)^2}{90} + \frac{(180-325)^2}{325} + \frac{(80-50)^2}{50}$$

$$= \cdots = 836.62$$

Critical value for $\chi^2(df=3)$ @ $\alpha = 0.01$:

11.34

$\Rightarrow$ Reject the null!