University of Texas at Austin

Quiz # 5

The binomial distribution.

Problem 5.1. (5 pts) Let X denote the number of $1's$ in 100 throws of a fair die. Find $\mathbb{E}[X^2]$.

(a) 125/9
(b) 50/3
(c) 875/3
(d) 1585/9
(e) None of the above

Solution: (c)
Evidently, $X \sim b(100, 1/2)$. So,

$$\mathbb{E}[X^2] = Var[X] + (\mathbb{E}[X])^2 = 100 \cdot \frac{1}{6} \cdot \frac{5}{6} + (100 \cdot \frac{1}{6})^2 = \frac{500 + 10000}{36} = \frac{875}{3}.$$

Problem 5.2. (10 points) *Source: Problem 2.1.5. from Pitman.*
Given that there are 12 heads in 20 independent tosses, find the probability that at least two of the first five tosses landed heads.
Note: It is OK to leave binomial coefficients in your answer; simplify the remainder of your expression as much as you can.

Solution: The number of heads in 20 has the binomial distribution $b(20, 1/2)$. Let

$$E = \{\text{there are 12 heads}\}, \quad F = \{\text{at least two of the first five tosses landed heads}\}.$$

Immediately $\mathbb{P}[E] = \binom{20}{12} \cdot \frac{1}{2^{20}}$. We need to find

$$\mathbb{P}[F|E] = \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[E]} = \frac{\mathbb{P}[E] - \mathbb{P}[E \cap F^c]}{\mathbb{P}[E]} = 1 - \frac{\mathbb{P}[E \cap F^c]}{\mathbb{P}[E]}.$$

Let

$$H_0 = \{\text{there are no heads in the first five tosses}\},$$
$$H_1 = \{\text{there is exactly one head in the first five tosses}\}.$$

Then, $F^c = H_0 \cup H_1$, and H_0 and H_1 are mutually exclusive. Also, with

$$G_0 = \{\text{there are exactly 12 heads in last 15 tosses}\},$$
$$G_1 = \{\text{there are exactly 11 heads in last 15 tosses}\},$$

we have, due to independence of trials,

$$\mathbb{P}[E \cap H_0] = \mathbb{P}[H_0 \cap G_0] = \mathbb{P}[H_0]\mathbb{P}[G_0],$$
$$\mathbb{P}[E \cap H_1] = \mathbb{P}[H_1 \cap G_1] = \mathbb{P}[H_1]\mathbb{P}[G_1].$$

Moreover, the number of heads in the first five trials is $b(5, 1/2)$, and the number of heads in the remaining 15 trials is $b(15, 1/2)$. So,

$$\mathbb{P}[E \cap H_0] = \binom{5}{0}\binom{15}{12} \cdot \frac{1}{2^{20}},$$
$$\mathbb{P}[E \cap H_1] = \binom{5}{1}\binom{15}{11} \cdot \frac{1}{2^{20}}.$$

and

$$\mathbb{P}[F|E] = 1 - \frac{\binom{15}{12}}{\binom{20}{12}} - 5 \cdot \frac{\binom{15}{11}}{\binom{20}{12}}.$$

Note that the fact that the coin is (by default) fair is completely irrelevant in the above calculation.