UNIVERSITY OF TEXAS AT AUSTIN

Quiz # 6

Binomial distribution. The Central Limit Theorem.

Provide your **complete solution** for the following problems.

Problem 6.1. (9 points) *Source: "Probability" by Jim Pitman.*

Let a and b be constants such that $a < b$. A random variable is said to be uniformly distributed over the interval $[a, b]$ if its probability density function is given by

$$f(x) = \frac{1}{b-a} \quad \text{for } x \in [a, b].$$

Consider the sample average $\bar{X}_n$ of n independent random variables, each of them uniformly distributed on $[0, 1]$. Find the smallest sample size n such that $\mathbb{P}[\bar{X}_n < 0.51]$ is at least 90%. Use the Central Limit Theorem.

Solution: From the CLT we know that approximately

$$\bar{X}_n \sim N(mean = 1/2, var = 1/(12n)), \quad \text{i.e.,} \quad \frac{\bar{X}_n - 0.5}{\frac{1}{\sqrt{12n}}} \sim N(mean = 0, var = 1).$$

So, with $Z \sim N(0, 1)$, the probability we are looking for is

$$\mathbb{P}[\bar{X}_n < 0.51] \approx \mathbb{P}\left[Z < \frac{0.51 - 0.5}{\frac{1}{2\sqrt{3n}}}\right] = \Phi(0.02\sqrt{3n})$$

Consulting the standard normal tables, we get that the sufficient condition for the above probability to be at least 0.90 is

$$0.02\sqrt{3n} \geq 1.29 \quad \Rightarrow \quad 3n \geq 4160.25.$$

Our final answer is $n \geq 1387$.

Problem 6.2. *Source: "Probability" by Pitman.*

Suppose that on average 1 in 100 people gave a certain genetic marker.

(i) (3 points) Suppose a simple random sample of 50 is selected and tested. What is the probability that **at least** one of them will have the genetic marker?
Solution: The number of people in the random sample who have the genetic marker Y has the binomial distribution with the number of trials equal to 50 and with probability of success in every trial equal to 1/100.

$$\mathbb{P}[Y \geq 1] = 1 - \mathbb{P}[Y = 0] = 1 - (99/100)^{50} = 0.395.$$

(ii) (3 points) About how many people would have to be tested so that the probability of having at least one positive result would be at least 99%? If this number of people were tested, what is the expected number of positives?
Solution:

$$0.99 \geq 1 - 0.99^n \quad \Leftrightarrow \quad 0.99^n \geq 0.01 \quad \Leftrightarrow \quad n \geq \frac{\ln(0.01)}{\ln(0.99)} = 458.11$$

Hence, $n \geq 459$. The expected number of positive results is $459 \times 0.01 = 4.59$.