More on $\mathbb{C}$ manifolds

Ex 1) $\mathbb{C}^n$ is a complex mfd of dim n. 1 chart $U = \mathbb{C}^n$, $\varphi: U \to \mathbb{C}^n$ is $\mathbb{1}$.

2) $\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{(0,\dots,0)\}\right)/\sim$ [with quotient topology] is complex manifold.

[for each $\lambda \in \mathbb{C}^\times$, $\vec{x} \sim \lambda \vec{x}$ i.e. $\{(x_0,\dots,x_n)\} \sim \{(\lambda x_0,\dots,\lambda x_n)\}$]

$n+1$ charts: $U_i = \left(\{x_i \neq 0\} \subset \mathbb{C}^{n+1} \setminus 0\right)/\sim$

$$\varphi_i : U_i \to \mathbb{C}^n$$
$$(x_0,\dots,x_n) \mapsto \left(\frac{x_0}{x_i},\dots,\widehat{\frac{x_i}{x_i}},\dots,\frac{x_n}{x_i}\right)$$

[Check: $\varphi_i \circ \varphi_j^{-1}$ holomorphic]

$\mathbb{CP}^n$ is a $\mathbb{C}$ mfd of dim n.

[Rk: $\mathbb{CP}^n$ is <u>compact</u>. $\left(\simeq S^{2n+1}/U(1)\right)$
$\mathbb{CP}^1 \simeq S^2$]

3) Torus: for any τ w/ $\mathrm{Im}\,\tau > 0$, $X_\tau = \mathbb{C}/\{a + b\tau : a, b \in \mathbb{Z}\}$

$\pi : \mathbb{C} \to X_\tau$

Cover $\mathbb{C}$ by small open sets $\{U_\alpha\}_{\alpha \in I}$; $\left\{\left(\pi(U_\alpha), \pi^{-1}\right)\right\}_{\alpha \in I}$ is hol. atlas on X_τ

3') For any rank-$2n$ lattice Λ in $\mathbb{C}^n$, take $X_\Lambda = \mathbb{C}^n/\Lambda$.
X_Λ is a $\mathbb{C}$ mfd of dim n.

4) Affine hypersurfaces: $f: \mathbb{C}^n \to \mathbb{C}$ s.t. 0 is regular value
(i.e. $df \neq 0$ at every point of $f^{-1}(0)$)

$X = f^{-1}(0)$ is a $\mathbb{C}$ mfd of dimension $n-1$
(produce coordinate charts using implicit f^n theorem)
[But e.g. $\{x_1^2 - x_2^3 = 0\}$ is <u>not</u> a $\mathbb{C}$ mfd in a natural way!]

5) Projective hypersurfaces: $f: \mathbb{C}^{n+1} \setminus \{\vec{0}\} \to \mathbb{C}$ homogeneous polynomial deg k s.t. 0 is regular value,

$$f(\lambda \vec{x}) = \lambda^k f(\vec{x})$$

$$X = \left(f^{-1}(0)\right) / \mathbb{C}^\times \subset \mathbb{CP}^n$$

X is a complex manifold of dimension $n-1$

(again, use implicit f^n thm, in each patch of $\mathbb{CP}^n$)

NB, X is compact.

6) Complete intersections: Given $f: \mathbb{C}^n \to \mathbb{C}^k$ s.t. 0 is a regular value ($df(x): T\mathbb{C}^n \to T\mathbb{C}^k$ surjective $\forall x \in f^{-1}(0)$), $f^{-1}(0)$ is a complex manifold of dim $= n-k$.

7) Any open subset of a complex manifold is a complex manifold.

8) $Gr(k,n) = \{k\text{-dimensional subspaces of } \mathbb{C}^n\}$ is a complex manifold. [Ex]

9) If X, Y are complex manifolds then so is $X \times Y$.

Holomorphic objects on $\mathbb{C}$ manifolds

On a C^∞ mfd can define C^∞ functions, C^∞ vector bundle, ...

On a $\mathbb{C}$ mfd can define holomorphic objects.

Def A holomorphic function $f: X \to \mathbb{C}$ is a function s.t. $f \circ \varphi_\alpha^{-1}$ is holomorphic $\forall$ charts $(U_\alpha, \varphi_\alpha)$.

Prop If X is compact connected, any holomorphic function $f: X \to \mathbb{C}$ is constant.

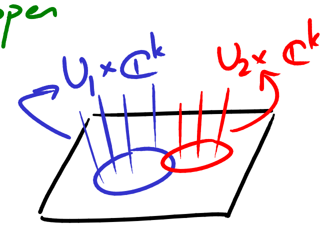
Pf X compact $\Rightarrow |f|$ attains its maximum, at some $f(x_0) = c$. $f^{-1}(c)$ is closed, nonempty.
If $f(x) = c$ for $x \in U_\alpha$ then $f \circ \varphi_\alpha^{-1}$ is constant on nbhd of x (maximum principle).
So $f^{-1}(c)$ is open in X, $f^{-1}(c) \neq \phi$, and X connected $\Rightarrow f^{-1}(c) = X$. ∎

Def A holomorphic map $f: X \to Y$ is a map $f: X \to Y$ s.t. $\forall$ charts $(U_\alpha, \varphi_\alpha)$ on X, $(U'_\beta, \varphi'_\beta)$ on Y, $\varphi'_\beta \circ f \circ \varphi_\alpha^{-1}$ is holomorphic

Say $X \simeq Y$ if $\exists$ holomorphic homeomorphism $X \xrightarrow{f} Y$.
(Then f^{-1} is holomorphic; call f biholomorphic)

[Exercise: if $X_\tau = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, $X_\tau \simeq X_{\tau+1}$, $X_\tau \simeq X_{-1/\tau}$]

Def A rank-r holomorphic vector bundle over X is a complex manifold E equipped with a holomorphic projection $\pi: E \to X$, s.t. each fiber is an r-dimensional complex <u>vector space</u>, and $\exists$ an open covering $\{U_\alpha\}$ of X w/ bihol. maps $\psi_\alpha: \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^r$ which are <u>linear</u> on each $\pi^{-1}(x)$. ("local trivializations")



NB: hol v.b. is <u>not</u> the same as a <u>complex</u> v.b. (For the latter, the $\psi_{\alpha\beta}$ would be just <u>C^∞</u> maps, not <u>holomorphic</u>.)

Def E hol v.b. A <u>holomorphic section</u> of E is a hol map $s: X \to E$ with $\pi_E \circ s = \mathbb{1}_X$. Let $\Gamma(U, E)$ denote the complex vector space of sections of $E|_U$.

Rk Define <u>transition functions</u>: hol maps

$$\Phi_{\alpha\beta}: U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$$

s.t. $\psi_\alpha \circ \psi_\beta^{-1}(x, v) = (x, \Phi_{\alpha\beta}(x) \cdot v)$

$$\begin{tikzcd}
 & (U_\alpha \cap U_\beta) \times \mathbb{C}^r \\
U_\alpha \cap U_\beta \arrow[ur, "\psi_\alpha"] \arrow[dr, "\psi_\beta"'] & \\
 & (U_\alpha \cap U_\beta) \times \mathbb{C}^r \arrow[uu, "{(\mathbb{1}, \Phi_{\alpha\beta})}"']
\end{tikzcd}$$

Can also <u>construct</u> E by gluing patches using transition functions.

Constructions of hol. vector bundles:

- Trivial bundle $E = X \times \mathbb{C}^r$ w/ obvious $\pi: \mathcal{O} \to X$. $\Gamma(X, \mathcal{O}) = \mathbb{C}^r$.

- If $X = \mathbb{CP}^n$, view X as space of lines in $\mathbb{C}^{n+1}$, $\mathcal{O}(-1) = \{(\ell, \vec{y}) \subset \mathbb{CP}^n \times \mathbb{C}^{n+1} : \vec{y} \in \ell\}$

  It's a $\mathbb{C}$ mfd; local charts $\tilde{U}_i = \{x_i \neq 0\}$ with $\varphi_i(\ell, \vec{y}) = \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}, y_i\right)$

  w/ obvious hol projection $\pi: \mathcal{O}(-1) \to \mathbb{CP}^n$

  and linear structure on the fibers $\pi^{-1}(\ell) = \{(\ell, \vec{x}) : \vec{x} \in \ell\}$

  Local trivializations $\psi_i: \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ $\quad \psi_i(\ell, \vec{y}) = (\ell, y_i)$

  $\Gamma(X, \mathcal{O}(-1)) = \{0\}$. (Because $s \in \Gamma(X, \mathcal{O}(-1))$ induces a nonconst hol map $\mathbb{CP}^n \to \mathbb{C}^{n+1}$)

Rk 1) holomorphic map $p: \mathcal{O}(-1) \to \mathbb{C}^{n+1}$, $(\ell, \vec{y}) \mapsto \vec{y}$ is onto, $p^{-1}(\vec{y}) = \begin{cases} 1 \text{ point if } \vec{y} \neq \vec{0} \\ \mathbb{CP}^n \text{ if } \vec{y} = \vec{0} \end{cases}$ (blow-up)

2) If $n=1$, just 2 patches: U_1 $(z = \frac{x_0}{x_1}, y_1)$, U_0 $(w = \frac{x_1}{x_0}, y_0)$ and $y_0 = \frac{x_0}{x_1} y_1 = z y_0$. Thus, a section of $\mathcal{O}(-1)$ means hol. function $y_0(z)$ on U_1 and $y_1(w)$ on U_0, with $y_1(w = \frac{1}{z}) = z y_0(z)$. But $z y_0(z)$ blows up as $z \to \infty$! So $\Gamma(X, \mathcal{O}(-1)) = \{0\}$.

- Any holomorphic functor on the category of vector spaces gives a functor on the category of hol vector bundles: e.g. from E, F make E^*, $E \oplus F$, $E \otimes F$, $\Lambda^i(E)$, ... [see Exercise]

- Over $\mathbb{CP}^n$: define $\mathcal{O}(1) = \mathcal{O}(-1)^*$.

  $\Gamma(X, \mathcal{O}(1)) \simeq (\mathbb{C}^{n+1})^*$ (Any elt of $(\mathbb{C}^{n+1})^*$ induces a section of $\mathcal{O}(1)$ by restriction; will prove later that this is all the sections)

- Also define $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$ $(m > 0)$, $\mathcal{O}(-m) = \mathcal{O}(-1)^{\otimes m}$ $(m > 0)$; then $\mathcal{O}(p) \otimes \mathcal{O}(q) = \mathcal{O}(p+q)$. [needs Ex: $\mathcal{O}(-1) \otimes \mathcal{O}(1) = \mathcal{O}$]

- For a hol. map $f: X \to Y$ and a hol. vector bundle $E \to Y$, get a hol. vector bundle $f^*E \to X$

- Holomorphic tangent bundle TX: $r = \dim X$, patches the U_α of a hol. atlas, transition functions $\Phi_{\alpha\beta}: U_{\alpha\beta} \to GL(n, \mathbb{C})$ given by the Jacobian of $\varphi_{\alpha\beta}$, ie $(\Psi_{\alpha\beta})_{ij} = \frac{\partial(\varphi_\alpha)_i}{\partial(\varphi_\beta)_j}$

  (We'll give a more "intrinsic" interpretation shortly.)

Def A hol vector bundle <u>homomorphism</u> is a holomorphic map $\varphi : E \to F$ with $\pi_E = \pi_F \circ \varphi$, linear on each fiber.
φ is an <u>isomorphism</u> if each $\varphi(x) : E_x \to F_x$ is an isomorphism.

Rk A hol v.b. hom $\varphi : E \to F$ is equivalent to a hol section of $F \otimes E^* = \mathrm{Hom}(E, F)$.