Hodge theory refresher

The p-form Laplacian

M oriented Riem manifold:

$$d: \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$$

uses two $\star$, so $\exists$ even for M not oriented!

Def 1) $d^*: \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$ given by $d^*\omega = (-1)^{n(p+1)+1} \star d \star \omega$

2) $\Delta: \Omega^p(M) \longrightarrow \Omega^p(M)$ given by $\Delta = dd^* + d^*d$

Prop If M compact, for L^2 pairing $\langle \alpha, d^*\beta \rangle_{L^2} = \langle d\alpha, \beta \rangle_{L^2}$

Pf $\langle \alpha, \beta \rangle_{L^2} = \int \langle \alpha, \beta \rangle \, \mathrm{vol} = \int \alpha \wedge \star\beta$

so $\langle d\alpha, \beta \rangle_{L^2} = \int d\alpha \wedge \star\beta$

$$= (-1)^{1+|\alpha|} \int \alpha \wedge d\star\beta$$

$$= (-1)^{1+|\alpha|+|\alpha|(n-|\alpha|)} \int \alpha \wedge \star(\star d \star \beta)$$

$$= \langle \alpha, d^*\beta \rangle_{L^2}$$

[since $1+|\alpha|+|\alpha|(n-|\alpha|) = n(|\beta|+1)+1 \mod 2$
using $|\alpha|+1 = |\beta|$]

Thus we call d^* a "formal adjoint" to d.

Cor If M compact, $\langle \alpha, \Delta\beta \rangle_{L^2} = \langle d\alpha, d\beta \rangle_{L^2} + \langle d^*\alpha, d^*\beta \rangle_{L^2} = \langle \Delta\alpha, \beta \rangle_{L^2}$.

Cor If M compact, $\Delta\alpha = \lambda\alpha$, then $\lambda \geq 0$; if $\lambda = 0$ then $d\alpha = 0$, $d^*\alpha = 0$.

Pf $\lambda \|\alpha\|^2_{L^2} = \langle \alpha, \Delta\alpha \rangle_{L^2} = \|d\alpha\|^2_{L^2} + \|d^*\alpha\|^2_{L^2}$

Rk This really needs M compact — e.g. if $M = \mathbb{R}$ and $f(x) = e^x$, $\Delta f = -f$.

Def $\mathcal{H}^p(M) = \ker(\Delta: \Omega^p(M) \rightarrow \Omega^p(M))$

Cor $\dim \mathcal{H}^0(M) = \#$ connected components of $M = b^0(M)$.

This fact has an important refinement:

Def (de Rham cohomology) M smooth manifold: $H^p_{dR}(M) = \dfrac{\ker\left(d: \Omega^p(M) \to \Omega^{p+1}(M)\right)}{\operatorname{im}\left(d: \Omega^{p-1}(M) \to \Omega^p(M)\right)}$

$$b^p(M) = \dim_{\mathbb{R}} H^p(M)$$

So this is another way of thinking about the "usual" cohomology of M.

Thm (Hodge) If M compact Riemannian,

Then each class in $H^p_{dR}(M)$ contains a unique element of $\mathcal{H}^p(M)$

Rk Note $H^p_{dR}(M)$ is defined without a metric, while $\mathcal{H}^p(M)$ depends on one a priori.

Pf Sketch If $\omega \in \mathcal{H}^p(M)$ then $d\omega = 0$, so have a map $\mathcal{H}^p(M) \longrightarrow H^p_{dR}(M)$.

- Injective: suppose $\omega \in \mathcal{H}^p(M)$, $\omega = d\alpha$; then $\|\omega\|^2 = \langle \omega, d\alpha \rangle = \langle d^*\omega, \alpha \rangle = 0$.
- Surjective: first note $\operatorname{Im} d$, $\operatorname{Im} d^*$, and $\mathcal{H}^p$ are all mutually orthogonal.

  Suppose we knew $\Omega^p = d\Omega^{p-1} \oplus d^*\Omega^{p+1} \oplus \mathcal{H}^p$. (see below)

  Then, given γ with $d\gamma = 0$, $\gamma = d\alpha + d^*\beta + \delta \qquad \delta \in \mathcal{H}^p$

  $$d\gamma = dd^*\beta = 0$$

  but then $\langle \beta, dd^*\beta \rangle_{L^2} = 0$ so $\|d^*\beta\|^2 = 0$, i.e. $d^*\beta = 0$.

  So, $\gamma = d\alpha + \delta$.

  But then $[\gamma] = [\delta]$ in H^p.

So, what we need is to prove

Lemma $\Omega^p = d\Omega^{p-1} \oplus d^*\Omega^{p+1} \oplus \mathcal{H}^p$.

Pf Sketch It would be enough to show $\Omega^p = \Delta\Omega^p \oplus \mathcal{H}^p$.

(since $\mathrm{Im}\,\Delta \subset \mathrm{Im}\,d \oplus \mathrm{Im}\,d^*$)

Note this would be easy in finite-dimensional setting: just diagonalize Δ to see $\exists\, G: \Omega^p \to \Omega^p$ s.t. for $\omega \in (\mathcal{H}^p)^\perp$, $(\Delta \circ G)\omega = \omega$.

(So in p^{tic} $\omega \in \mathrm{Im}\,\Delta$.)

In infinite-dimensional setting, need to develop theory of "ellipticity" to show that G exists. (This theory also shows that $\mathcal{H}^p$ is finite-dimensional.)

(Very rough idea: on $(S^1)^n$, write $\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$, then $\widetilde{\Delta f}(k) = \|k\|^2 \tilde{f}(k)$ $k \in \mathbb{Z}^n$

thus $\widetilde{Gf} = \frac{\tilde{f}}{\|k\|^2}$. No problem, as long as $\tilde{f}(0) = 0$. A version of this idea really works.

It depends crucially on $\|k\|^2 \neq 0$ when $k \neq 0$. This is ellipticity of Δ.)

Rk Warning: α, β harmonic $\not\Rightarrow$ $\alpha \wedge \beta$ harmonic!

So $\wedge$ does not reproduce the "cup product".