Hopf surface

Fix $\lambda \in (0,1)$. Recall we can build torus as $\mathbb{C}^\times/\mathbb{Z}$ with $\mathbb{Z}$ action $z \mapsto \lambda z$.

Define Hopf surface $X = \dfrac{(\mathbb{C}^2 \setminus \{0\})}{\mathbb{Z}}$ w/ $\mathbb{Z}$ action $(z_1, z_2) \mapsto (\lambda^k z_1, \lambda^k z_2)$

Evidently X is complex manifold (since $\mathbb{Z}$ acts holomorphically, and prop. discontinuous)

In fact, $X \simeq S^1 \times S^3$.

Thus $H^k_{dR}(X) = \bigoplus_{a+b=k} H^a_{dR}(S^1) \otimes H^b_{dR}(S^3)$ (Künneth)

[use a product metric, $\Delta_X = \Delta_{S^1} + \Delta_{S^3}$]

And S^1 has $\begin{matrix} b_1 = 1 \\ b_0 = 1 \end{matrix}$ (checked this before) S^3 has $\begin{matrix} b_3 = 1 \\ b_2 = 0 \\ b_1 = 0 \\ b_0 = 1 \end{matrix}$ (exercise, or wait a few weeks)

So X has Betti #s $\begin{matrix} b_4 = 1 \\ b_3 = 1 \\ b_2 = 0 \\ b_1 = 1 \\ b_0 = 1 \end{matrix}$ In particular, X isn't Kähler, although it's compact complex!