Connections in complex vector bundles

Def (E,h) Hermitian v.b. over X.
A conn. ∇ in E, $\nabla : \Omega^0(E) \to \Omega^1(E)$,
is called <u>Hermitian</u> if $\forall s_1, s_2 \in \Omega^0(E)$ we have

$$d(h(s_1,s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$$

[where we defined h on $\Omega^1 \otimes \Omega^0$ by $h(\alpha \otimes s_1, s_2) = \alpha\, h(s_1,s_2)$
on $\Omega^0 \otimes \Omega^1$ by $h(s_1, \alpha \otimes s_2) = \bar{\alpha}\, h(s_1,s_2)$]

Rk If ∇ is Hermitian then parallel xport along a path preserves h.

Def (E,h) Hermitian v.b. over X. Define a v.b. $\mathrm{End}(E,h)$

$$\mathrm{End}(E,h)_x = \{ a \in \mathrm{End}\, E_x \mid h(a(s_1), s_2) + h(s_1, a(s_2)) = 0 \}$$

Rk In a local unitary trivialization (one for which h is represented by the identity matrix), $a \in \mathrm{End}\, E$ is rep. by a $\mathfrak{gl}(n)$-valued f^n ($n \times n$ matrix)
$a \in \mathrm{End}(E,h)$ " " " " $\mathfrak{u}(n)$-valued f^n ($n \times n$ <u>skew-adjoint matrix</u>)

Rk $\mathrm{End}(E,h)$ is a <u>real</u> vector bundle, not a complex one.

Prop {Hermitian connections on (E,h)} is an affine space for $\Omega^1(X, \mathrm{End}(E,h))$.

Pf Exercise

Recall that hol. structure on $E \longleftrightarrow$ operator $\bar\partial : \Omega^0(E) \to \Omega^{0,1}(E)$ whose ext^n to $\Omega^\bullet(E)$ has $\bar\partial^2 = 0$

Def A connection ∇ on hol. v.b. E is compatible w/ hol. structure if $\nabla^{0,1} = \bar\partial$.

Prop $\{\nabla$ on E compat w/ hol str$\}$ is affine space over $\Omega^{1,0}(X, \operatorname{End} E)$.

Pf Exercise.

Def/Prop (E,h) hol v.b. w/ Hermitian metric.
$\exists$ unique Hermitian conn. ∇ compatible w/ the holomorphic structure.
("Chern connection")

Pf Local hol triv: $\nabla = d + A \qquad A \in \Omega^{1,0}(\mathrm{Mat}_{r\times r})$

$$A = (a_{ij})$$

pos def. Hermitian matrix H, $\qquad h(v,w) = v^t H \bar w$

$$dh(e_i, e_j) = h(\nabla e_i, e_j) + h(e_i, \nabla e_j)$$
$$= h\Big(\sum_k a_{ik} e_k, e_j\Big) + h\Big(e_i, \sum_\ell a_{j\ell} e_\ell\Big)$$

ie $\qquad dH = A^t \cdot H + H \cdot \bar A$

take $(0,1)$ parts $\Rightarrow \bar\partial H = H \cdot \bar A \qquad$ i.e. $A = \bar H^{-1} \partial \bar H$

This works locally, and since it's unique, it must glue correctly on overlaps.
(Or, just check explicitly that it glues) $\blacksquare$

Ex E hol Herm line bundle: then h is just a $\mathbb{R}_+$-valued function (locally defined),

$$\nabla = d + \partial h / h = d + \partial \log h \quad \text{(locally)}$$

$$F = d(\partial \log h) = \bar\partial \partial \log h$$

(cf. our discussion of Kähler forms)

Ex X a Hermitian manifold, $E = T_{hol}X$: ∇ is some connection in the complex bundle $T_{hol}X$, in partic a connection in the underlying real bundle TX.
∇ coincides with the Levi-Civita connection $\iff$ X is Kähler.
(Appendix 4.A in Huybrechts)

One should carefully distinguish these notions from the next:

Def E hol v.b.
A hol connection on E is a $\mathbb{C}$-linear map of sheaves

$$D : E \to \Omega^1 \otimes E$$

with

$$D(fs) = \partial(f) \otimes s + f D(s)$$

$\forall f \in \mathcal{O}(U),\ s \in E(U)$

Rk The $(1,0)$ part of a gen'l connection does not give a hol. connection!

Indeed there is an obstruction to existence of hol connections:

Def E hol v.b. w/ local trivializations $\psi_i : E \xrightarrow{\sim} \mathbb{C}^r$ over U_i
Atiyah class $A(E) \in H^1(X, \Omega^1 \otimes \mathrm{End}\, E)$ is given by
Čech cocycle $\quad \psi_j^{-1} \circ (\psi_{ij}^{-1} d\psi_{ij}) \circ \psi_j$.

(To see it's really cocycle:

$$\psi_j^{-1} \circ (\psi_{ij}^{-1} d\psi_{ij}) \circ \psi_j + \psi_k^{-1} \circ (\psi_{jk}^{-1} d\psi_{jk}) \circ \psi_k + \psi_i^{-1} \circ (\psi_{ki}^{-1} d\psi_{ki}) \circ \psi_i$$

$$\psi_{ij} = (\psi_{jk}\psi_{ki})^{-1} \Rightarrow d\psi_{ij} = \dots)$$

Rk: $A(E)$ is well defined — doesn't depend on the choice of the ψ_i (Exercise)

Prop E admits hol. connection $\iff A(E)=0$.

Pf $(\Longrightarrow)$ $\psi_i^{-1}\circ(\partial+A_i)\circ\psi_i = \psi_j^{-1}\circ(\partial+A_j)\circ\psi_j$

i.e. $\psi_i^{-1}\circ\partial\circ\psi_i - \psi_j^{-1}\circ\partial\circ\psi_j = \psi_j^{-1}A_j\psi_j - \psi_i^{-1}A_i\psi_i$

$$\underbrace{\psi_i^{-1}\circ\partial\circ\psi_i - \psi_j^{-1}\circ\partial\circ\psi_j}_{=\ \psi_j^{-1}\circ(\psi_{ij}^{-1}\partial(\psi_{ij}))\circ\psi_j}$$

But $\psi_j^{-1}\circ A_j\circ\psi_j - \psi_i^{-1}\circ A_i\circ\psi_i$ is coboundary of the cochain $B_i = \psi_i^{-1}\circ A_i\circ\psi_i \in C^0(X,\Omega^1\otimes \mathrm{End}\,E)$

$(\Longleftarrow)$ read the above from bottom to top. ∎

As we'll see, $A(E)$ is a kind of "universal characteristic class" for E.