Now consider $X = \mathbb{CP}^n$.

Euler sequence $\quad 0 \to \mathcal{O} \xrightarrow{\varphi} \bigoplus_{j=0}^{n} \mathcal{O}(1) \xrightarrow{\psi} T_{hol}X \to 0$

$$\varphi : f \mapsto (z_0 f, \ldots, z_n f)$$
$$\psi : (s_0, \ldots, s_n) \mapsto s_0 \frac{\partial}{\partial z_0} + \cdots + s_n \frac{\partial}{\partial z_n}$$

[Here we think of $\mathcal{O}(1)$ as bundle of functions on $\mathbb{C}^{n+1}$, linear on each line thru 0. The RHS of ψ is a vector field on $\mathbb{C}^{n+1}$ invariant under $\mathbb{C}^\times$ action $\vec{z} \mapsto \lambda \vec{z}$ which thus gives a vector field on $\mathbb{CP}^n$ [exercise]]

Def X complex: canonical bundle $K(X) = \det(T^*X)$.

Prop For $X = \mathbb{CP}^n$, $K \simeq \mathcal{O}(-n-1)$.

Pf Given an exact seq $0 \to E \to F \to G \to 0$ of hol vb,

we have $\det(F) \simeq \det(E) \otimes \det(G)$. $\leftarrow$ [Exercise: boils down to $\det \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right] = \det A \det C$]

So $\det(\mathcal{O}) \otimes \det(T_{hol}X) \simeq \det(\mathcal{O}(1)^{\oplus n+1})$.

$$\det(T_{hol}X) \simeq \det(\mathcal{O}(1)^{\oplus n+1}).$$

Generally, given a local section s of $\mathcal{L}$, trivialize $\det(\mathcal{L}^{\oplus k})$ locally by $(s, 0, \ldots, 0) \wedge (0, s, \ldots, 0) \wedge \cdots \wedge (0, 0, \ldots, s)$

This gives an isomorphism $\det(\mathcal{L}^{\oplus k}) \simeq \mathcal{L}^{\otimes k}$.

So $\det(T_{hol}X) \simeq \mathcal{O}(n+1)$. ∎

Cor For $X = \mathbb{CP}^n$, $c_1(TX) = \mathcal{O}(n+1)$.

Pf Recall $c_1(TX) = c_1(\det TX)$. ∎

Alternatively: on $\mathbb{CP}^n$, choose the local patch U_0 and write $dw_1 \wedge \cdots \wedge dw_n$. This is a section of K over U_0. It extends globally to a meromorphic section, with a pole of order $(n+1)$ along the divisor $D = \{z_0 = 0\}$ i.e. $\mathbb{CP}^n \setminus U_0$.

This shows $K \otimes \mathcal{O}((n+1)D)$ is trivial. And $D = \mathrm{div}(z_0)$, z_0 a section of $\mathcal{O}(1)$, so $\mathcal{O}(D) = \mathcal{O}(1)$. So $K \simeq \mathcal{O}(-n-1)$.