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β-functions and the Exact Renormalization Group *

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Abstract

We relate β-functions to the flow of relevant couplings in the exact renormalization group. The specific case of a cutoff $\lambda\phi^4$ theory in four dimensions is discussed in detail. The underlying idea of convergence of the flow of effective lagrangians is developed to identify the β-functions. A perturbative calculation of the β-functions using the exact flow equations is sketched. The analysis may be extended to any system with a cutoff.

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1. Introduction

The renormalization group analysis is by now a classical tool for studying a variety of systems with many degrees of freedom such as quantum field theory, matter near a critical point, and turbulent fluids. In field theory, there are two distinguishable developments of the renormalization group.

The approach that is most popular was introduced by Gell-Mann and Low [1]. We will refer to it as ORG for old-fashioned renormalization group. In a given theory the bare Greens functions have infinities which arise from fluctuations of large momentum modes. These infinities can be absorbed into redefinitions of coupling constants and field normalizations. One way this is done[f1] is by choosing an arbitrary energy scale, μ, and requiring that appropriate Greens functions, evaluated at scale μ, have given finite values. Couplings and field normalizations are then functions of this scale. One then asks how these couplings change as μ is continuously varied. This information is valuable for many uses. For example, once this variation is known, perturbation theory can avoid potentially large logarithms by choosing μ equal to the energy scale of the process being calculated. Perhaps the most significant discovery to emerge from the ORG is the QCD is asymptotically free, thereby explaining the scaling that is observed in strong interactions.

A less understood approach to field theory which uses renormalization group ideas was pioneered by Wilson [2]. We refer to it as the exact RG. A theory (L_0, Λ_0) is defined by a Lagrangian, L_0, and some arbitrary momentum cutoff, Λ_0. Now take some different cutoff, say Λ, and ask what new Lagrangian, $L(\Lambda)$, will reproduce the same Greens functions that (L_0, Λ_0) generates. The differential change in $L(\Lambda)$ with Λ is known as the exact renormalization group equation. This equation then describes the flow of the effective Lagrangian. It is believed that a typical flow will rapidly converge to a finite dimensional submanifold of the space of Lagrangians. In perturbation theory, "typical" can be made more precise. Take, for example, a scalar field theory. If it is free then each coupling in the Lagrangian flows according to its canonical dimensions. Now, the exact RG equation describes infinitesimal changes in the cutoff; only modes with momentum in this small range are integrated out. Thus the qualitative features of the solutions of the exact RG equation are thought to vary smoothly as the interactions at scale Λ_0 are turned on. In particular, imagine a theory at Λ_0 such that all of its terms have coefficients equal to Λ_0^{4-d} times couplings of order 1 (d is the dimension of the term). Then as Λ is reduced we expect these couplings will collapse down to the relevant surface by powers of $(\Lambda/\Lambda_0)^{d-4}$. As Λ is further reduced the flow is determined entirely in terms of the values of the relevant, or renormalizable, couplings. This idea of flow convergence is the key to applications of the exact RG. For example, Polchinski [3] derived an exact RG equation for $\lambda\phi^4$ and used it to describe the flow precisely enough to give a short, simple and illuminating proof of

perturbative renormalizability. More recently, Warr [4] has extended this approach to prove renormalizability and unitarity of gauge theories.

Both RG approaches have been applied to string theory. Callan et. al. [5] make use of the ORG in their analysis of massless background field in 2-dimensional conformal field theory. Banks and Martinec [6], on the other hand, propose that an exact RG equation for a 2-dimensional field theory on the sphere be interpreted as the classical equation for the string field. Both approaches appear promising. The former yields, at lowest order in perturbation theory, classical equations for the massless background fields. From the latter approach, one can extract string S-matrix elements [7], and there is even hope that it will provide a useful framework for a string field theory.

It is natural, then, to wonder what relations exist between the ORG and the exact RG. The β-functions describe the flow of the relevant couplings once the Lagrangian has collapsed by power laws down to the relevant surface. The remaining flow on the surface proceeds more slowly.[f2] So it becomes a matter of matching definitions to identify conventional ORG β-functions in terms of the relevant coupling flows. The remainder of this paper is devoted to clarifying these ideas in the specific case of a cutoff $\lambda\phi^4$ theory in four dimensions. We use the notations of reference [3] hereafter. In the next section we briefly review [3] with emphasis given to those results which are directly relevant to our problem. This includes writing down the exact RG equation and restating the convergence of flow more precisely. An approximate flow equation for the theory's Greens functions is then presented. From it the ORG β-functions[f3] are identified in terms of the flow of relevant couplings in the exact RG. In section 3, the flow equations are solved perturbatively giving correctly the lowest order values of the β-functions. Appendix A presents lowest order calculations of γ_ϕ and γ_{m^2} from the expressions given in section 3. The paper concludes with observations on the calculations and summarizing remarks.

2. Exact RG and ORG β-functions

To make the above discussion more precise we study the $\lambda\phi^4$ model defined by the partition function

$$Z[J,\Lambda] = \int d\phi e^{\int \frac{d^4p}{(2\pi)^4}[-\frac{1}{2}\phi(p)\phi(-p)(p^2+m^2)K^{-1}(\frac{p^2}{\Lambda^2})+J(p)\phi(-p)]+L(\phi,\Lambda)} \tag{1}$$

$K(x)$ is a cutoff function which we require to be smooth and to satisfy

$$K(x) = \begin{cases} 1 & 0 \le x \ll 1 \\ 0 & 1 \ll x \end{cases} \tag{2}$$

We see from (1) that Λ is a momentum cutoff. If we impose

$$\Lambda \frac{d}{d\Lambda} L(\phi,\Lambda) = -\int d^4p \frac{(2\pi)^4}{2}(p^2+m^2)^{-1}\Lambda\frac{\partial}{\partial\Lambda}K\left(\frac{p^2}{\Lambda^2}\right) \left\{\frac{\delta L}{\delta\phi(p)}\frac{\delta L}{\delta\phi(-p)} + \frac{\delta^2 L}{\delta\phi(p)\delta\phi(-p)}\right\} \tag{3}$$

then, it is easy to prove that

$$\Lambda\frac{d}{d\Lambda}Z[J,\Lambda] = (\text{field independent function of } \Lambda) \times Z[J,\Lambda] \tag{4}$$

If the interacting Lagrangian, $L(\phi,\Lambda)$, flows according to (3) then the Greens functions derived from $Z[J,\Lambda]$ are independent of Λ. As Λ is lowered, modes no longer propagate. Thus, if the physics is to remain unchanged then the interacting Lagrangian must change correspondingly; equation (3) effectively integrates out modes. Refer to [3] for a simple graphical interpretation of the two terms on the right hand side of (3).

Equation (3) is one example of an exact RG equation. Even if L is simple at some scale Λ_0, the L that results from it at lower scales will be complicated. Indeed, we expect all possible terms to appear in $L(\phi,\Lambda)$ that are allowed by the symmetries of the system. Nevertheless, it is believed that as Λ is made small enough the flow of $L(\phi,\Lambda)$ will converge to a finite dimensional subspace of the space of all lagrangians. The number of dimensions is equal to the number of relevant and marginal operators[f4] in the theory. In perturbation theory these are simply identified by power counting.

Consider the example of $\lambda\phi^4$ described above. Expand $L(\Lambda)$ as

$$L(\phi,\Lambda) = \sum_{m=1}^{\infty}\frac{1}{(2m)!}\Lambda^{4-2m}\int\frac{d^4p_1\cdots d^4p_{2m}}{(2\pi)^{8m-4}}\phi(p_1)\cdots\phi(p_{2m}) \delta\left(\sum_{i-1}^{2m}p_i\right)A_{2m}(p_1,\cdots,p_{2m},\Lambda) \tag{5}$$

Inspection of the exponent of Λ in (5) for each term suggests

$$\begin{aligned} \rho_1(\Lambda) &\equiv -\Lambda^2 A_2(0,0,\Lambda) \\ \rho_2(\Lambda) &\equiv -\frac{1}{8}\Lambda^2 \frac{\partial^2}{\partial p^2} A_2(p,-p,\Lambda)\Big|_{p=0} \\ \rho_3(\Lambda) &\equiv -A_4(0,0,0,0,\Lambda) \end{aligned} \tag{6}$$

are the relevant couplings in this theory. Roughly, $\rho_1(\Lambda)$ contributes to a running mass shift, $\rho_2(\Lambda)$ corresponds to a field renormalization, and $\rho_3(\Lambda)$ is the coupling constant at scale Λ. Inserting (5) into (3) yields

$$(\Lambda\frac{\partial}{\partial\Lambda} + 4 - 2m)A_{2m}(p_1,\cdots,p_{2m},\Lambda) = -\sum_{\ell=1}^{m}\Big\{\frac{1}{P^2+m^2}\Lambda^3\frac{\partial}{\partial\Lambda}K\left(\frac{P^2}{\Lambda^2}\right)$$

$$A_{2\ell}(p_1,\cdots,p_{2\ell-1}P,\Lambda)\times A_{2m+2-2\ell}(p_{2\ell},\cdots,p_{2m},-P,\Lambda)+\left[\frac{1}{2}\binom{2m}{2\ell-1}-1\right]\text{perms}\Big\}$$

$$-\frac{1}{2}\int\frac{d^4p}{(2\pi\Lambda)^4}A_{2m+2}(p_1,\cdots,p_{2m},p,-p,\Lambda)\frac{1}{p^2+m^2}\Lambda^3\frac{\partial}{\partial\Lambda}K\left(\frac{p^2}{\Lambda^2}\right) \tag{7}$$

where $P = \sum_{i=1}^{2\ell-1} p_i$. Using flow equations similar to and including (7), Polchinski proved perturbative renormalizability of $\lambda\phi^4$. Refer to [3] for details and a precise statement of the results. The analysis proves the following. Take two scales $\Lambda_R \ll \Lambda_0$. The theory is initially defined at scale Λ_0. Assume it is natural in the sense that all the dimensionless couplings, $A_{2m}(\Lambda_0)$ are of order unity. Beyond this, imagine that all we know of the theory are the values of the relevant couplings at the scale Λ_R– that is, $\rho_i(\Lambda_R)$. By suitable redefinitions of the squared mass and of the scale of

the field we may take

$$\begin{aligned} \rho_1(\Lambda_R) &= 0 \\ \rho_2(\Lambda_R) &= 0 \end{aligned} \tag{8}$$

and write

$$\rho_3(\Lambda_R) = \lambda \tag{9}$$

Then, using the flow equations, the couplings $A_{2m}(\Lambda)$ at any intermediate scale $\Lambda_R \ll \Lambda \ll \Lambda_0$ may be solved perturbatively in terms of a power series in λ. The whole flow of the effective Lagrangian may thus be reconstructed. At any finite order in this perturbation expansion the error is of order $(\frac{\Lambda}{\Lambda_0})^2 \times$ finite order polynomial in $\ell n(\Lambda_0/\Lambda)$. This arises from ignorance of the values of the irrelevant couplings at scale Λ_0.

Now we relate this exact RG development to the ORG and write β-functions in terms of the flow of the relevant couplings. The Greens functions, denoted G_N, may be computed at any scale $\Lambda \ll \Lambda_0$ in terms of the relevant couplings $\rho_i(\Lambda)$ at that scale. We know that

$$G_N(\rho(\Lambda),\Lambda) \tag{10}$$

is Λ independent. Therefore

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \Lambda\frac{\partial\rho_i}{\partial\Lambda}\frac{\partial}{\partial\rho_i}\right)G_N = 0 \tag{11}$$

Note (11) includes the flows of ρ_1 and ρ_2. In order to identify conventional ORG β-functions, we must shuffle these flows into running redefinitions of the free propagator pole and the field normalization.

Consider the evolution of ρ_2 first. Upon rescaling, $\phi = a\phi'$, gives

$$G_N(\rho) = a^N G_N(\rho') \tag{12}$$

where

$$\begin{aligned} \rho_1' &= a^2\rho_1 + (a^2-1)m^2 \\ \rho_2' &= a^2\rho_2 + (a^2-1) \\ \rho_3' &= a^4\rho_3 \end{aligned} \tag{13}$$

Therefore

$$\begin{aligned} 0 &= a\frac{d}{da}(a^N G_N(\rho'))\Big|_{a=0} \\ &= \left[N + 4\rho_3\frac{\partial}{\partial\rho_3} + 2(\rho_2+1)\frac{\partial}{\partial\rho_2} + 2(\rho_1+m^2)\frac{\partial}{\partial\rho_1}\right] G_N(\rho) \end{aligned} \tag{14}$$

Equation (14) may be used to eliminate the flow of ρ_2 in terms of the overall rescaling of G_N. Multiplying (14) by $\frac{1}{2(\rho_2+1)}\Lambda\frac{\partial\rho_2}{\partial\Lambda}$ and subtracting it from (11) implies

$$\begin{aligned} 0 = \Bigg[\Lambda\frac{\partial}{\partial\Lambda} &+ \left[\Lambda\frac{\partial\rho_3}{\partial\Lambda} - \frac{2\rho_3}{\rho_2+1}\Lambda\frac{\partial\rho_2}{\partial\Lambda}\right]\frac{\partial}{\partial\rho_3} + \left[\Lambda\frac{\partial\rho_1}{\partial\Lambda} - \frac{m^2(\Lambda)}{\rho_2+1}\Lambda\frac{\partial\rho_2}{\partial\Lambda}\right]\frac{\partial}{\partial m^2(\Lambda)} \\ &- N\frac{1}{2(\rho_2+1)}\Lambda\frac{\partial\rho_2}{\partial\Lambda}\Bigg] G_N(\rho) \end{aligned} \tag{15}$$

where

$$m^2(\Lambda) = m^2 + \rho_1(\Lambda) \tag{16}$$

The flow equation is now in the form typical of ORG flow equations and we may tentatively identify the β-functions in terms of the relevant couplings

$$\begin{aligned} \beta &= \Lambda\frac{\partial\rho_3}{\partial\Lambda} - 2\frac{\rho_3}{\rho_2+1}\Lambda\frac{\partial\rho_2}{\partial\Lambda} \\ \gamma_{m^2} &= -\frac{1}{m^2}\left[\Lambda\frac{\partial\rho_1}{\partial\Lambda} - \frac{m^2(\Lambda)}{\rho_2+1}\Lambda\frac{\partial\rho_2}{\partial\Lambda}\right] \\ \gamma_\phi &= -\frac{1}{\rho_2+1}\Lambda\frac{\partial\rho_2}{\partial\Lambda} \end{aligned} \tag{17}$$

Definitions of ORG β-functions vary even within a given regularization scheme. We chose the above ones in order to compare the lowest order values with those written in Collins' book [8].

The above renormalization scheme, with momentum integral cutoff Λ and general interacting Lagrangian $L(\Lambda)$ is very different from the mass independent subtraction used in [8]. For example, there is no simple relation between Λ and the arbitrary renormalization scale, μ, of the ORG. In both cases it is convenient to equate them with a typical energy scale of the problem being studied. Indeed, μ is analogous to the cutoff Λ in that it effectively shuffles the effects of fluctuations above scale μ into the renormalization of the relevant couplings. Both schemes make it possible to focus attention on fluctuations at energies less than any given arbitrary scale. But the method of focus differs very much in detail. Nevertheless, general arguments [9],[8] suggest that in the limit $m^2 \to 0$ and $\Lambda_0 \to \infty$

the leading terms in λ of β and γ_ϕ should be the same for both schemes. γ_{m^2} differs in the two schemes, due to the quadratic divergence in the self energy of the scalar field. In a cutoff scheme we expect

$$m^2_{\rm bare} = m^2 Z_m + \Lambda_0^2 Y \tag{18}$$

The additive renormalization, $\Lambda_0^2 Y$, requires additional treatment and is beyond the scope of the usual scheme independence arguments. This is discussed further in the Appendix. Referring to [8] for ORG results, we expect that a perturbative calculation based on (7) and (17) will yield the following lowest order results

$$\begin{aligned} \beta &= \frac{3\lambda^2}{16\pi^2} + 0(\lambda^3) \\ \gamma_\phi &= \frac{\lambda^2}{6}\frac{1}{(16\pi^2)^2} + 0(\lambda^3) \end{aligned} \tag{19}$$

This calculation is outlined in the next section and some additional details are given in the Appendix.

3. Sample Calculation

Renormalizability implies that the theory at Λ_0 may be taken to be

$$\begin{aligned} A_2(p,-p,\Lambda_0) &= -\frac{\rho_1^0}{\Lambda_0^2} - \frac{p^2}{\Lambda_0^2}\rho_2^0 \\ A_4(p_1,p_2,p_3,p_4,\Lambda_0) &= -\rho_3^0 \\ A_{2m}(\Lambda_0) &= 0 \qquad m \geq 3 \end{aligned} \tag{20}$$

Then the flow equations (7) together with the renormalization conditions

$$\begin{aligned} \rho_1(\Lambda_R) &= 0 = \rho_2(\Lambda_R) \\ \rho_3(\Lambda_R) &= \lambda \end{aligned} \tag{21}$$

define, by iterations in λ, a perturbation theory. More specifically, expanding the $A_{2m}(\Lambda)$ in a power series in Λ yields solutions for the coefficients of each power. The leading powers for the $A_{2m}(\Lambda)$ are of the following orders

$$\begin{aligned} A_2(\Lambda) &\sim \lambda \\ A_4(\Lambda) &\sim \lambda \\ A_{2m}(\Lambda) &\sim \lambda^{m-1} \qquad m \geq 3 \end{aligned} \tag{22}$$

It is well know that field renormalization in $\lambda\phi^4$ first occurs at the two-loop order. Thus $\rho_2(\Lambda) \sim \lambda^2$. Therefore to compute the lowest order terms in the β-functions, (22) implies we need to solve the flow equations for A_2, A_4 and A_6.

The method of solution of (7) is straightfoward but unwieldly. Even at low orders the results for the $A_{2m}(\Lambda)$ are complicated and depend sensitively on the form of the cutoff function $K(x)$. For example

$$
\begin{aligned}
A_4(p_1,p_2,p_3,p_4,\Lambda) &= -\lambda + \lambda^2\Bigg\{\frac{1}{2}\sum_{i=1}^{4}\frac{1}{p_i^2+m^2} \\
\times\left[K\left(\frac{p^2}{\Lambda^2}\right)\right. &- \left.K\left(\frac{p^2}{\Lambda_0^2}\right)\right]\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\left[K\left(\frac{q^2}{\Lambda^2}\right)-K\left(\frac{q^2}{\Lambda_R^2}\right)\right] \\
&+ \frac{1}{2}\sum_{i<j}\int\frac{d^4q}{(2\pi)^4}\frac{1}{(p_i+p_j+q)^2+m^2}\frac{1}{q^2+m^2}\int_\Lambda^{\Lambda_0}d\Lambda' \\
&\times \frac{\partial}{\partial\Lambda'}K\left(\frac{q^2}{\Lambda'^2}\right)\left[K\left(\frac{(p_i+p_i+q)^2}{\Lambda_0^2}\right)-K\left(\frac{(p_i+p_i+q)^2}{\Lambda'^2}\right)\right] \\
&- \frac{3}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\left[K\left(\frac{q^2}{\Lambda_R^2}\right)-K\left(\frac{q^2}{\Lambda_0^2}\right)\right]\Bigg\}+0(\lambda^3)
\end{aligned}
\tag{23}
$$

However, we only need the functions $A_2(0,0,\Lambda)$, $\frac{\partial^2}{\partial p^2}A_2(p,-p,\Lambda)|_{p=0}$ and $A_4(0,0,0,\Lambda)$. We expect that in the limit $m^2\to 0, \Lambda_0\to\infty$ these are independent of the precise form of $K(x)$. The arguments for scheme independence of the leading coefficients of the ORG β-functions, and for the convergence of flow in the exact RG could be used to simplify the calculation. They point out those parts of the flow equation solutions that determine the β-functions. For example, the initial conditions, (20), are irrelevant to the calculation. We choose to retain all the terms because the calculation is straightfoward at lowest order and the results illustrate some of the above general arguments about the flow.

Equations (7), (18), and (20) through (22) imply

$$
\begin{aligned}
\beta &= -\Lambda\frac{\partial}{\partial\Lambda}A_4(0,0,0,0,\Lambda)+0(\lambda^3) \\
&= -\Lambda\frac{\partial}{\partial\Lambda}\left[3\lambda^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)^2}\right. \\
&\quad\times\left.\int_\Lambda^{\Lambda_0}d\Lambda'\left[K\left(\frac{q^2}{\Lambda_0^2}\right)-K\left(\frac{q^2}{\Lambda'^2}\right)\right]\frac{\partial}{\partial\Lambda'}K\left(\frac{q^2}{\Lambda'^2}\right)\right]+0(\lambda^3) \\
\gamma_{m^2} &= -\frac{1}{m^2}\Lambda\frac{\partial}{\partial\Lambda}\left[-\Lambda^2A_2(0,0,\Lambda)\right]+0(\lambda^2) \\
&= \Lambda\frac{\partial}{\partial\Lambda}\frac{\lambda}{2m^2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\left[K\left(\frac{q^2}{\Lambda^2}\right)-K\left(\frac{q^2}{\Lambda_R^2}\right)\right]+0(\lambda^2) \\
\gamma_\phi &= -\Lambda\frac{\partial}{\partial\Lambda}\left[-\frac{1}{8}\Lambda^2\frac{\partial^2}{\partial p^2}A_2(p,-p,\Lambda)\Big|_{p=0}\right]+0(\lambda^3) \\
&= -\frac{\lambda^2}{8}\Lambda\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\frac{\partial}{\partial\Lambda}K\left(\frac{q^2}{\Lambda^2}\right)\int\frac{d^4r}{(2\pi)^4}\frac{1}{r^2+m^2} \\
\times\int_\Lambda^{\Lambda_0}d\Lambda'\frac{\partial}{\partial\Lambda'}K\left(\frac{r^2}{\Lambda'^2}\right)&\frac{\partial^2}{\partial p^2}\frac{1}{(p+q+r)^2}\left[K\left(\frac{(p+q+r)^2}{\Lambda_0^2}\right)-K\left(\frac{(p+q+r)^2}{\Lambda'^2}\right)\right]\Bigg|_{p=0}
\end{aligned}
\tag{24}
$$

The calculations of γ_{m^2} and γ_ϕ from the expressions in (24) are presented in the Appendix. In the limit $m^2\to 0, \Lambda_0\to\infty$ the answers agree with (19), independent of the precise form of $K(x)$. The only feature of $K(x)$ that is used is $K(0)=1$ and $K(\infty)=0$. This is simply illustrated below in the calculation of β.

$$
\begin{aligned}
\beta &= 3\lambda^2\frac{1}{(2\pi)^4}\int d^4q\frac{1}{q^4}\left(-\frac{1}{2}\right)\Lambda\frac{\partial}{\partial\Lambda}\left[K\left(\frac{q^2}{\Lambda^2}\right)-1\right]^2 \\
&= \frac{3\lambda^2}{(2\pi)^4}\pi^2\int_0^\infty dq\frac{\partial}{\partial q}\left[K\left(\frac{q^2}{\Lambda^2}\right)-1\right]^2 \\
&= \frac{3\lambda^2}{16\pi^2}
\end{aligned}
\tag{25}
$$

4. Conclusions

ORG β-functions are simply related to the flow of the relevant couplings ρ_i. The basic idea is that, as the cutoff is reduced, the effective Lagrangian flow rapidly converges to a submanifold parametrized by the values of the ρ_i. The flow down to this relevant surface depends sensitively on the initial Lagrangian, whereas the remaining flow on the surface is a universal property of the theory. The irrelevant couplings do not vanish at lower scales. Rather, the contributions to their values from the nonrenormalizable couplings in $L(\Lambda_0)$ are suppressed by powers of Λ/Λ_0. Thus the flows of the ρ_i determine everything. The β-functions describe this flow.

Such observations can be used to simplify calculations with the flow equations. At lowest order, the calculations are no more difficult than in the ORG. However, calculating the next-to-leading terms directly from the flow is very tedious. The flow equations provide a simple route to prove renormalizability and explain why renormalization works. But they are awkward when performing detailed calculations. In contrast, the usual Feynman graph calculations in ORG are simple, but renormalizability appears as a combinatoral miracle.

In the Introduction we claimed that the flow on the relevant surface proceeds slowly. We have also not distinguished between relevant and marginal couplings. Both of these things reflect a particle physics bias. Particle masses have given fixed values. To account for them in a field theory, en-

visioned to have a cutoff at the Planck mass, the corresponding couplings of positive canonical dimension must be fine tuned at the much lower scale Λ_R. This is a nuisance, but once it is done the flow at scales around Λ_R proceeds more slowly. More precisely, the relevant couplings, with their canonical $(\Lambda_R/\Lambda)^2$ dependence on scale Λ, will collapse down arbitrarily close to the critical surface as Λ is increased above Λ_R. Thereafter, the flow creeps along until the irrelevant couplings begin to contribute, forcing $L(\Lambda)$ away from the relevant surface. In this scenario, the distinction between relevant and marginal is moot.

In the theory of critical phenomena the situation is quite different. The relevant couplings at the original lattice scale Λ_0 are taken to be smooth functions of the temperature difference $T - T_c$, where T_c is the critical temperature. This difference, unlike particle masses, may be continuously varied in experiments. Fine tuning is no longer an issue. Instead, the full range of RG flows needs to be sampled. For these flows the relevant couplings diverge as Λ is reduced like $(\Lambda_0/\Lambda)^2$. This fact then explains the divergence of the correlation length as T approaches T_c, and so admits a calculation of critical exponents.

The flow equations are quadratic in the couplings, but the flow of the irrelevant couplings feeds into the flow of the ρ_i, and vice versa. To solve the flow, the irrelevant couplings are expanded in terms of the ρ_i thereby defining a perturbation. This procedure was carried out in detail for the specific case of a $\lambda\phi^4$ theory in four dimensions. Of course, it applies to more general field theories. Consider string theory at the (string-) tree level[/5]. The "coupling constants" are reinterpreted as background fields corresponding to expectation values of the various string modes. The relations between these fields implied by the exact RG are taken to be the classical equations of motion. They are quadratic in the fields. The naively relevant couplings include the tachyon and the massless fields. Just as in the case of $\lambda\phi^4$ we imagine using the massive-field equations to expand the massive (irrelevant) fields in terms of the relevant ones. This is then inserted into the massless-field equations which will then be written entirely in terms of the massless fields. Expanding these about small momentum should be equivalent to the results on [5]. This sensible procedure of integrating out the massive fields is just that outlined in [6].

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Appendix

We evaluate the expressions for γ_ϕ and γ_{m^2} in (24). Begin with γ_ϕ. It is a two loop expression and is considerable more complicated than those

for β and γ_{m^2}. Nevertheless in the limit $m^2 \to 0$ and $\Lambda_0 \to \infty$ it is $K(x)$-independent just like the lowest order expression for β is. In this limit (24) becomes

$$\lambda_\phi = -\frac{\lambda^2}{8}\Lambda \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2}\frac{\partial}{\partial\Lambda}K\left(\frac{q^2}{\Lambda^2}\right)\int\frac{d^4r}{(2\pi)^4}\frac{1}{r^2}\int_\Lambda^\infty d\Lambda'\frac{\partial}{\partial\Lambda'}K\left(\frac{r^2}{\Lambda'^2}\right)$$
$$\Box_p\frac{1}{(p+q+r)^2}\left[1-K\left(\frac{(p+q+r)^2}{\Lambda'^2}\right)\right]\Bigg|_{p=0} \qquad (A1)$$

Use

$$\frac{\partial}{\partial\Lambda}K\left(\frac{q^2}{\Lambda^2}\right) = -\frac{2q^2}{\Lambda^3}K'\left(\frac{q^2}{\Lambda^2}\right) \qquad (A2)$$

and rescale the momentum $q^\mu \to \frac{q^\mu}{\Lambda'}$ to rewrite (A1) as

$$\gamma_\phi = -\frac{\lambda^2}{2}\frac{1}{(2\pi)^8}\frac{1}{\Lambda^2}\int_\Lambda^\infty d\Lambda'\Lambda'\int d^4qK'\left(\left(\frac{\Lambda'}{\Lambda}\right)^2 q^2\right)\int d^4qK'(r^2)$$
$$\times\Box_p\frac{1}{(p+q+r)^2}\left[1-K((p+q+r)^2)\right]\Big|_{p=0} \qquad (A3)$$

Now

$$\int_\Lambda^\infty d\Lambda'\Lambda'K'\left(\left(\frac{\Lambda'}{\Lambda}\right)^2 q^2\right) = \frac{1}{2}\frac{\Lambda^2}{q^2}\int_{q^2}^\infty dy\frac{\partial}{\partial y}K(y) = -\frac{1}{2}\frac{\Lambda^2}{q^2}K(q^2) \qquad (A4)$$

so

$$\gamma_\phi = \frac{\lambda^2}{4}\frac{1}{(2\pi)^8}\int d^4q\frac{1}{q^2}K(q^2)\int d^4rK'(r^2)$$
$$\Box_p\frac{1}{(p+q+r)^2}\times\left[1-K((p+q+r)^2)\right]\Big|_{p=0} \qquad (A5)$$

Next observe

$$\begin{aligned}
&\Box_p\frac{1}{(p+q+r)^2}\left[1-K((p+q+r)^2)\right]\Big|_{p=0}\\
=&\left\{\Box_p\frac{1}{(p+q+r))^2}\Big|_{p=0}\right\}\left[1-K((q+r)^2)\right]-4K''((q+r)^2)\\
=&-4K''((q+r)^2)
\end{aligned} \qquad (A6)$$

since $K(0)=1$. Therefore

$$\gamma_\phi = -\frac{\lambda^2}{(2\pi)^8}\int d^4q\,d^4r\frac{1}{q^2}K(q^2)K'(r^2)K''((q+r)^2) \qquad (A7)$$

Now Fourier transform as in

$$K(q^2) = \int_{-\infty}^{\infty}dx\;\; e^{ixq^2}L(x) \qquad (A8)$$

so

$$\gamma_\phi = \lambda^2\frac{1}{(2\pi)^8}\int dxdydz(-iyz^2)\int d^4qd^4r\frac{1}{q^2}e^{ixq^2+iyr^2+iz(q+r)^2}L(x)L(y)L(z) \qquad (A9)$$

The integrals over q^μ and r^μ yield

$$\gamma_\phi = \frac{\lambda^2}{(2\pi)^8}\pi^4\int dxdydz\;yz^2\frac{1}{(y+z)(xy+xz+zy)}L(x)L(y)L(z) \qquad (A10)$$

Upon symmetrizing the integrand in x,y,z and using

$$\begin{aligned}
\int_{-\infty}^{\infty}dxL(x) &= K(0)\\
&= 1
\end{aligned} \qquad (A11)$$

we see

$$\begin{aligned}
\gamma_\phi &= \frac{\lambda^2}{6}\frac{1}{(16\pi^2)^2}\left[\int_{-\infty}^{\infty}dxL(x)\right]^3\\
&= \frac{\lambda^2}{6}\frac{1}{(16\pi^2)^2}
\end{aligned} \qquad (A12)$$

Next consider γ_{m^2}. Observe that in the limit $m^2\to 0$

$$\gamma_{m^2} = \frac{\lambda^2}{16\pi^2m^2}\int_0^\infty dq\frac{q^3}{q^2+m^2}\Lambda\frac{\partial}{\partial\Lambda}K\left(\frac{q^2}{\Lambda^2}\right) \qquad (A13)$$

diverges. Of course, only the product $m^2\gamma_{m^2}$ appears in the ORG flow equation for the Greens functions (15), and this is finite but nonzero at $m^2=0$. This is due to the Λ_0^2Y term in (18). Using dimensional regularization this divergence is absent. But in other schemes it is present and must be dealt with [8]. If we impose

$$\Lambda_R\frac{\partial}{\partial\Lambda_R}Y = 0 \qquad (A14)$$

then γ_{m^2} is finite as $m^2\to 0$ and the ORG flow equation has the same form as in dimensionally regularized schemes. At lowest order in λ this amounts to ignoring the term proportional to Λ^2/m^2 in (A13). To isolate it expand

$$\frac{1}{q^2+m^2} = \frac{1}{q^2}\sum_{n=0}^{\infty}\left(-\frac{m^2}{q^2}\right)^n \qquad (A15)$$

inserting (A15) into (A13) gives

$$\gamma_{m^2} = -\frac{\lambda}{16\pi^2 m^2}\int_0^\infty dq\, q^2 \frac{\partial}{\partial q}K\left(\frac{q^2}{\Lambda^2}\right) - \frac{\lambda}{16\pi^2}m^2\int_0^\infty dq \frac{1}{q^2}\sum_{n=0}^{\infty}\left(-\frac{m^2}{q^2}\right)^n \frac{\partial}{\partial q}K\left(\frac{q^2}{\Lambda^2}\right)$$

$$\frac{\lambda}{16\pi^2}\int_0^\infty dq \frac{\partial}{\partial q}K\left(\frac{q^2}{\Lambda^2}\right) \qquad (A16)$$

The first term diverges as $m^2 \to 0$. Equation (A14) is equivalent to choosing a cutoff function such that

$$\int_0^\infty dq q^2 \frac{\partial}{\partial q}K\left(\frac{q^2}{\Lambda^2}\right) = 0 \qquad (A17)$$

The second term vanishes as $m^2 \to 0$ so that all that remains in this limit is the third term. Note that it only depends on the values $K(0)$ and $K(\infty)$ since

$$\begin{aligned}\gamma_{m^2} &= \frac{\lambda}{16\pi^2}[K(\infty) - K(0)] \\ &= -\frac{\lambda}{16\pi^2}\end{aligned} \qquad (A18)$$

This is the lowest order value for γ_{m^2} found in [8].

Footnotes

f1. ORG refers to a set of ideas that can be implemented in a broad class of renormalization schemes. We choose a mass independent scheme to exemplify the ideas. Similarly, we refer to a particular realization of the exact RG.

f2. The mass term coupling flows canonically as $(\Lambda_R/\Lambda)^2$. By slowly we refer to the additional quantum corrections to this canonical flow.

f3. The name β-functions is meant to include the usual β-functions, anomalous dimensions of the fields and the anomalous flow of masses. For the $\lambda\phi^4$ we thus refer to β, γ_ϕ and γ_{m^2}.

f4. From now on both relevant and marginal will be referred to as relevant.

f5. In [6] the cutoff is not just reduced. Instead, scale invariance is demanded of the theory. Thus, in place of the exact RG equation, the equation involves an additional term which counts the canonical dimensions. For massless fields this term vanishes so that in [5] scale invariance implies $\beta = 0$.

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