Coordinate computations

TM = tangent bundle to M. $\mathcal{T}(M) = C^\infty(TM) = \{\text{vector fields on } M\}$

$\mathcal{T}^k_\ell(M) = C^\infty(TM^{\otimes \ell} \otimes T^*M^{\otimes k}) = \{(k,\ell)\text{-tensors on } M\}$

Given any basis $\{e_i\}_{i=1}^n$ of sections of TM, we may expand any vector field $X \in \mathcal{T}(M) = \mathcal{T}^1(M)$: $X = \sum_{i=1}^n X^i e_i = X^i e_i$ $\quad X^i \in C^\infty(M)$

or tensor field $\omega \in \mathcal{T}^k(M)$: $\quad \omega = \omega^{i_1 i_2 \cdots i_k}(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) \quad \omega^{i_1 \cdots i_k} \in C^\infty(M)$

The basis $\{e_i\}$ also determines canonically a dual basis $\{e^j\}$ of sections of T^*M (characterized by $e^j \cdot e_i = \delta_i^j$)

so we can also expand covector fields $\omega \in \mathcal{T}_1(M)$: $\omega = \omega_j e^j$

and most generally tensors $\eta \in \mathcal{T}^k_\ell(M)$: $\quad \eta = \eta^{i_1 \cdots i_k}_{j_1 \cdots j_\ell}(e_{i_1} \otimes \cdots \otimes e_{i_k})(e^{j_1} \otimes \cdots \otimes e^{j_\ell})$

- Pairing between vectors and covectors: if $\omega \in \mathcal{T}^1(M)$, $X \in \mathcal{T}_1 M = TM$ then

  $\omega(X) = \omega_i X^i \qquad [\omega \cdot X = (\omega_i e^i)(X^j e_j) = (\omega_i X^j)\, e^i(e_j) = (\omega_i X^j)(\delta^i_j) = \omega_i X^i]$

- The contraction of a vector field X with an n-tensor $\omega \in \mathcal{T}^n(M)$ is $\iota_X \omega \in \mathcal{T}^{n-1}(M)$, given in any frame by

  $$(\iota_X \omega)_{i_1 \cdots i_{k-1}} = X^i \omega_{i i_1 \cdots i_{k-1}}$$

- Traces: $\omega \in \mathcal{T}_1^{\,1}(M) = C^\infty(\mathrm{End}\, TM)$ has $\mathrm{Tr}(\omega) = \omega_i^i$

Change of basis: if $\quad e_{i'} = e_i C^i_{\ i'} \qquad C^i_{\ i'} \in C^\infty(M)$

then we can relate the expansion of η in the bases e_i and e_i':

$$\eta_{i'} = \eta_i C^i_{\ i'} \qquad X^{i'} = X^i (C^{-1})^{i'}_{\ i}$$

$$\eta^{i_1' \cdots i_k'}_{j_1' \cdots j_\ell'} = \eta^{i_1 \cdots i_k}_{j_1 \cdots j_\ell}\left((C^{-1})^{i_1'}_{i_1} (C^{-1})^{i_2'}_{i_2} \cdots (C^{-1})^{i_k'}_{i_k}\right)\left(C^{j_1}_{j_1'} \cdots C^{j_\ell}_{j_\ell'}\right)$$

An especially convenient choice: pick a local coord sys x^i and take $e_i = \partial/\partial x^i$. ("coordinate frame")

On a change of coords to $x^{i'}$ you get $C^i_{\ i'} = \dfrac{\partial x^i}{\partial x^{i'}}$

Ex On $M = \mathbb{R}^2$, take 2 coord sys:

Cartesian (x^1, x^2) Polar $(x^{1'} = r,\ x^{2'} = \theta)$ related by $x^1 = r\cos\theta$, $x^2 = r\sin\theta$

The 1-form $\eta = dx^1$ has expansion coeff. $\eta_1 = 1, \eta_2 = 0$

also $\eta = d(r\cos\theta) = \cos\theta\, dr - r\sin\theta\, d\theta$ so $\eta_{1'} = \cos\theta$, $\eta_{2'} = -r\sin\theta$

[Could also get this from our general rule:

$$\begin{pmatrix} C^1_{\ 1'} & C^2_{\ 1'} \\ C^1_{\ 2'} & C^2_{\ 2'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ r\sin\theta & -r\cos\theta \end{pmatrix}$$ so $\eta_{1'} = \eta_1 C^1_{\ 1'} + \eta_2 C^2_{\ 1'} = 1\cdot\cos\theta + 0 = \cos\theta$

NB: This example already shows that there is no coord-inv't notion of a "constant 1-form": the components may be constant in one coord sys but not in another!

Coordinate formulas:

- Action of a vector field $X \in \mathcal{T}(M)$ on a function $f: M \to \mathbb{R}$ is given in a coordinate frame by

$$Xf = X^i \partial_i f$$

- Recall: the differential of a function $f: M \to \mathbb{R}$ is a 1-form $df \in \Omega^1(M)$ determined by

$$df(X) = X(f) \quad \text{for every } X \in \mathcal{T}(M)$$

It also has the representation

$$df = \sum_{i=1}^{n} \partial_i f\, dx^i$$

ie, in a coord frame, $(df)_i = \dfrac{\partial f}{\partial x^i} = \partial_i f$

- The Lie bracket of 2 vector fields $X, Y \in T(M)$ is $[X,Y] \in T(M)$ defined by

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$

It's given in a coord. frame by

$$[X,Y]^i = X^j \partial_j Y^i - Y^j \partial_j X^i$$

- The differential of a 1-form $\omega \in \Omega^1(M)$ is $d\omega \in \Omega^2(M)$ defined by

$$(d\omega)(X,Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X,Y])$$

[Why does this formula actually define an element of $\Omega^2(M)$? *A priori* the RHS looks like just a map $TM \times TM \longrightarrow C^\infty(M)$. To see it actually comes from an elt in $\Omega^2(M)$, need linearity principle: for any $f \in C^\infty(M)$,

$$d\omega(fX, Y) = fX[\omega(Y)] - Y[\omega(fX)] - \omega([fX, Y])$$
$$= \cdots$$
$$= f \cdot d\omega(X,Y)$$
]

In coord frame, $(d\omega)_{ij} = \partial_i \omega_j - \partial_j \omega_i$

[Pf: $(d\omega)_{ij} = d\omega(\partial_i, \partial_j) = \partial_i(\omega(\partial_j)) - \partial_j(\omega(\partial_i)) - \omega(\cancel{[\partial_i, \partial_j]}^{\,0})$

$$= \partial_i \omega_j - \partial_j \omega_i$$
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