Basic constructions with Riemannian metrics

Say (M,g) Riemannian.

① Induced map "flat": $TM \to T^*M$, $X \mapsto X^\flat$

by $X^\flat(Y) = g(X,Y)$ i.e. $(X^\flat)_i Y^i = g_{ij} X^j Y^i$

$$(X^\flat)_i = g_{ij} X^j$$

Inverse map: "sharp": $T^*M \to TM$

$(\omega^\#)^i = g^{ij}\omega_j$ where g^{ij} is short for $(g^{-1})^{ij}$

So, "on a Riemannian manifold vectors and covectors are the same."

Can use this e.g. to define gradient: $\text{grad}\, f = (df)^\#$
(direction of fastest decrease per unit length)

(And more generally all T^k_l with fixed $k+l$ are the same, e.g.
$B_{ijk} \in T^3_0 \longleftrightarrow B_i{}^l{}_k = B_{ijk}\, g^{jl} \in T^2_1$)

② All $T^k_l M$ get induced inner products. Characterized by:
If $(E_1, \dots, E_n)$ is an ON-basis for TM
then $\otimes$ products of the E_i and E^i give ON bases for $T^k_l M$.

③ In p^{te}, $\Lambda^n T^*M$ has 2 elements of norm 1. If M oriented, can single one out:
volume form $dV \in \Omega(M)$, characterized by $dV(E_1, \dots, E_n) = 1$
if $(E_1, \dots, E_n)$ is an oriented orthonormal basis for T_pM.
If M compact oriented, define $\text{vol}(M) = \int_M dV$.